Models of Adding Edges between the Root and Descendants in a Complete Binary Tree Minimizing Total Path Length

Kiyoshi Sawada *

Abstract—**This paper proposes two models of adding relations to an organization structure which is a complete binary tree of height H: (i) a model of adding an edge between the root and one node of depth N and (ii) a model of adding edges between the root and all nodes of depth N. For each of the two models, an optimal depth N^* is obtained by minimizing the total path length which is the sum of lengths of shortest paths between every pair of all nodes.**

Keywords: graph theory, organization structure, complete binary tree, shortest path length

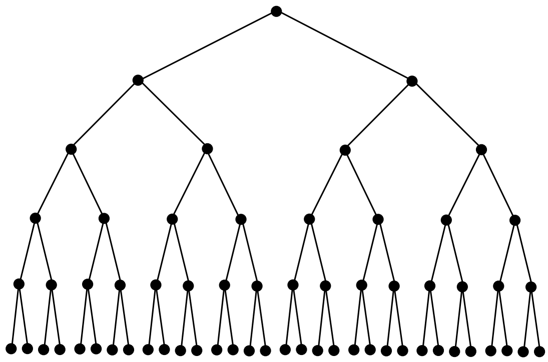

Figure 1: An example of a complete binary tree of $H = 5$.

1 Introduction

A pyramid organization [8] is the basic type of formal organization structure which is a hierarchical structure based on the principle of unity of command [3] that every unit except the top in the organization should have a single superior. In the pyramid organization there exist relations only between each superior and his subordinates. However, it is desirable to have formed additional relations other than that between each superior and his subordinates in advance in case they need communication with other departments in the organization. In companies, the relations with other departments are built by meetings, group training, internal projects, and so on. Personal relations exceeding departments are also considered to be useful for the communication of information in the organization.

The pyramid organization structure can be expressed as a rooted tree, if we let nodes and edges in the rooted tree correspond to units and relations between units in the organization respectively [5, 7]. Then the path between each node in the rooted tree is equivalent to the route of communication of information between each unit in the organization. Moreover, adding edges to the rooted tree is equivalent to forming additional relations other than that between each superior and his subordinates.

The problem of adding the minimum number of edges to a given graph so as to satisfy a given edge- or vertex-connectivity is called the graph augmentation problem [2, 4], which has applications such as network construction problems. On the other hand, the purpose of our study is to obtain an optimal set of additional relations to the pyramid organization such that the communication of information between every unit in the organization becomes the most efficient. This means that we obtain a set of additional edges to the rooted tree minimizing the sum of lengths of shortest paths between every pair of all nodes. Therefore, the problem treated in this paper is different from the graph augmentation problem described above.

We have obtained an optimal set for each of the following three models of adding relations in the same level to an organization structure which is a complete binary tree of height H: (i) a model of adding an edge between two nodes with the same depth, (ii) a model of adding edges between every pair of nodes with the same depth and (iii) a model of adding edges between every pair of siblings with the same depth [6]. A complete binary tree is a rooted tree in which all leaves have the same depth and all internal nodes have two children [1]. Figure 1 shows an example of a complete binary tree of $H = 5$. In Figure 1 the value of N expresses the depth of each node.

This paper proposes two models of adding edges between the root and descendants in a complete binary tree of height $H\,(H = 2, 3, \cdots)$: (i) a model of adding an edge between the root and one node of depth N and (ii) a

*University of Marketing and Distribution Sciences, Kobe 651-2188, Japan, Email: sawada@umds.ac.jp

model of adding edges between the root and all nodes of depth N. Model (i) corresponds to the formation of an additional relation between the top and one unit of an organization. Model (ii) is equivalent to additional relations between the top and all units in the same level.

If $l_{i,j}(= l_{j,i})$ denotes the path length, which is the number of edges in the shortest path from a node v_i to a node v_j $(i, j = 1, 2, \cdots, 2^{H+1} - 1)$ in the complete binary tree of height H, then $\sum_{i<j} l_{i,j}$ is the total path length. Furthermore, if $l'_{i,j}$ denotes the path length from v_i to v_j after adding edges in the above models, $l_{i,j} - l'_{i,j}$ is called the shortening path length between v_i and v_j, and $\sum_{i<j}(l_{i,j} - l'_{i,j})$ is called the *total shortening path length.*

In Section 2 and Section 3, for each of the two models of adding edges respectively, we formulate the total shortening path length and obtain an optimal depth N^* which maximizes the total shortening path length.

