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The Reliability Analysis of the Process Loss Index Based on Control Chart Samples

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***Abstract*—Process loss index, L_e, provides measures to determine the quality performance of a process. In real situations where the actual value of L_e is generally unknown one may estimate it by its corresponding sample counterparts. Most of the results obtained regarding the distributional and inferential properties of estimated expected loss indices were based on one single sample. In practice, however, process information is often derived from multiple samples rather than from one single sample. In this paper, we first investigate the relationship between L_e and process yield, and the distributional and inferential properties of the estimator of process loss index based on $\bar{X}$ and R control chart samples. We then investigate the performance of the estimator of L_e, based on the absolute relative bias, the relative mean square error and the α-level confidence relative error for various combinations of sample size, and implement the hypothesis testing procedure. The developed step-by-step procedure for practitioner to use in determining whether the given process is capable, then the decision making will be reliable. The technique provided in this paper will be applicable when the process measurements are taken from $\bar{X}$ and R control chart.**

***Index Terms*—Process expected loss, Process yield, Non-central chi-square distribution, α-level confidence relative error.**

I. Introduction

Under the assumption that the process is in control, the process capability indices can be estimated reliably, process capability indices, including C_p, C_a, C_{pk} and C_{pm}, etc., provide numerical measures to determine whether this process is capable of producing items within the established specification limits present by the product engineer or manufacturing engineer. The process capability indices were first applied to the automatic industry in Japan and America. In these widely used indices, the precision index C_p measures the magnitude of process variation, the accuracy index C_a measures the departure of process mean from the midpoint of the specification interval. The index C_{pk} takes the process mean into consideration but it can fail to distinguish between on-target processes. The index C_{pm} takes the proximity of process mean from the target value into account, and is more sensitive to process departure than C_p and C_{pk}.

Hsiang and Taguchi [3] first used the loss function to improve process quality, focusing on reducing the process variation around the target value. Johnson [4] introduced the process loss index L_e for processes with symmetric tolerances. A process is said to have a symmetric tolerance if the target value is set to be the midpoint of the specification interval, i.e. $\mathrm{T} = (\mathrm{USL} - \mathrm{LSL})/2$. The index L_e is defined as the ratio of the expected quadratic loss to the square of the half of the specification width. The advantage of using L_e over C_{pm} is that the estimator of the former has better statistical properties than that of the latter, as the former does not involve a reciprocal transformation of the process mean and variance. Also it provides an uncontaminated separation between information concerning the process precision and process accuracy.

It is known that process capability indices are the functions of process mean and process standard deviation. The quality and statistics literatures discussed the estimations of these capability indices based on a single sample ([2], [4], [5], [11], [12], [13]). In practice, process information about process measurements is often derived from multiple samples rather than from one single sample, particularly, when a daily-based production control plan is implemented for monitoring process stability. Finley [6] stressed the importance of using control charts first to determine if a process is in control, before estimating process capability indices. For process information came from multiple samples, particularly, came from $\bar{X}$ and R control chart samples, Li *et al.* [7] gave tables of lower confidence bounds on C_p and C_{pk} where the sample range was substituted for the population standard deviation in the definition formula. Pearn *et al*, [14] considered the problem of estimating and testing process precision based on $\bar{X}$ and R control chart and $\bar{X}$ and S control chart samples. They provided the statistical properties of the natural estimator of C_p and implement the hypothesis testing procedure.

In this paper, we investigate the relationship between index L_e and process yield, and the distributional and inferential properties of the estimator of index L_e when using $\bar{X}$ and R control chart samples. We then investigate the performance of the estimator of L_e based on the absolute relative bias, the relative mean square error and the α-level confidence relative error for various combinations of sample size. The results obtained for the accuracy of the estimated process loss index which is widely used in the manufacturing industry, relative to the control chart samples, is useful to the practitioner in determining the combination of sample size required in his application for its estimation good to the desired accuracy. We also develop a step-by-step hypothesis testing procedure for practitioner to use in determining whether the given process is capable. The technique provided in this paper will be applicable when the process measurements are taken from $\bar{X}$ and R control chart.

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Table 1. The five ranges of L_e and corresponding C_{pm} for various quality conditions

Condition	Range of L_e	Range of corresponding C_{pm}
Inadequate	$0.11 \le L_e$	$C_{pm} \le 1.00$
Capable	$0.06 \le L_e \le 0.11$	$1.00 \le C_{pm} \le 1.33$
Satisfactory	$0.05 \le L_e \le 0.06$	$1.33 \le C_{pm} \le 1.50$
Excellent	$0.03 \le L_e \le 0.05$	$1.50 \le C_{pm} \le 2.00$
Super	$L_e \le 0.03$	$2.00 \le C_{pm}$

