Proceedings of the International MultiConference of Engineers and Computer Scientists 2008 Vol II
IMECS 2008, 19-21 March, 2008, Hong Kong

Computational Solution for Rendezvous on a Line of Four or Five Points

William H. Ruckle*

Abstract–**In a rendezvous problem on a discrete line two players are placed at points on the line. At each moment of time each player can move to an adjacent point or remain at the point at which it stands. The goal is for both players to reach the same point in the least time. There are optimal strategy pairs for which both players tend toward the center. Using this result and a matrix representation for the situation we employ a symbolic program (Maxima) to determine all possible solutions to searches on lines having four or five points, the cases of 1, 2, or 3 points being trivial.**

Keywords: rendezvous, search

Introduction

We consider the problem in which two teams called Player I and Player II are placed at locations i and j respectively with probability $p_{i,j}$ on a discrete line. Thereafter the two players move to adjacent locations until they finally meet by arriving at the same location. The goal is for the players to meet in the shortest time. Thus if Player I starting at i chooses a path f_i and Player II starting at j chooses a path g_j the goal is to minimize the quantity

$$E(\{f_i\},\{g_j\}) = \sum_{i,j} p_{i,j}\,[f_i, g_j]$$

where $[f_i, g_j]$ denotes the time before the two paths are at the same location.

The problem described above is known as the rendezvous problem on the discrete line. A description of results for this problem on the line and other lattices is described in [1], and some results for lines of arbitrary length appear in [2] and [3]. In this paper we first show that there is always an optimal pair of paths that tend toward the center. Next we show how to represent a pair of paths and its result using matrix calculations. Finally we apply the calculation to the cases of a four and five point line.

The Restriction Theorem

The main result in this section is that in every rendezvous game on the line there is always a pair of optimal strategies that are within increasingly shorter lines as the search proceeds. A more general theorem of this type is found in [4], but the present result is not a special case since there we defined a meeting to be in the same or *adjacent* locations at the same time. It is convenient to represent the set L of locations on a line by

$$L = \{-n, -(n-1), ..., -1, 0, 1, ..., n\}$$

if the line has an odd number $(2n+1)$ of locations and by

$$L = \{-n, -(n-1), ..., -1, 1, ..., n\},$$

omitting 0, if the line has an even number $(2n)$ of locations.

A **path** f_i beginning at location i is a function from the set of positive integers into L such that (1) $f_i(1) = i$ and (2) $f_i(t+1) \in \{f_i(t)-1, f_i(t), f_i(t)+1\}$ for each positive integer k. We express this briefly by saying that $f_i(t+1)$ is **adjacent** to $f_i(t)$. If f_i and g_j are two paths then $[f_i, g_j]$ is the smallest integer k for which $f_i(k) = g_j(k)$ or ∞ if the paths never meet. A path f_i is said to be k **restricted** where $0 \le k \le n$ after time T if $-k \le f_i(t) \le k$ for $t > T$. If f_i is a k restricted path after time T we define $P_{k-1}(f_i)$ to be the function g from the positive integers into L defined by

$$g(k) = \begin{array}{l} f_i(t) \text{ if } t \le T+1 \\ f_i(t) \text{ if } t > T+1 \text{ and } -(k-1) \le f_i(t) \le k-1 \\ k-1 \text{ if } t > T+1 \text{ and } f_i(t) = k \\ -(k-1) \text{ if } t > T+1 \text{ and } f_i(t) = -k \end{array}$$

Thus g coincides with f_i until time $T+1$ coincides with f_i except it stays at $-(k-1)$ when f_i goes to $-k$ or at $(k-1)$ when f_i goes to k.