2 Adding an Edge between the Root and One Node of Depth N

This section obtains an optimal depth N^* by maximizing the total shortening path length, when a new edge between the root and one node of depth $N\,(N = 2, 3, \cdots, H\,)$ is added to a complete binary tree of height H.

2.1 Formulation of Total Shortening Path Length

Let $S_{1,H}(N)$ denote the total shortening path length by adding a new edge between the root and one node of depth N.

Let v_0 denote the node of depth N which gets adjacent to the root. The set of descendants of v_0 is denoted by V_1. (Note that every node is a descendant of itself [1].) Let V_2 denote the set obtained by removing V_1 from the set of all nodes of the complete binary tree.

The sum of shortening path lengths between every pair of nodes in V_1 and nodes in V_2 is given by

$$\begin{aligned} &A_{1,H}(N) \\ &\quad= M(H-N)\sum_{i=1}^{\lfloor \frac{N}{2} \rfloor}\Big(M(H-i)+1\Big)(N-2i+1)\,, \end{aligned} \tag{1}$$

where $M(h)$ denotes the number of nodes of a complete binary tree of height $h\,(h = 0, 1, 2, \cdots)$, and $\lfloor x \rfloor$ denotes the maximum integer which is equal to or less than x. The sum of shortening path lengths between every pair of nodes in V_2 is given by

$$\begin{aligned} &B_{1,H}(N) \\ &\quad= \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor - 1}\Big(M(H-N+i-1)+1\Big) \\ &\qquad\times \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor - i}\Big(M(H-j)+1\Big)(N-2i-2j+1)\,, \end{aligned} \tag{2}$$

where we define

$$\sum_{i=1}^{0} \cdot = 0\,. \tag{3}$$

From the above equations, the total shortening path length $S_{1,H}(N)$ is given by

$$\begin{aligned} &S_{1,H}(N) \\ &\quad= A_{1,H}(N) + B_{1,H}(N) \\ &\quad= M(H-N)\sum_{i=1}^{\lfloor \frac{N}{2} \rfloor}\Big(M(H-i)+1\Big)(N-2i+1) \\ &\qquad+ \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor - 1}\Big(M(H-N+i-1)+1\Big) \\ &\qquad\times \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor - i}\Big(M(H-j)+1\Big)(N-2i-2j+1)\,. \end{aligned} \tag{4}$$

Since the number of nodes of a complete binary tree of height h is

$$M(h) = 2^{h+1} - 1\,, \tag{5}$$

$S_{1,H}(N)$ of Equation (4) becomes

$$\begin{aligned} &S_{1,H}(N) \\ &\quad= \Big(2^{H-N+1} - 1\Big)\sum_{i=1}^{\lfloor \frac{N}{2} \rfloor} 2^{H-i+1}(N-2i+1) \\ &\qquad+ \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor - 1} 2^{H-N+i} \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor - i} 2^{H-j+1}(N-2i-2j+1)\,. \end{aligned} \tag{6}$$

2.2 An Optimal Depth

In this subsection, we seek $N = N^*$ which maximizes $S_{1,H}(N)$ in Equation (6).

Lemma 1.

$$S_{1,H}(2L) \;>\; S_{1,H}(2L+1)\,, \tag{7}$$

for $L = 1, 2, \cdots, \lfloor \frac{H-1}{2} \rfloor$.