II. Process Loss Index And Process Yield

Under the assumption that the process measurements X arises from a normal distribution with a mean μ and a variance σ^2, and has a symmetric tolerance, then the index L_e is defined as the ratio of $\sigma^2 + (\mu - \mathrm{T})^2$ (the expected quadratic loss) to d^2 (the square of the half of the specification width):

$$L_e = \int_{-\infty}^{\infty} \left[\frac{(x-\mathrm{T})^2}{\mathrm{d}^2} \right] dF(x) = \left(\frac{\sigma}{\mathrm{d}} \right)^2 + \left(\frac{\mu - \mathrm{T}}{\mathrm{d}} \right)^2, \quad (1)$$

where $\mathrm{d} = (\mathrm{USL} - \mathrm{LSL})/2$ denotes the half of the specification width, LSL and USL are the lower specification limit and upper specification limit, respectively, and T denotes the target value, $F(x)$ is the cumulative distribution function of the process measurements X. Tsui [15] rewrote $L_e = L_{pe} + L_{ot}$ to provide an uncontaminated separation between information concerning process relative inconsistency loss L_{pe} and process relative off-target loss L_{ot}. In fact, the subindex L_{pe} is defined as the first term $(\sigma/\mathrm{d})^2$ and the second term $((\mu - \mathrm{T})/\mathrm{d})^2$ as subindex L_{ot}. We note that the mathematical relationships $L_{pe} = 1/(C_p)^2$, $L_{ot} = (1 - C_a)^2$ and $L_e = 1/(3C_{pm})^2$ can be established.

While the subindex L_{pe} measures process variation relative to the specification tolerance and the subindex L_{ot} measures the relative process departure. For processes with bilateral specification limits, process yield can be calculated as $F(\mathrm{USL}) - F(\mathrm{LSL})$. On the assumption of normality, process yield can be expressed as $\Phi((\mathrm{USL} - \mu)/\sigma) - \Phi((\mathrm{LSL} - \mu)/\sigma)$, where $\Phi(\cdot)$ is the cumulative function of the standard normal distribution. For cases with symmetric tolerances, since $\mathrm{T} = (\mathrm{USL} - \mathrm{LSL})/2$, the process yield can be calculated as:

$$\%\,\mathrm{Yield} = \Phi\left[\frac{\mathrm{d} - (\mu - \mathrm{T})}{\sigma} \right] - \Phi\left[\frac{-\mathrm{d} - (\mu - \mathrm{T})}{\sigma} \right]$$
$$= \Phi\left(\frac{1 - \sqrt{L_{ot}}}{\sqrt{L_{pe}}} \right) - \Phi\left(\frac{-1 - \sqrt{L_{ot}}}{\sqrt{L_{pe}}} \right). \quad (2)$$

For example, suppose the process is perfectly accurate (that is, $L_{ot} = 0$), then the process yield can be expressed as $2\Phi(1/\sqrt{L_{pe}}) - 1$ for various L_{pe} values.

In current practice, a process is called "inadequate" if $0.11 \le L_e$; it indicates that the process is not adequate with respect to the required specifications, either process mean needs to be shifted closer to the target value or process variation needs to be reduced. A process is called "capable" if $0.06 \le L_e \le 0.11$; it indicates that caution needs to be taken regarding process distribution, some stringent quality improvement is required. A process is called "satisfactory" if $0.05 \le L_e \le 0.06$; it indicates that process quality is satisfactory, and no stringent quality improvement is required. A process is called "excellent" if $0.03 \le L_e \le 0.05$; it indicates that process quality exceeds "satisfactory". Finally, a process is called "super" if $L_e \le 0.03$. Table 1 displays some commonly used L_e values and the corresponding C_{pm} values. For example, if the quality requirement is $L_e \le 0.06$, then from Table 1, we can find that the equivalent quality requirement for the C_{pm} index is $C_{pm} \ge 1.33$.

III. Estimating Process Loss Index

Many statistical quality control textbooks recommend the use of subgroup ranges for estimating the standard deviation of a normally distributed random variable. Suppose the combination of sample size has m independent subgroups, each of size n, from a normal distribution with standard deviation σ. We denote this sequence of independent samples as $\{X_{i1}, X_{i2}, \ldots, X_{in}\}$, $i = 1, 2, \ldots, m$, $N = mn$. Let $\bar{X}_i = \sum_{j=1}^{m} X_{ij}/n$ and $R_i = \max\{X_{ij}\} - \min\{X_{ij}\}$ be the ith subgroup mean and the subgroup range, respectively. $\bar{\bar{X}} = \sum_{i=1}^{m} \bar{X}_i/m$ and $\bar{R} = \sum_{i=1}^{m} R_i/m$ are the grand mean and the mean of subgroup range, respectively. The mean and variance of the statistic $\bar{R}/\sigma$ are respectively given as $\mathrm{E}(\bar{R}/\sigma) = d_2$ and $\mathrm{Var}(\bar{R}/\sigma) = d_3^2/m$, as well as the values of d_2 and d_3 as determined from n. The coefficients d_2, and d_3 are tabulated and referred in Montgomery [9].