Proposition 1 *If f_i is a k restricted path after time T then $g = P_{k-1}(f_i)$ is a path that begins at i and is $(k-1)$ restricted after time $T+1$ as well as k restricted after time T.*

Proof. Since $1 \le T+1$ it follows that $g(1) = f_i(1) = i$. If $t \le T+1$ or $-(k-1) < f_i(t) < k-1$ then $g(t+1)$ is adjacent to $g(t)$ because it coincides with f_i. If $t > T+1$ and $f_i(t) = k$ then $g(t)$ can be k (if $t = T+1$) or $k-1$ and $f_i(t+1)$ can be k or $k-1$ so $g(t+1)$ has to be $k-1$ which is adjacent to $g(t)$. If $t > T+1$ and $f_i(t) = k-1$ then $g(t)$ is $k-1$ and $f_i(t+1)$ can be k or $k-1$ so $g(t+1)$ has to be $k-1$ which is adjacent to $g(t)$. We omit the similar argument for $f_i(t)$ equal $-k$ or $-(k-1)$. That g is k restricted after time T follows since $g(T+1) = f_i(T+1)$. □

*Professor Emeritus, Clemson University 106 Whippoorwill Drive, Seneca,SC 29672, USA ruckle@excite.com

Proposition 2 *If f_i and g_j are both k restricted paths after time T then*

$$[P_{k-1}f_i, P_{k-1}g_j] \leq [f_i, g_j].$$

Proof. Denote $P_{k-1}f_i$ by f_i^* and $P_{k-1}g_j$ by g_j^*, and let $[f_i, g_j] = t_0$. If $t_0 \leq T+1$ or $f_i(t_0) \notin \{k, -k\}$ then $f_i^*(t_0) = f_i(t_0) = g_j(t_0) = g_j^*(t_0)$. If $t_0 > T+1$ and $f_i(t_0) = k$ then $f_i^*(t_0)$ and $g_i^*(t_0)$ are both $k-1$. If $t_0 > T+1$ and $f_i(t_0) = -k$ then $f_i^*(t_0)$ and $g_i^*(t_0)$ are both $-(k-1)$. Thus f_i^* and g_i^* both meet at time t_0 and possibly before. □

Definition 3 *A path f_i on L is called* **restricted** *if it is $n-T$ restricted after time T for $T = 0, 1, 2, .., n-1$ and $f_i(n) = 0$ when L has an odd number ($2n+1$) locations or $f_i(n) = 1$ when L has an even number ($2n$) locations.*

Proposition 4 *If f_i and g_j is any pair of paths on L, there are restricted paths f_i^* and g_j^* such that $\left[f_i^*, g_j^*\right] \leq [f_i, g_j]$.*

Proof. Since f_i and g_i are paths on L, they are n restricted so by Proposition $P_{n-1}f_i$ and $P_{n-1}g_j$ are n restricted after time 0 and $n-1$ restricted after time 1 with $[P_{n-1}f_i, P_{n-1}g_j] \leq [f_i, g_j]$. We can iterate this process $n-1$ times to obtain the desired f_i^* and g_i^*. □

Proposition 5 *Suppose Players I and II begin at locations i and j respectively with probability $p_{i,j}$. If $\{f_i : i \in L\}$ is any set of paths for Player I and $\{g_j : j \in L\}$ is any set of paths for Player II then there are sets of restricted paths If $\{f_i^* : i \in L\}$ and $\left\{g_j^* : j \in L\right\}$ such that*

$$\sum_{i,j} p_{i,j}\left[f_i^*, g_j^*\right] \leq \sum_{i,j} p_{i,j}[f_i, g_j].$$

Proof. For each i, j let f_i^*, g_j^* satisfy the conclusion of Proposition 4. □

Theorem 6 *Suppose Players I and II begin at locations i and j respectively with probability $p_{i,j}$. There are restricted paths $\{f_i^* : i \in L\}$ and $\left\{g_j^* : j \in L\right\}$ such that*

$$\sum_{i,j} p_{i,j}\left[f_i^*, g_j^*\right] \leq \sum_{i,j} p_{i,j}[f_i, g_j]$$

for any pair of sets of paths $\{f_i : i \in L\}$ and $\{g_j : j \in L\}$.