Proof. Since

$$S_{1,H}(2L) - S_{1,H}(2L+1)$$

$$\begin{aligned} &= 2^{H-2L}\sum_{i=1}^{L} 2^{H-i+1}(2L-2i) + \sum_{i=1}^{L} 2^{H-i+1} \\ &\quad + \sum_{i=1}^{L-1} 2^{H-2L+i-1} \sum_{j=1}^{L-i} 2^{H-j+1}(2L-2i-2j) \\ &> 0\,, \end{aligned} \tag{8}$$

we have $S_{1,H}(2L) > S_{1,H}(2L+1)$. □

Let $R_H(L) \equiv S_{1,H}(2L)$, so that we have

$$\begin{aligned} R_H(L) &= 2^{2H-3L+2} - 2^{2H-2L+3} + 2^{2H-L+2} \\ &\quad - 3\cdot 2^{H-L+1} - L\cdot 2^{H+2} + 3\cdot 2^{H+1}\,, \end{aligned} \tag{9}$$

for $L = 1, 2, \cdots, \lfloor \frac{H}{2} \rfloor$. Let $\Delta R_H(L) \equiv R_H(L+1) - R_H(L)$, so that we have

$$\begin{aligned} \Delta R_H(L) &= \left(-7\cdot 2^{-3L-1} + 3\cdot 2^{-2L+1} - 2^{-L+1}\right) 2^{2H} \\ &\quad + \left(3\cdot 2^{-L} - 4\right) 2^{H}\,, \end{aligned} \tag{10}$$

for $L = 1, 2, \cdots, \lfloor \frac{H}{2} \rfloor - 1$.

Lemma 2.
(1) *If* $L = 1$*, then* $\Delta R_H(1) < 0$ *for* $H = 4, 5$ *and* $\Delta R_H(1) > 0$ *for* $H = 6, 7, \cdots$.
(2) *If* $L = 2, 3, \cdots$*, then* $\Delta R_H(L) < 0$.

Proof. Let us define x as

$$x = 2^H\,, \tag{11}$$

then $\Delta R_H(L)$ in Equation (10) becomes

$$\begin{aligned} T_L(x) &= \left(-7\cdot 2^{-3L-1} + 3\cdot 2^{-2L+1} - 2^{-L+1}\right) x^2 \\ &\quad + \left(3\cdot 2^{-L} - 4\right) x\,, \end{aligned} \tag{12}$$

which is a quadratic function of the continuous variable x.

From the sign of the coefficient of x^2 in Equation (12), the following two cases can be discussed:
(i) When $L = 1$, then $-7\cdot 2^{-3L-1} + 3\cdot 2^{-2L+1} - 2^{-L+1} > 0$ which indicates that $T_L(x)$ is convex downward.
(ii) When $L = 2, 3, \cdots$, then $-7\cdot 2^{-3L-1} + 3\cdot 2^{-2L+1} - 2^{-L+1} < 0$ which means that $T_L(x)$ is convex upward.

In the case of (i), $T_L(x)$ becomes

$$T_L(x) = \frac{1}{16}x^2 - \frac{5}{2}x\,. \tag{13}$$

Since $T_L(x) < 0$ for $0 < x < 40$ and $T_L(x) > 0$ for $x > 40$ in Equation (13), we have $\Delta R_H(1) < 0$ for $H = 4, 5$ and $\Delta R_H(1) > 0$ for $H = 6, 7, \cdots$.

In the case of (ii), by differentiating $T_L(x)$ in Equation (12) with respect to x, we obtain

$$\begin{aligned} T_L'(x) &= \left(-7\cdot 2^{-3L} + 3\cdot 2^{-2L+2} - 2^{-L+2}\right) x \\ &\quad + 3\cdot 2^{-L} - 4\,. \end{aligned} \tag{14}$$

Since

$$T_L(0) = 0 \tag{15}$$

and

$$T_L'(0) = 2^{-L+1}\left(\frac{3}{2} - 2^{L+1}\right) < 0\,, \tag{16}$$

we have $T_L(x) < 0$ for $x > 0$. Therefore, we have $\Delta R_H(L) < 0$ for $H = 4, 5, \cdots$. □

From Lemma 2, the optimal depth N^* can be obtained and is given in Theorem 3.

Theorem 3.
(1) *If* $H = 2, 3, 4, 5$*, then the optimal depth is* $N^* = 2$.
(2) *If* $H = 6, 7, \cdots$*, then* $N^* = 4$.