Several authors have provided approximations for the distribution of the range. Patnaik [10] showed that $\bar{R}/\sigma$ is distributed approximately as $c\chi_v/\sqrt{v}$ for sample size of $n \le 10$, where χ_v is the chi distribution with v degrees of freedom and c is constant, was most accurate. It is noted that c and v are determined from m and n. Table 2 displays the corresponding c and v for $m = 5(5)30$ and $n = 2(1)8$. In fact,

$$\mathrm{E}(\bar{R}/\sigma) = \sqrt{2}c\Gamma((v+1)/2)/(\sqrt{v}\Gamma(v/2)),$$
$$\mathrm{Var}(\bar{R}/\sigma) = (c^2/v)(v - 2(\Gamma((v+1)/2)/\Gamma(v/2))^2).$$

Therefore, we can derive the constant $c = \sqrt{(d_3^2/m) + d_2^2}$. In this case, Montgomery [9] recommended estimator of σ is $\bar{R}/d_2$, ($\bar{R}/d_2$ is the unbiased estimator of σ) and the grand mean $\bar{\bar{X}}$ is used as an estimator for the process mean associated with $\bar{X}$ and R control chart samples. In this paper, we choice the estimator $\bar{R}/c$ to estimate σ, since $(\bar{R}/c)^2$ is the unbiased estimator of σ^2. Thus, $(\bar{R}/(c\sigma))^2$ is distributed as χ_v^2/v, and the estimator of process loss index L_e as the following:

Table 2. The coefficients of the distribution of $\overline{R}/\sigma$ with $m = 5(5)30$ and $n = 2(1)8$

n	m = 5		m = 10		m = 15	
	c	v	c	v	c	v
2	1.191	4.591	1.160	8.989	1.150	13.376
3	1.739	9.305	1.716	18.389	1.708	27.467
4	2.096	13.927	2.078	27.623	2.071	41.315
5	2.358	18.353	2.342	36.471	2.337	54.586
6	2.563	22.567	2.549	44.896	2.544	67.223
7	2.730	26.581	2.717	52.923	2.713	79.262
8	2.871	30.399	2.859	60.557	2.855	90.713
n	m = 20		m = 25		m = 30	
	c	v	c	v	c	v
2	1.144	17.760	1.141	22.142	1.139	26.523
3	1.704	36.544	1.702	45.619	1.700	54.695
4	2.068	55.006	2.066	68.697	2.065	82.387
5	2.334	72.700	2.332	90.714	2.331	108.927
6	2.542	89.549	2.540	111.875	2.539	134.200
7	2.711	105.600	2.710	131.939	2.709	158.277
8	2.853	120.869	2.852	151.024	2.851	181.179

Table 3. $\text{BIAS}_R(\hat{L}_e)$ for $m = 25$, $n = 2(1)8$, various L_{pe}, and L_{ot}

n	L_{pe}			
	0.11	0.06	0.05	0.03
$L_{ot} = 0.56$				
2	0.0033	0.0019	0.0016	0.0010
3	0.0022	0.0013	0.0011	0.0007
4	0.0016	0.0010	0.0008	0.0005
5	0.0013	0.0008	0.0007	0.0004
6	0.0011	0.0006	0.0005	0.0003
7	0.0009	0.0006	0.0005	0.0003
8	0.0008	0.0005	0.0004	0.0003
$L_{ot} = 0.25$				
2	0.0061	0.0039	0.0033	0.0021
3	0.0041	0.0026	0.0022	0.0014
4	0.0031	0.0019	0.0017	0.0011
5	0.0024	0.0015	0.0013	0.0009
6	0.0020	0.0013	0.0011	0.0007
7	0.0017	0.0011	0.0010	0.0006
8	0.0015	0.0010	0.0008	0.0005
$L_{ot} = 0.06$				
2	0.0129	0.0100	0.0091	0.0067
3	0.0086	0.0067	0.0061	0.0044
4	0.0065	0.0050	0.0045	0.0033
5	0.0052	0.0040	0.0036	0.0027
6	0.0043	0.0033	0.0030	0.0022
7	0.0037	0.0029	0.0026	0.0019
8	0.0032	0.0025	0.0023	0.0017
$L_{ot} = 0.00$				
2	0.0200	0.0200	0.0200	0.0200
3	0.0133	0.0133	0.0133	0.0133
4	0.0100	0.0100	0.0100	0.0100
5	0.0080	0.0080	0.0080	0.0080
6	0.0067	0.0067	0.0067	0.0067
7	0.0057	0.0057	0.0057	0.0057
8	0.0050	0.0050	0.0050	0.0050

$$\hat{L}_e = \frac{(\overline{R}/c)^2}{d^2} + \frac{(\overline{\overline{X}} - T)^2}{d^2}. \qquad (3)$$