Proof. Since the set of restricted paths is finite so is the set of pairs of restricted paths. Thus there is a pair $\{f_i^* : i \in L\}$ and $\left\{g_j^* : j \in L\right\}$ of restricted paths for which

$$\sum_{i,j} p_{i,j}\left[f_i^*, g_j^*\right]$$

is a minimum. If $\{f_i : i \in L\}$ and $\{g_j : j \in L\}$ is any pair of paths, by Proposition 5 there is a pair of restricted paths $\left\{\hat{f_i} : i \in L\right\}$ and $\left\{\hat{g_j} : j \in L\right\}$ such that

$$\sum_{i,j} p_{i,j}\left[\hat{f_i}, \hat{g_j}\right] \leq \sum_{i,j} p_{i,j}[f_i, g_j]$$

but we also have

$$\sum_{i,j} p_{i,j}\left[f_i^*, g_j^*\right] \leq \sum_{i,j} p_{i,j}\left[\hat{f_i}, \hat{g_j}\right]$$

because $\sum_{i,j} p_{i,j}\left[f_i^*, g_j^*\right]$ is minimal over restricted paths. □

Matrix Representation

In this section it is convenient to represent the locations on the line L by $\{1, 2, ..., n\}$ where n can be odd or even. A collection of n motions to other locations can be represented by an $n \times n$ matrix D where the j^{th} column (d_{ij}) has $d_{kj} = 1$ to represent a motion from j to k and 0's elsewhere. The transpose $D^\top$ of such a matrix also represents such a motion.

Proposition 7 *If $Q = (q_{i,j})$ is a matrix for which $q_{i,j}$ denotes the probability that Player I is at i and Player II is at j then $DQE^\top = (r_{i,j})$ is a matrix in which $r_{i,j}$ is the probability that Player I is at i and Player II is at j given that Player I performs the motions represented by D and Player II performs the motions represented by $E^\top$.*

Proof. If $DQ = (s_{i,j})$ then for each i we have

$$s_{i,j} = \sum_{h \in A} p_{h,j}$$

where A is the set of all h that Player I moved to i from h. Thus $s_{i,j}$ represents the probability that Player I is at i and Player II is at j after the move. A similar argument applies for $DQE^\top$. □

In the situation we are studying moves are restricted to adjacent locations so we shall take d_{kj} to be 1 for $k \in \{j-1, j, j+1\}$ and 0 elsewhere. We denote by $\mathbf{e}_i$ the column matrix that has 1 in the i^{th} row and 0's elsewhere. A path for Player I is represented by a sequence of matrices $D_t : t = 1, 2, ...$

Proposition 8 *For $n = 2m$, or $n = 2m+1$ a sequence of matrices (D_t) represents a restricted path for Player I if and only if for each $h = 1, 2, ..., m-1$ (1) the h^{th} column of D_h is $\mathbf{e}_{h+1}$, (2) the $h+1^{th}$ column of D_h is $\mathbf{e}_{h+1}$ or $\mathbf{e}_{h+2}$, (3) the $n-h^{th}$ column of D_h is $\mathbf{e}_{n-(h+1)}$, (4) the $n-(h+1)^{th}$ column of D_h is $\mathbf{e}_{n-(h+1)}$ or $\mathbf{e}_{n-(h+2)}$, and (5) For n even, the m^{th} and $(m+1)^{th}$ columns of D_m are both $\mathbf{e}_{m+1}$, and for n odd the $m^{th}, (m+1)^{th}$ and $(m+2)^{th}$ are all $\mathbf{e}_{m+1}$.*

Proof. Conditions (1) and (2) hold if and only if on the h^{th} turn Player I moves toward the center from location h. Conditions (3) and (4) hold if and only if on the h^{th} turn Player I moves toward the center from location $n-h$. If these conditions are satisfied by matrices D_t for $t < h$ the probability that Player I is outside of the interval $\{h, h+1, ..., n-h\}$ on turn h is zero. That is the first and last h rows of the matrix

$$D_1 D_2 ... D_h P$$

are zero. The last condition holds if and only if Player I moves to $m+1$ on move m. □

For a matrix $A=[a_{i,j}]$ we denote by $\Delta(A)$ the matrix $D=[d_{i,j}]$ for which $d_{i,i}=a_{i,i}$ and $d_{i,j}=0$ for $i\neq j$; we denote by $Tr(A)$ the sum $\sum_i a_{i,i}$.