Proof.
(1) If $H = 2, 3$, then $L^* = 1$; that is, $N^* = 2$ trivially. If $H = 4, 5$, then $L^* = 1$; that is, $N^* = 2$ since $\Delta R_H(L) < 0$ for $L = 1, 2, \cdots, \lfloor \frac{H}{2} \rfloor - 1$.
(2) If $H = 6, 7, \cdots$, then $L^* = 2$; that is, $N^* = 4$ since $\Delta R_H(1) > 0$ and $\Delta R_H(L) < 0$ for $L = 2, 3, \cdots, \lfloor \frac{H}{2} \rfloor - 1$. □

3 Adding Edges between the Root and All Nodes of Depth N

This section obtains an optimal depth N^* by maximizing the total shortening path length, when new edges between the root and all nodes of depth $N\,(N = 2, 3, \cdots, H\,)$ are added to a complete binary tree of height H.

3.1 Formulation of Total Shortening Path Length

Let $S_{2,H}(N)$ denote the total shortening path length by adding new edges between the root and all nodes of depth N.

The sum of shortening path lengths between every pair of nodes whose depths are equal to or more than N is given by

$$A_{2,H}(N) = \Big(M(H-N)\Big)^2 2^N \sum_{i=1}^{N-1} i\,2^i\,, \tag{17}$$

where $M(h)$ is as before. The sum of shortening path lengths between every pair of nodes whose depths are less than N and those whose depths are equal to or more than N is given by

$$\begin{aligned} &B_{2,H}(N) \\ &= M(H-N)\,2^{N+1} \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor - 1} \sum_{j=1}^{N-i-1} j\,2^j \\ &\quad + M(H-N)\,2^N \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor} \left(M\left(\left\lfloor \frac{N+1}{2} \right\rfloor - i \right) + 1 \right) \\ &\quad \times (N-2i+1)\,, \end{aligned} \tag{18}$$

and the sum of shortening path lengths between every pair of nodes whose depths are less than N is given by

$$\begin{aligned}
& C_{2,H}(N) \\
= \; & 2^N \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor - 1} \sum_{j=1}^{N-2i-1} j\, 2^j \\
& + 2^{N+1} \sum_{h=1}^{\lfloor \frac{N}{2} \rfloor - 2} \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor - h - 1} \sum_{j=1}^{N-2h-i-1} j\, 2^j \\
& + \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor - 1} 2^{N-i} \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor - i} \left(M \left(\left\lfloor \frac{N+1}{2} \right\rfloor - j \right) + 1 \right) \\
& \times (N - 2i - 2j + 1)\,, \qquad (19)
\end{aligned}$$

where $\lfloor x \rfloor$ is as before and where Equation (3) is applied and furthermore we define

$$\sum_{i=1}^{-1} \cdot = 0\,. \qquad (20)$$

From these equations, the total shortening path length $S_{2,H}(N)$ is given by

$$\begin{aligned}
& S_{2,H}(N) \\
= \; & A_{2,H}(N) + B_{2,H}(N) + C_{2,H}(N) \\
= \; & \left\{ M(H-N) \right\}^2 2^N \sum_{i=1}^{N-1} i\, 2^i \\
& + M(H-N)\, 2^{N+1} \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor - 1} \sum_{j=1}^{N-i-1} j\, 2^j \\
& + M(H-N)\, 2^N \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor} \left(M \left(\left\lfloor \frac{N+1}{2} \right\rfloor - i \right) + 1 \right) \\
& \times (N - 2i + 1) \\
& + 2^N \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor - 1} \sum_{j=1}^{N-2i-1} j\, 2^j \\
& + 2^{N+1} \sum_{h=1}^{\lfloor \frac{N}{2} \rfloor - 2} \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor - h - 1} \sum_{j=1}^{N-2h-i-1} j\, 2^j \\
& + \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor - 1} 2^{N-i} \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor - i} \left(M \left(\left\lfloor \frac{N+1}{2} \right\rfloor - j \right) + 1 \right) \\
& \times (N - 2i - 2j + 1)\,. \qquad (21)
\end{aligned}$$