Since $\overline{X}_i$ and R_i are mutually independent as has been shown by Lord [8], this will then equally hold for $\overline{\overline{X}}$ and $\overline{R}$. If the process measurement is normally distributed, then we can shown that the statistic $N(\overline{\overline{X}} - T)^2)/\sigma^2$ is distributed as non-central chi-square with 1 degrees of freedom and non-centrality parameter λ, we denote as $\chi_1^2(\lambda)$, where $\lambda = N(\mu - T)^2/\sigma^2 = NL_{ot}/L_{pe}$. If we define the statistic $\zeta = (v(\overline{R}/c)^2 + (v/N)N(\overline{\overline{X}} - T)^2)/\sigma^2$, then ζ is distributed as $Q_{v+1}^2(\lambda)$ and is a linear combination of two independent chi-square distributions, $\chi_v^2 + (v/N)\chi_1^2(\lambda)$. Thus, the estimator $\hat{L}_e$ is distributed as $(L_{pe}/v)Q_{v+1}^2(\lambda)$. the probability density function of $\hat{L}_e$ can be expressed as (see Appendix):

$$f_{\hat{L}_e}(x) = \frac{v}{L_{pe}} \sum_{j=0}^{\infty} P_j(\lambda) \int_0^{Nx/L_{pe}} f_K\left[\left(\frac{v}{N}\right)\left(\frac{Nx}{L_{pe}} - y\right)\right] f_{Y_j}(y)dy, \qquad (4)$$

for $x \geq 0$, where $P_j(\lambda) = (\lambda/2)^j \exp(-\lambda/2)/j!$, K and Y_j are distributed as χ_v^2 and χ_{1+2j}^2, respectively.

Since $E(\chi_v^2) = v$, $Var(\chi_v^2) = 2v$, $E(\chi_1^2(\lambda)) = 1 + \lambda$, and $Var(\chi_1^2(\lambda)) = 2(1 + 2\lambda)$, the expected value, the variance and the mean square error of $\hat{L}_e$ are

$$E(\hat{L}_e) = \left(\frac{N+1}{N}\right)L_{pe} + L_{ot}, \qquad (5)$$

$$Var(\hat{L}_e) = \frac{2L_{pe}^2}{v} + \frac{2L_{pe}^2 + 4NL_{pe}L_{ot}}{N^2}, \qquad (6)$$

$$MSE(\hat{L}_e) = Var(\hat{L}_e) + [E(\hat{L}_e) - L_e]^2. \qquad (7)$$

To evaluate the performance of the estimator $\hat{L}_e$, the first criterion, the absolute relative bias of $\hat{L}_e$ is defined as $\text{BIAS}_R(\hat{L}_e) = |E(\hat{L}_e) - L_e|/L_e$, which presents the absolute relative deviation of the average value of $\hat{L}_e$ from the true value of L_e. Since the bias of $\hat{L}_e$, $\text{BIAS}(\hat{L}_e) = E(\hat{L}_e) - L_e = L_{pe}/N >$ 0, then the estimator $\hat{L}_e$ overestimates L_e. Table 3 displays $\text{BIAS}_R(\hat{L}_e)$ for $m = 25$, $n = 2(1)8$, and some commonly used values of $L_{pe} = 0.11$, 0.06, 0.05 and 0.03, equivalent to L_{ot} = 0.56, 0.25, 0.06 and 0.00. For example, if $(m, n) = (25, 5)$, $L_{pe} = 0.11$ and $L_{ot} = 0.06$, then $\text{BIAS}_R(\hat{L}_e) = 0.0052$. We expect that the deviation of $\hat{L}_e$, on the average, would be no greater than 0.52% of the true L_e.

The second criterion, the relative mean square error presents the average of the squared relative deviation of $\hat{L}_e$ from the true value of L_e, and defined as $\text{MSE}_R(\hat{L}_e) = E[((\hat{L}_e - L_e)/L_e)^2]$. Thus, the value of the relative mean square error is a function of L_{pe} and L_{ot}. The statistic $\sqrt{\text{MSE}_R(\hat{L}_e)}$, a more direct measurement of the deviation, can be easily calculated and used to measure the performance of the estimator $\hat{L}_e$, which presents the average of the relative deviation of $\hat{L}_e$ from the true L_e. Table 4 displays $\sqrt{\text{MSE}_R(\hat{L}_e)}$ for $m = 25$, $n = 2(1)8$, and some commonly

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used values of $L_{\text{pe}} = 0.11$, 0.06, 0.05 and 0.03, equivalent to $L_{\text{ot}} = 0.56, 0.25, 0.06$ and 0.00. For example, if $(m, n) = (25, 5)$, $L_{\text{pe}} = 0.11$ and $L_{\text{ot}} = 0.06$, then $\sqrt{\text{MSE}_R(\hat{L}_{\text{e}})} = 0.1289$. We expect that the value of $\hat{L}_{\text{e}}$, on the average, would be no greater than 12.89% of the true L_{e}.