Proposition 9 *Suppose $P=[p_{i,j}]$ is the matrix for which $p_{i,j}$ is the probability that Player I begins at location i and Player II begins at location j. Suppose the number of locations is either $n=2m$ or $n=2m+1$ and Player I uses the restricted paths $\{f_i\}$ described by the matrices $\{D_t : t=1,...,m\}$ while Player II uses the restricted paths $\{g_j\}$ described by the matrices $\left\{E_t^\top : t=1,...,m\right\}$. Let*

$$P_1 = D_1(P-\Delta(P))E_1^\top$$

and for $t=2,...,m$ let

$$P_t = D_t(P_{t-1}-\Delta(P_{t-1}))E_1^\top$$

Then the probability that Player I and II meet after turn t is $Tr(P_t)$.

Proof. Each element $p_{i,j}$ of $P-\Delta(P)$ is the probability that I is at i and II is at j and they did not meet at the start. Thus the diagonal elements of P_1 are the probabilities that I and II meet immediately after the first move (Proposition 7). If $P_{t-1}=[s_{i,j}]$ then $s_{i,j}$ is the probability that after turn $t-1$ I is at i and II is at j and they have not previously met. Thus $Tr(P_{t-1})$ is the probability that they meet after turn $t-1$, $P_{t-1}-\Delta(P_{t-1})$ is the matrix of probabilities that they have not yet met and are at different locations after turn $t-1$ and $D_t(P_{t-1}-\Delta(P_{t-1}))E_1^\top$ is the matrix of probabilities that they are at their various locations after I and II make their moves. □

If we write $b_0 = Tr(P)$ and $b_t = Tr(P_t)$ then assuming Player I uses the restricted paths $\{f_i\}$ described by the matrices $\{D_t : t=1,...,m\}$ while Player II uses the restricted paths $\{g_j\}$ described by the matrices $\left\{E_t^\top : t=1,...,m\right\}$ we have

$$E(\{f_i\},\{g_j\}) = \sum_{t=1}^{m} tb_t$$

since the players will certainly meet at the end of turn m it follows that $\sum_{t=0}^{m} b_t = \sum_{i,j} p_{i,j} = 1$ so that

$$E(\{f_i\},\{g_j\}) = \sum_{t=1}^{m-1} tb_t + m\left(1-\sum_{t=0}^{m-1} b_t\right)$$

$$= m(1-b_0) - \sum_{t=1}^{m-1}(m-t)b_t$$

The quantity $m(1-b_0)$ is fixed so minimizing $E(\{f_i\},\{g_j\})$ is equivalent to maximizing

$$\sum_{t=1}^{m-1}(m-t)b_t.$$

Solutions for Four and Five Locations

We have used the matrix method described in the previous section to completely solve the rendezvous problem for a line of four or five locations. To do this we have employed the symbolic computational program MAXIMA, which is a decendent of the Macsyma program maintained at the U. S. Department of Energy, and now available without charge on the internet [5].

First observe that the solutions for the one, two or three point line are obvious. In the case of one point the players meet at time 0. For two points the players decide before on a point to end at if they do not meet at time 0 and both go (or remain) there. For three points both players go to the center if they do not meet at time 0.

When there are four or five locations the players can meet after no more than two moves using restricted strategies. The terminal point in the four point case on the line $[1,2,3,4]$ being 2 or 3 (chosen beforehand by the players or their controller) and the terminal point in the five point case on the line $[1,2,3,4,5]$ being 3. If the players have not met after the first move they both move to the terminal point on the second.

Four Locations

When there are four locations each player has only two possible tactics on the first move. They are described by the vectors

$$\mathbf{e}_2 = [0,1,0,0] \text{ and } \mathbf{e}_3 = [0,0,1,0]$$

The strategy described by $\mathbf{e}_2$ in row i is to move to 2 from i and the strategy described by $\mathbf{e}_3$ in row i is to move to 3 from i. The first column must be $\mathbf{e}_2$ (if the strategy is restricted) and the fourth column must be $\mathbf{e}_3$ while the two middle columns can be either. Thus on the first move there are four strategy matrices for each player resulting in a total of sixteen strategy pairs for the two teams. In formula the sum that has to be maximized is simply b_1. We have calculated the quantities b_1 for all sixteen strategy pairs and have found that each pair can be optimal in the appropriate situation.