From Equations (5) and (21), we have that

$$\begin{aligned}
& S_{2,H}(N) \\
= \; & (N-2)\, 2^{2H+2} + 2^{2H-N+3} - 2^{H+N+3} \\
& + \left(-N + 2 \left\lfloor \frac{N}{2} \right\rfloor + 3 \right) 2^{H+N-\lfloor \frac{N}{2} \rfloor + 2} \\
& + \left(-N + 2 \left\lfloor \frac{N}{2} \right\rfloor + 3 \right) 2^{H+N-2\lfloor \frac{N}{2} \rfloor + 2} \\
& + \left(\left\lfloor \frac{N}{2} \right\rfloor - 2 \right) 2^{H+3} \\
& + \left(-N \left\lfloor \frac{N}{2} \right\rfloor + 2 \left\lfloor \frac{N}{2} \right\rfloor^2 + 2N - \left\lfloor \frac{N}{2} \right\rfloor - 4 \right) \\
& \times 2^{2N - 2\lfloor \frac{N}{2} \rfloor + 1} \\
& + \left(\left\lfloor \frac{N}{2} \right\rfloor^2 - 4 \left\lfloor \frac{N}{2} \right\rfloor + 4 \right) 2^{N+1}\,. \qquad (22)
\end{aligned}$$

3.2 An Optimal Depth

In this subsection, we seek $N = N^*$ which maximizes $S_{2,H}(N)$ in Equation (22).

Let $S_{even}(N)$ denote the total shortening path length when N is an even number. From Equation (22), we have that

$$\begin{aligned}
S_{even}(N) \; = \; & (N-2)\, 2^{2H+2} + 2^{2H-N+3} - 2^{H+N+3} \\
& + 3 \cdot 2^{H+\frac{1}{2}N+2} + (N-1)\, 2^{H+2} \\
& + \left(N^2 - 2N \right) 2^{N-1}\,, \qquad (23)
\end{aligned}$$

for $N = 2, 4, 6, \cdots, 2\left\lfloor \frac{H}{2} \right\rfloor$. Let $S_{odd}(N)$ denote the total shortening path length when N is an odd number. From Equation (22), we have that

$$\begin{aligned}
S_{odd}(N) \; = \; & (N-2)\, 2^{2H+2} + 2^{2H-N+3} - 2^{H+N+3} \\
& + 2^{H+\frac{1}{2}N+\frac{7}{2}} + (N-1)\, 2^{H+2} \\
& + \left(N^2 - 2N + 1 \right) 2^{N-1}\,, \qquad (24)
\end{aligned}$$

for $N = 3, 5, 7, \cdots, 2\left\lfloor \frac{H-1}{2} \right\rfloor + 1$.

Let $\Delta S_{even}(N) \equiv S_{odd}(N+1) - S_{even}(N)$ when N is even, so that we have

$$\begin{aligned}
& \Delta S_{even}(N) \\
& = \; 2^{2H+2} - 2^{2H-N+2} - 2^{H+N+3} + 2^{H+\frac{1}{2}N+2} \\
& \quad + 2^{H+2} + \left(N^2 + 2N \right) 2^{N-1}\,, \qquad (25)
\end{aligned}$$

for $N = 2, 4, 6, \cdots, 2\left\lfloor \frac{H-1}{2} \right\rfloor$. Let $\Delta S_{odd}(N) \equiv S_{even}(N+1) - S_{odd}(N)$ when N is odd, so that we have

$$\begin{aligned}
& \Delta S_{odd}(N) \\
& = \; 2^{2H+2} - 2^{2H-N+2} - 2^{H+N+3} + 2^{H+\frac{1}{2}N+\frac{5}{2}} \\
& \quad + 2^{H+2} + \left(N^2 + 2N - 3 \right) 2^{N-1}\,, \qquad (26)
\end{aligned}$$

for $N = 3, 5, 7, \cdots, 2\left\lfloor \frac{H}{2} \right\rfloor - 1$.

Lemma 4.
(1) $\Delta S_{even}(N) > 0$ *for* $N = 2, 4, 6, \cdots, 2\left\lfloor \frac{H-1}{2} \right\rfloor$.
(2) $\Delta S_{odd}(N) > 0$ *for* $N = 3, 5, 7, \cdots, 2\left\lfloor \frac{H}{2} \right\rfloor - 1$.

Proof.