IV. The Reliability Analysis

To evaluate the reliability of the estimator $\hat{L}_{\text{e}}$, we consider the criterion called α-level confidence relative error. Pearn and Lin [13] noted that the α-level confidence relative error which is obtained from the same approach as used for finding the confidence interval, provides the practitioners with more direct and easily understood information than the confidence interval approach regarding the accuracy of their estimations and suggests a clear range on the true value of the process performance measure using the process capability index. The α-level confidence relative error of $\hat{L}_{\text{e}}$, which is defined as $\text{CRE}_\alpha(\hat{L}_{\text{e}}) = \max_\alpha\{|\hat{L}_{\text{e}} - L_{\text{e}}|/L_{\text{e}}\} = \max_\alpha |(\hat{L}_{\text{e}}/L_{\text{e}}) - 1| = \max_\alpha\{|L_{\alpha/2} - 1|, |U_{\alpha/2} - 1|\}$, where the percentiles $L_{\alpha/2}$ and $U_{\alpha/2}$ satisfy the probability equation $\Pr\{L_{\alpha/2} \le \hat{L}_{\text{e}}/L_{\text{e}} \le U_{\alpha/2}\} = 1-\alpha$, which can be obtained as

Table 4. $\sqrt{\text{MSE}_R(\hat{L}_{\text{e}})}$ for $m = 25$, $n = 2(1)8$, various L_{pe}, and L_{ot}

n	L_{pe} 0.11	0.06	0.05	0.03
$L_{\text{ot}} = 0.56$				
2	0.1160	0.0886	0.0815	0.0640
3	0.0923	0.0713	0.0657	0.0519
4	0.0793	0.0614	0.0566	0.0448
5	0.0706	0.0548	0.0506	0.0400
6	0.0644	0.0500	0.0461	0.0365
7	0.0596	0.0463	0.0427	0.0338
8	0.0557	0.0433	0.0399	0.0316
$L_{\text{ot}} = 0.25$				
2	0.1598	0.1262	0.1168	0.0933
3	0.1243	0.0999	0.0930	0.0749
4	0.1060	0.0857	0.0798	0.0645
5	0.0942	0.0763	0.0711	0.0576
6	0.0857	0.0695	0.0648	0.0525
7	0.0792	0.0643	0.0600	0.0486
8	0.0741	0.0602	0.0561	0.0455
$L_{\text{ot}} = 0.06$				
2	0.2379	0.2071	0.1968	0.1672
3	0.1754	0.1563	0.1496	0.1295
4	0.1465	0.1317	0.1265	0.1103
5	0.1289	0.1164	0.1119	0.0979
6	0.1168	0.1057	0.1017	0.0890
7	0.1077	0.0976	0.0939	0.0823
8	0.1007	0.0913	0.0878	0.0770
$L_{\text{ot}} = 0.00$				
2	0.3025	0.3025	0.3025	0.3025
3	0.2107	0.2107	0.2107	0.2107
4	0.1715	0.1715	0.1715	0.1715
5	0.1491	0.1491	0.1491	0.1491
6	0.1342	0.1342	0.1342	0.1342
7	0.1235	0.1235	0.1235	0.1235
8	0.1154	0.1154	0.1154	0.1154

$$\Pr\left\{\frac{\nu(L_{\text{pe}} + L_{\text{ot}})}{L_{\text{pe}}} L_{\alpha/2} \le \zeta \le \frac{\nu(L_{\text{pe}} + L_{\text{ot}})}{L_{\text{pe}}} U_{\alpha/2}\right\} = 1-\alpha, \qquad (8)$$

where ζ is distributed as $Q^2_{\nu+1}(\lambda)$. Therefore, the percentiles $L_{\alpha/2}$ and $U_{\alpha/2}$ may be obtained by finding the corresponding percentiles of the distribution. Thus,

$$L_{\alpha/2} = \frac{L_{\text{pe}} Q^2_{\nu+1,\alpha/2}(\lambda)}{\nu(L_{\text{pe}} + L_{\text{ot}})}, \text{ and } U_{\alpha/2} = \frac{L_{\text{pe}} Q^2_{\nu+1,1-\alpha/2}(\lambda)}{\nu(L_{\text{pe}} + L_{\text{ot}})}, \qquad (9)$$

where $Q^2_{\nu+1,\alpha}(\lambda)$ is the lower αth percentile of $Q^2_{\nu+1}(\lambda)$.