Example 10 *The pair of matrices*

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

describes the pair of strategies for which Player I moves to 2 if it begins at 1, to 3 if it begins at 2, to 2 if it begins at 3 and to 3 if it begins at 4 while Player II moves to 2 if it begins at 1 to remains at 2 if it begins at 2 moves to 2 if it begins at 3 and to 3 if it begins at 4. Since

$$A\begin{bmatrix} 0 & p_{1,2} & p_{1,3} & p_{1,4} \\ p_{2,1} & 0 & p_{2,3} & p_{2,4} \\ p_{3,1} & p_{3,2} & 0 & p_{3,4} \\ p_{4,1} & p_{4,2} & p_{4,3} & 0 \end{bmatrix}B$$

has trace equal to

$$b_1 = p_{3,2} + p_{3,1} + p_{1,3} + p_{1,2} + p_{2,4}$$

it follows that if the players use this strategy the expected time will be

$$b_1 + 2\left(1 - b_1 - \sum_{j=1}^{4} p_{j,j}\right).$$

Since no other strategy results in all of these terms in b_1 *it follows that if* $p_{3,2} = p_{3,1} = p_{1,3} = p_{1,2} = p_{2,4} = \frac{1}{5}$ *then the expected time will be 1 and this strategy and no other will be optimal.*

The results of the calculations are displayed in the following table. The first four columns are interpreted as follows: first column - Player I strategy at location 2, 0 means stay, 1 means move to 3; second column - Player I strategy at location 3, 0 means stay, -1 means move to location 2; third column - Player II strategy at location 2 , 0 means stay, 1 means move to 3; fourth column - Player II strategy at location 3, 0 means stay, -1 means move to 2. Since we are dealing with restricted paths if a player is at an endpoint it will move to the adjacent point. The last column denotes quantity b_1. If each of the quantities appearing in the last column are equal and have sum 1 then no strategy will do as well as that described in the previous row. For example if $p_{4,3} = p_{4,2} = p_{3,4} = p_{3,2} = p_{2,1} = \frac{1}{5}$ then no strategy will do as well as that depicted in row 2: Player I remains in place at location 2 or 3 while player moves to 3 if at location 2 and remains in place if at location 3.

0	0	0	0	$p_{4,3} + p_{3,4} + p_{2,1} + p_{1,2}$
0	0	1	0	$p_{4,3} + p_{4,2} + p_{3,4} + p_{3,2} + p_{2,1}$
0	0	0	-1	$p_{3,4} + p_{2,3} + p_{2,1} + p_{1,3} + p_{1,2}$
0	0	1	-1	$p_{4,2} + p_{3,4} + p_{3,2} + p_{2,3} + p_{2,1} + p_{1,3}$
1	0	0	0	$p_{4,3} + p_{3,4} + p_{2,4} + p_{2,3} + p_{1,2}$
1	0	1	0	$p_{4,3} + p_{4,2} + p_{3,4} + p_{3,2} + p_{2,4} + p_{2,3}$
1	0	0	-1	$p_{3,4} + p_{2,4} + p_{1,3} + p_{1,2}$
1	0	1	-1	$p_{4,2} + p_{3,4} + p_{3,2} + p_{2,4} + p_{1,3}$
0	-1	0	0	$p_{4,3} + p_{3,2} + p_{3,1} + p_{2,1} + p_{1,2}$
0	-1	1	0	$p_{4,3} + p_{4,2} + p_{3,1} + p_{2,1}$
0	-1	0	-1	$p_{3,2} + p_{3,1} + p_{2,3} + p_{2,1} + p_{1,3} + p_{1,2}$
0	-1	1	-1	$p_{4,2} + p_{3,1} + p_{2,3} + p_{2,1} + p_{1,3}$
1	-1	0	0	$p_{4,3} + p_{3,2} + p_{3,1} + p_{2,4} + p_{2,3} + p_{1,2}$
1	-1	1	0	$p_{4,3} + p_{4,2} + p_{3,1} + p_{2,4} + p_{2,3}$
1	-1	0	-1	$p_{3,2} + p_{3,1} + p_{2,4} + p_{1,3} + p_{1,2}$
1	-1	1	-1	$p_{4,2} + p_{3,1} + p_{2,4} + p_{1,3}$

Five Locations

When there are five locations, there are 12 matrices describing restricted strategies resulting in 144 strategy pairs. The following matrix describes these strategies. The first column is a number used to name the strategy. The action of the strategy at location 2,3,4 are given in the columns marked 2,3,4 respectively. For example, Strategy 6 is that of staying in place at location 2 moving to 4 from location 3 and staying in place at location 4.