(1) If $N \leq H-2$, then

$$\begin{aligned}\Delta S_{even}(N) &\\ = \;& 2^{2H+2}\left(1-2^{-N}-2^{-H+N+1}\right)+2^{H+\frac{1}{2}N+2}\\ &+2^{H+2}+\left(N^2+2N\right)2^{N-1}\\ > \;& 0\,. \end{aligned} \tag{27}$$

If H is odd and $N = H-1$, then

$$\begin{aligned}\Delta S_{even}(N) &\\ = \;& 2^{H+2}\left(-2+2^{\frac{1}{2}(H-1)}+1\right)+\left(H^2-1\right)2^{H-2}\\ > \;& 0\,. \end{aligned} \tag{28}$$

(2) If $N \leq H-2$, then

$$\begin{aligned}\Delta S_{odd}(N) &\\ = \;& 2^{2H+2}\left(1-2^{-N}-2^{-H+N+1}\right)+2^{H+\frac{1}{2}N+\frac{5}{2}}\\ &+2^{H+2}+\left(N^2+2N-3\right)2^{N-1}\\ > \;& 0\,. \end{aligned} \tag{29}$$

If H is even and $N = H-1$, then

$$\begin{aligned}\Delta S_{odd}(N) &\\ = \;& 2^{H+2}\left(-2+2^{\frac{1}{2}H}+1\right)+\left(H^2-4\right)2^{H-2}\\ > \;& 0\,. \end{aligned} \tag{30}$$

□

From Lemma 4, the optimal depth N^* can be obtained and is given in Theorem 5.

Theorem 5. *The optimal depth is $N^* = H$.*

Proof. If $H = 2$, then $N^* = 2$ trivially; that is $N^* = H$. If $H = 3, 4, \cdots$, then $N^* = H$ since $\Delta S_{even}(N) > 0$ for $N = 2, 4, 6, \cdots, 2\left\lfloor \frac{H-1}{2} \right\rfloor$ and $\Delta S_{odd}(N) > 0$ for $N = 3, 5, 7, \cdots, 2\left\lfloor \frac{H}{2} \right\rfloor - 1$. □

4 Conclusions

This study considered the addition of relations to an organization structure such that the communication of information between every unit in the organization becomes the most efficient. For each of two models of adding edges between the root and nodes of depth N to a complete binary tree of height H which can describe the basic type of a pyramid organization, we obtained an optimal depth N^* which maximizes the total shortening path length.

For the first model of adding an edge between the root and one node of depth N, we showed Theorem 3 that the depth of node so as to maximize the total shortening path length is $N^* = 2$ for complete binary trees of small height ($H = 2, 3, 4, 5$) and $N^* = 4$ for higher complete binary trees ($H = 6, 7, \cdots$). These results indicate the most efficient manner of adding a single relation between the top and one unit and this depends on the number of levels in the organization structure.

For the second model of adding edges between the root and all nodes of depth N, we showed Theorem 5 that the depth of nodes so as to maximize the total shortening path length is $N^* = H$. This means that the most efficient way to add relations between the top and all units at the same level is to get adjacent to the lowest level.

References

[1] Cormen, T.H., Leiserson, C.E., Rivest, R.L., Stein, C., *Introduction to Algorithms*, 2nd Edition, MIT Press, 2001.

[2] Eswaran, K.P., Tarjan, R.E., "Augmentation Problems," *SIAM Journal on Computing*, V5, N4, pp.653-665, 12/76.

[3] Koontz, H., O'Donnell, C., Weihrich, H., *Management*, 7th Edition, McGraw-Hill, 1980.

[4] Nagamochi, H., "Recent Development of Graph Connectivity Augmentation Algorithms," *IEICE Trans on Information & Systems*, VE83-D, N3, pp.372-383, 3/00.

[5] Robbins, S.P., *Essentials of Organizational Behavior*, 7th Edition, Prentice Hall, 2003.

[6] Sawada, K., Wilson, R., "Models of Adding Relations to an Organization Structure of a Complete K-ary Tree," *European Journal of Operational Research*, V174, pp.1491-1500, 11/06.

[7] Takahara, Y., Mesarovic, M., *Organization Structure: Cybernetic Systems Foundation*, Kluwer Academic / Plenum Publishers, 2003.

[8] Takahashi, N., "Sequential Analysis of Organization Design: a Model and a Case of Japanese Firms," *European Journal of Operational Research*, V36, N3, pp.297-310, 9/88.