Thus, $\text{CRE}_\alpha(\hat{L}_{\text{e}}) = e$ presents that with at least $1-\alpha$ confidence the relative deviation (relative error) of $\hat{L}_{\text{e}}$ will be no greater than e. In this case, because the value of ν/N is reasonably constant and not dramatically different from 1, it is suggested that $Q^2_{\nu+1,\alpha}(\lambda)$ be replaced by $\chi^2_{\nu+1,\alpha}(\lambda)$. Table 5 displays $\text{CRE}_\alpha(\hat{L}_{\text{e}})$ for $m = 25$, $n = 2(1)8$, $\alpha = 0.05$, and some commonly used values of $L_{\text{pe}} = 0.11$, 0.06, 0.05 and 0.03, equivalent to $L_{\text{ot}} = 0.56$, 0.25, 0.06 and 0.00. The values of $\text{CRE}_\alpha(\hat{L}_{\text{e}})$, for other values of L_{pe} and L_{ot}, and the combinations of sample size (m, n) are available from the authors. We find under equal total sample size, the larger n, we can obtain smaller value of $\text{CRE}_\alpha(\hat{L}_{\text{e}})$. If m or n is increasing, then the value of $\text{CRE}_\alpha(\hat{L}_{\text{e}})$ is decreasing. For example, if $(m, n) = (25, 5)$, $L_{\text{pe}} = 0.11$ and $L_{\text{ot}} = 0.06$, then $\text{CRE}_\alpha(\hat{L}_{\text{e}}) = 0.4579$, which indicates that with at least 95% confidence the obtained $\hat{L}_{\text{e}}$ value will be within 45.79% of the true L_{e} value, for the described condition. If the estimated value $\hat{L}_{\text{e}}$ based on $(m, n) = (25, 5)$ and $\alpha = 0.05$ is equal to 0.02, then the true value of L_{e} would be between 0.0137 $(= 0.02/(1+45.79\%))$ and 0.0369 $(= 0.02/(1-45.79\%))$ with at least 95% confidence.

V. A Decision Making Procedure

We can consider the following statistical testing hypothesis for L_{e}: $H_0 : L_{\text{e}} \ge l_0$ (the process is incapable) and $H_1 : L_{\text{e}} < l_0$ (the process is capable), which judges whether a given process meets the preset capability requirement and runs under the desired quality condition. Process fails to meet the capability requirement if $L_{\text{e}} \ge l_0$, and meets the capability requirement if $L_{\text{e}} < l_0$. We define the test $\phi(x)$ as: $\phi(x) = 1$ if $\hat{L}_{\text{e}} < l$, and $\phi(x) = 0$ otherwise. Thus, the test $\phi(x)$ rejects the null hypothesis H_0 if $\hat{L}_{\text{e}} < l$, with type I error $\alpha(l) = \alpha$, the probability of incorrectly judging an incapable process as capable. The critical value, l, can be determined from $\Pr\{\hat{L}_{\text{e}} \le l \mid L_{\text{e}} = l_0\} = \alpha$. Thus,

$$\Pr\left\{\zeta \le \frac{l\nu[1 + (L_{\text{ot}}/L_{\text{pe}})]}{l_0}\right\} = \alpha. \qquad (10)$$

Hence, we have the critical value as:

Table 5. $\mathrm{CRE}_{\alpha}(\hat{L}_e)$ for $m = 25$, $n = 2(1)8$, $\alpha = 0.05$, various L_{pe}, and L_{ot}

n	L_{pe}			
	0.11	0.06	0.05	0.03
$L_{ot} = 0.56$				
2	1.5595	1.5313	1.5192	1.4817
3	0.8385	0.8152	0.8065	0.7814
4	0.6101	0.5899	0.5828	0.5625
5	0.5094	0.4919	0.4858	0.4685
6	0.4562	0.4412	0.4359	0.4208
7	0.4293	0.4167	0.4121	0.3988
8	0.4169	0.4065	0.4026	0.3908
$L_{ot} = 0.25$				
2	1.5392	1.5622	1.5599	1.5382
3	0.8415	0.8431	0.8390	0.8203
4	0.6177	0.6147	0.6106	0.5942
5	0.5167	0.5134	0.5098	0.4956
6	0.4617	0.4596	0.4565	0.4444
7	0.4321	0.4319	0.4296	0.4194
8	0.4165	0.4187	0.4171	0.4089
$L_{ot} = 0.06$				
2	1.2683	1.4152	1.4519	1.5271
3	0.7251	0.7915	0.8074	0.8375
4	0.5443	0.5878	0.5979	0.6158
5	0.4579	0.4930	0.5011	0.5152
6	0.4078	0.4396	0.4470	0.4602
7	0.3776	0.4089	0.4163	0.4303
8	0.3585	0.3907	0.3987	0.4143
$L_{ot} = 0.00$				
2	0.7279	0.7279	0.7279	0.7279
3	0.4766	0.4766	0.4766	0.4766
4	0.3781	0.3781	0.3781	0.3781
5	0.3239	0.3239	0.3239	0.3239
6	0.2887	0.2887	0.2887	0.2887
7	0.2638	0.2638	0.2638	0.2638
8	0.2452	0.2452	0.2452	0.2452