	2	3	4
1	0	-1	-1
2	0	-1	0
3	0	0	-1
4	0	0	0
5	0	1	-1
6	0	1	0
7	1	-1	-1
8	1	-1	0
9	1	0	-1
10	1	0	0
11	1	1	-1
12	1	1	0

Of the 144 possible values of b_i 97 result in values that are dominated by other values so there are 47 non dominated strategy pairs. We have listed below the 47 non dominated strategy pairs using the designations described in the matrix. The first number is the strategy used by I, the second by II and the third column is the resulting value of b_1.

1	8	$p_{5,4} + p_{4,2} + p_{3,1} + p_{2,3} + p_{2,1} + p_{1,3}$
1	10	$p_{5,4} + p_{4,3} + p_{4,2} + p_{3,1} + p_{2,1}$
1	12	$p_{5,4} + p_{5,3} + p_{4,2} + p_{3,1} + p_{2,1}$
2	2	$p_{5,4} + p_{4,5} + p_{3,2} + p_{3,1} + p_{2,3} + p_{2,1} + p_{1,3} + p_{1,2}$
2	6	$p_{5,4} + p_{5,3} + p_{4,5} + p_{4,3} + p_{3,2} + p_{3,1} + p_{2,1} + p_{1,2}$
3	7	$p_{4,2} + p_{3,4} + p_{3,2} + p_{2,3} + p_{2,1} + p_{1,3}$
3	8	$p_{5,4} + p_{4,2} + p_{3,2} + p_{2,3} + p_{2,1} + p_{1,3}$
3	9	$p_{4,3} + p_{4,2} + p_{3,4} + p_{3,2} + p_{2,1}$
3	10	$p_{5,4} + p_{4,3} + p_{4,2} + p_{3,2} + p_{2,1}$
3	11	$p_{5,3} + p_{4,2} + p_{3,4} + p_{3,2} + p_{2,1}$
3	12	$p_{5,4} + p_{5,3} + p_{4,2} + p_{3,2} + p_{2,1}$
4	7	$p_{4,5} + p_{3,4} + p_{3,2} + p_{2,3} + p_{2,1} + p_{1,3}$
4	11	$p_{5,3} + p_{4,5} + p_{4,3} + p_{3,4} + p_{3,2} + p_{2,1}$
5	8	$p_{5,4} + p_{4,2} + p_{3,5} + p_{3,4} + p_{2,3} + p_{2,1} + p_{1,3}$
5	10	$p_{5,4} + p_{4,3} + p_{4,2} + p_{3,5} + p_{3,4} + p_{2,1}$
5	12	$p_{5,4} + p_{5,3} + p_{4,2} + p_{3,5} + p_{3,4} + p_{2,1}$
6	2	$p_{5,4} + p_{4,5} + p_{3,5} + p_{3,4} + p_{2,3} + p_{2,1} + p_{1,3} + p_{1,2}$
6	6	$p_{5,4} + p_{5,3} + p_{4,5} + p_{4,3} + p_{3,5} + p_{3,4} + p_{2,1} + p_{1,2}$
7	3	$p_{4,3} + p_{3,2} + p_{3,1} + p_{2,4} + p_{2,3} + p_{1,2}$
7	4	$p_{5,4} + p_{4,3} + p_{3,2} + p_{3,1} + p_{2,3} + p_{1,2}$
7	7	$p_{4,2} + p_{3,1} + p_{2,4} + p_{1,3}$
7	9	$p_{4,3} + p_{4,2} + p_{3,1} + p_{2,4} + p_{2,3}$