Table 6. Critical values for $m = 25$, $n = 2(1)8$, various α-risk, and l_0

n	l_0			
	0.11	0.06	0.05	0.03
$\alpha = 0.05$				
2	0.0656	0.0358	0.0298	0.0179
3	0.0770	0.0420	0.0350	0.0210
4	0.0824	0.0450	0.0375	0.0225
5	0.0856	0.0467	0.0389	0.0234
6	0.0879	0.0479	0.0399	0.0240
7	0.0895	0.0488	0.0407	0.0244
8	0.0907	0.0495	0.0412	0.0247
$\alpha = 0.025$				
2	0.0586	0.0319	0.0266	0.0160
3	0.0715	0.0390	0.0325	0.0195
4	0.0777	0.0424	0.0353	0.0212
5	0.0814	0.0444	0.0370	0.0222
6	0.0839	0.0458	0.0382	0.0229
7	0.0858	0.0468	0.0390	0.0234
8	0.0872	0.0476	0.0397	0.0238
$\alpha = 0.01$				
2	0.0511	0.0279	0.0232	0.0139
3	0.0654	0.0357	0.0297	0.0178
4	0.0724	0.0395	0.0329	0.0197
5	0.0766	0.0418	0.0348	0.0209
6	0.0795	0.0434	0.0361	0.0217
7	0.0817	0.0446	0.0371	0.0223
8	0.0833	0.0455	0.0379	0.0227

$$l = \frac{l_0 Q^2_{\nu+1,\alpha}(\lambda)}{\nu[1+(L_{ot}/L_{pe})]}. \tag{11}$$

Therefore, if $\hat{L}_e < l$, then $\phi(x) = 1$ and we reject the null hypothesis H_0 and conclude that the process is capable. Otherwise, we can not conclude that the process is incapable. Here, we set $L_{ot}/L_{pe} = \hat{L}_{ot}/\hat{L}_{pe} = (\overline{\overline{X}} - \mathrm{T})^2/(\overline{R}/c)^2$, since generally $L_{ot}/L_{pe} = (\mu - \mathrm{T})^2/\sigma^2$ is unknown. This approach is similar to the one proposed by Johnson [4]. Such an approach introduces additional sampling errors from estimating L_{ot}/L_{pe}, and would be less reliable. Consequently, any decisions made would provide less quality assurance to the customers. However, to eliminate the need for further estimating L_{ot}/L_{pe}, as the description from above section, it is suggested that $Q^2_{\nu+1,\alpha}(\lambda)$ be replaced by $\chi^2_{\nu+1,\alpha}(\lambda)$, Pearn *et al.* [11] examined the sensitivity of the critical value l against the parameter L_{ot}/L_{pe}. The result indicated that the critical value l is increasing in L_{ot}/L_{pe} and reaches its minimum at $L_{ot}/L_{pe} = 0$ (that is, $\mu = \mathrm{T}$) in all cases. Hence, for practical purposes we may calculate the critical value l by setting $L_{ot}/L_{pe} = \hat{L}_{ot}/\hat{L}_{pe} = 0$, for given l_0, α and ν, without having to further estimate the parameter L_{ot}/L_{pe}. Thus, based on such an approach, the α-risk can be ensured and the decisions made are indeed more reliable.

Generally, the calculated ν is not always an integer. If ν is not an integer, then we may approximate $\chi^2_{\nu+1,\alpha}/\nu$ by interpolating values of $\chi^2_{\lceil\nu\rceil,\alpha}/\nu$ and $\chi^2_{\lceil\nu\rceil+1,\alpha}/\nu$, where $\lceil\nu\rceil$ means the least integer such that $\lceil\nu\rceil \geq \nu$. We may use the conservative value of $\chi^2_{\lceil\nu\rceil+1,\alpha}/\lceil\nu\rceil$, because $\chi^2_{\nu+1,\alpha}/\nu$ is a monotonic decreasing function of ν in the case of $\alpha < 0.5$ [1].

In the following, we develop a practical step-by-step procedure for testing process loss index. The practitioner can use the procedure in his in-plant application to obtain reliable decision.

Step 1: Decide the definition of "capable", (common requirement values of l_0 include 0.11, 0.06, 0.05, and 0.03) and the α-risk (normally set to 0.05, 0.025, or 0.01), the chance of wrongly concluding an incapable process as capable.

Step 2: Estimate the process expected loss L_e from the past "in control" data by using the sample range.

Step 3: Calculate the critical value $l = l_0\chi^2_{\nu+1,\alpha}/\nu$ based on the specified α-risk, l_0, and ν. (If ν is not an integer, then we may approximate $\chi^2_{\nu+1,\alpha}/\nu$ by interpolating method or using the conservative value of $\chi^2_{\lceil\nu\rceil+1,\alpha}/\lceil\nu\rceil$).

Step 4: Conclude that the process is capable ($L_e < l_0$) if the estimated $\hat{L}_e$ value is less than the critical value l ($\hat{L}_e < l$). Otherwise, we do not have enough information to conclude that the process is capable.

Table 6 displays critical values for $m = 25$, $n = 2(1)8$, $\alpha = 0.05$, 0.025 and 0.01, and some commonly used requirement values of "capable" of $l_0 = 0.11$, 0.06, 0.05 and 0.03. Suppose that the requirement for a process to be capable is that $L_e < 0.06$. We take a $\overline{X}$ and R control chart samples $m = 25$ and $n = 5$, and calculate $\hat{L}_e$. Using Table 2 based on $m = 25$ and $n = 5$, we obtain $\nu = 90.714$, We then can compute the critical value $l = 0.0467$. Thus, if the calculated $\hat{L}_e < 0.0467$, then we claim that the process is capable at least 95% of the time.

VI. Conclusion

Process loss index provides measures to determine the quality performance of a process. In fact, $L_e = L_{pe} + L_{ot}$, L_{pe} denotes the process relative inconsistency loss, L_{ot} is the relative off-target loss. In real situations where the actual value of L_e is unknown one may estimate it by its corresponding process samples. Most of the results obtained regarding the distributional and inferential properties of estimated process loss index were based on one single sample. In practice, however, process information is often derived from multiple samples rather than from one single sample. Particularly, process measurements come from control chart samples, since the importance of using control charts first to determine if a process is in control, before estimating process capability. In this paper, we first investigated the relationship between L_e and process yield, and the distributional and inferential properties of the estimator of process loss index L_e based on $\overline{X}$ and R control chart samples. We then investigated the performance of the estimator of L_e based on the absolute relative bias, the relative mean square error and the α-level confidence relative error for various combinations of sample size. The results obtained for the accuracy of the estimated process loss index which is widely used in the manufacturing industry, relative to the control chart samples, is useful to the practitioner in determining the sample size required in his application for its estimation good to the desired accuracy. We also developed a practical step-by-step hypothesis testing procedure for practitioner to use in determining whether the given process is capable, then the decision making will be reliable. The technique provided in this paper will be applicable when the process measurements are taken from $\overline{X}$ and R control chart.

Appendix

To derive the probability density function of the estimator $\hat{L}_e$, we define some statistics:

(1) $D = N/L_{pe}$,

(2) $K = \nu\overline{R}^2/(c\sigma)^2$, is distributed as χ^2_ν, and

(3) $Y = N(\overline{\overline{X}} - T)^2/\sigma^2$, is distributed as $\chi^2_1(\lambda)$.

Thus, $\hat{L}_e$ can be rewritten as $\hat{L}_e = ((N/\nu)K + Y)/D$. The cumulative distribution function of $\hat{L}_e$ is:

$$F_{\hat{L}_e}(x) = \Pr\left\{\left(\frac{N}{\nu}\right)K + Y \le Dx\right\}$$

$$= \int_0^{Nx/L_{pe}} \Pr\left\{K \le \left(\frac{\nu}{N}\right)\left(\frac{Nx}{L_{pe}} - y\right)\right\} f_Y(y)dy, \quad \text{(A1)}$$

for $x \ge 0$. The last equality is valid since $Nx/L_{pe} - y \ge 0$ for $0 \le y \le Nx/L_{pe}$. Thus, $\Pr\{K \le (\nu/N)(Nx/L_{pe} - y)\} = 0$ for $y > Nx/L_{pe}$, and the probability density function of Y is $f_Y(y) = \sum_{j=0}^{\infty} P_j(\lambda) f_{Y_j}(y)$, where $f_{Y_j}(y)$ is the probability density function of chi-square distribution χ^2_{1+2j}, we have

$$F_{\hat{L}_e}(x) = \sum_{j=0}^{\infty} P_j(\lambda) \int_0^{Nx/L_{pe}} F_K\left[\left(\frac{\nu}{N}\right)\left(\frac{Nx}{L_{pe}} - y\right)\right] f_{Y_j}(y)dy, \quad \text{(A2)}$$

for $x \ge 0$. Therefore, the probability density function of $\hat{L}_e$ as:

$$f_{\hat{L}_e}(x) = \frac{\nu}{L_{pe}} \sum_{j=0}^{\infty} P_j(\lambda) \int_0^{Nx/L_{pe}} f_K\left[\left(\frac{\nu}{N}\right)\left(\frac{Nx}{L_{pe}} - y\right)\right] f_{Y_j}(y)dy, \quad \text{(A3)}$$

for $x \ge 0$.

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