ISBN: 978-988-17012-1-3

7	10	$p_{5,4}+p_{4,3}+p_{4,2}+p_{3,1}+p_{2,3}$
7	11	$p_{5,3}+p_{4,2}+p_{3,1}+p_{2,4}$
8	1	$p_{4,5}+p_{3,2}+p_{3,1}+p_{2,4}+p_{1,3}+p_{1,2}$
8	3	$p_{4,5}+p_{3,2}+p_{3,1}+p_{2,4}+p_{2,3}+p_{1,2}$
8	5	$p_{5,3}+p_{4,5}+p_{4,3}+p_{3,2}+p_{3,1}+p_{2,4}+p_{1,2}$
9	3	$p_{4,3}+p_{3,4}+p_{2,4}+p_{2,3}+p_{1,2}$
9	7	$p_{4,2}+p_{3,4}+p_{3,2}+p_{2,4}+p_{1,3}$
9	9	$p_{4,3}+p_{4,2}+p_{3,4}+p_{3,2}+p_{2,4}+p_{2,3}$
9	10	$p_{5,4}+p_{4,3}+p_{4,2}+p_{3,2}+p_{2,3}$
9	11	$p_{5,3}+p_{4,2}+p_{3,4}+p_{3,2}+p_{2,4}$
10	1	$p_{4,5}+p_{3,4}+p_{2,4}+p_{1,3}+p_{1,2}$
10	3	$p_{4,5}+p_{3,4}+p_{2,4}+p_{2,3}+p_{1,2}$
10	5	$p_{5,3}+p_{4,5}+p_{4,3}+p_{3,4}+p_{2,4}+p_{1,2}$
10	7	$p_{4,5}+p_{3,4}+p_{3,2}+p_{2,4}+p_{1,3}$
10	9	$p_{4,5}+p_{3,4}+p_{3,2}+p_{2,4}+p_{2,3}$
10	11	$p_{5,3}+p_{4,5}+p_{4,3}+p_{3,4}+p_{3,2}+p_{2,4}$
11	3	$p_{4,3}+p_{3,5}+p_{2,4}+p_{2,3}+p_{1,2}$
11	4	$p_{5,4}+p_{4,3}+p_{3,5}+p_{3,4}+p_{2,3}+p_{1,2}$
11	7	$p_{4,2}+p_{3,5}+p_{2,4}+p_{1,3}$
11	9	$p_{4,3}+p_{4,2}+p_{3,5}+p_{2,4}+p_{2,3}$
11	10	$p_{5,4}+p_{4,3}+p_{4,2}+p_{3,5}+p_{3,4}+p_{2,3}$
11	11	$p_{5,3}+p_{4,2}+p_{3,5}+p_{2,4}$
12	1	$p_{4,5}+p_{3,5}+p_{2,4}+p_{1,3}+p_{1,2}$
12	3	$p_{4,5}+p_{3,5}+p_{2,4}+p_{2,3}+p_{1,2}$
12	5	$p_{5,3}+p_{4,5}+p_{4,3}+p_{3,5}+p_{2,4}+p_{1,2}$

Example 11 *The most obvious situation occurs when both players begin at a location with equal and independent probabilities so that for each* i,j*,* $p_{i,j}=\frac{1}{25}$*. Any strategy in which there is a maximal number of terms in the third column will then be optimal. These are* $(2,2),(2,6),(6,2),(6,6)$*, having 8 terms. Each of these strategies will give an expected time of* $\frac{8}{25}+2\left(1-\frac{8}{25}-\frac{5}{25}\right)=\frac{32}{25}$*. A similar situation is when both players are placed with equal probability at pairs of different locations. The same strategy pairs are optimal and the expected time is then* $\frac{8}{20}+2\left(1-\frac{8}{20}\right)=\frac{8}{5}$*.*

References

[1] S. Alpern and S. Gal, *The theory of search games and rendezvous*, International Series in Operations Research and Management Sciences, Kluwer, Boston, 2002.

[2] E. J. Anderson, R. R. Weber, "The rendezvous problem on discrete locations", *Journal of Applied Probability* 28(1990) 335-340.

[3] V. J. Baston, "Two rendezvous problems on the line", *Naval Research Logistics* 46 (1999) 335-340.

[4] Ruckle, W. H., "Rendez-vous Search on a Rectangular Lattice", *Naval Research Logistics* 54: 492-496, 2007

[5] http//maxima.sourceforge.net