On Multicomponent System Reliability with Microshocks - Microdamages Type of Components' Interaction

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Abstract–**Consider a two component parallel system. The defined new stochastic dependences of the component' life-times are understood as a stochastic reflection of rather complicated components' physical interactions. The interactions are assumed to obey some "continuous" pattern, valid until one of the components fails. According to that pattern some infinitesimal micro-damages in each component' physical structure (caused by the remaining component's "harmful" activity) are related to corresponding infinitesimal increments in the component' failure rate. Based on this association between the physical failure mechanism and its much simpler probabilistic equivalence, we describe the latter analytically in terms of the life-times' joint probability distribution; actually we constructed a wide class of such stochastic models. As a special case of the bivariate distributions we obtain the first bivariate Gumbel distribution along with its, perhaps first known, reliability interpretation.**

Index Terms–**Gumbel's bivariate exponential distribution as a reliability model, Stochastic dependence, System reliability models.**

I. INTRODUCTION

We construct a (new) **stochastic model** for reliability ([2]) of a two components parallel system. Both the components $\mathbf{u_1}$, $\mathbf{u_2}$, also called "units", are considered in two different situations that we call the **"off-system"** and the **"in-system"** conditions. While in the 'off -system conditions' both the units work separately of each other, i. e., with no mutual physical interactions. As a consequence of lack of physical contact, their life-times, say $\mathbf{T_1}$, $\mathbf{T_2}$ are stochastically **independent** random variables with the, given in advance, "original" probability distribution functions $\mathbf{F(t_1)}$, $\mathbf{F(t_2)}$ **.** The 'in-system' conditions are created when the components are installed into the system. It is assumed, that in the system some **mutual** interactions occur. As a result, in the considered system, side-effects of each component performance, influence physics of failure mechanism of the remaining unit. The components failure mechanism has two parallel aspects**: 1)** the **reality** aspect, based on engineering devices and / or underlying physical and chemical processes that eventually lead to each unit's failure, and **2)** the **stochastic reflection of** the system's **physics.** The associated physical phenomena are often too complex and complicated to be followed and efficiently handled for the reliability prediction purposes.

On the physical part of this (modeling) problem, the mutual impact of any component on the other, can be explained in the following manner. During the components 'in-system

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performance, either of the two creates such an (environmental) situation that the other unit is "constantly bombarded" by a string of small harmful (or beneficial) "micro-incentives" that we call, in this paper, **"micro-shocks".** Each such "micro-shock" causes a corresponding **"micro-damage"** in the affected unit's physical constitution. This **'micro-shock → micro-damage'** relation we classify as being the "physics of the problem". This part might be investigated by use of engineering methods as long as the underlying phenomena are not too complicated for an efficient physical analysis. Otherwise, in order to simplify investigations, we rather consider an associated '**stochastic mechanism',** basically ignoring the details of underlying physical phenomena and replacing the problem's "physics" by proper **statistical analysis** of the stochastic model now being constructed.

For the junction between the system's physics and its probabilistic behavior we will consider any possible relation between the micro-damages of the components and the resulting small **"probability -micro-changes"** in the original probability distributions of the unit' life-times. This stochastic part of the problem we may consider as the **'micro-damages → probability-micro-changes' relation** that in accordance to the chosen methodology of the research replaces the previous, more physical, relation.

In order to quantify these micro-changes in the life time' probability distributions we have chosen, among several other possibilities**, micro-changes (**in general increments**) in the components' failure rates.**

Denote the components' (original) **'off-system' failure rates,** associated with the independent life times T_1 , T_2 by the symbols $\boldsymbol{\lambda_1(t_1)}$ **,** $\boldsymbol{\lambda_2(t_2)}$, respectively.

It is quite obvious that the changes in $\lambda_1(x_1)$, $\lambda_2(x_2)$ will transform the independent 'off-system' life times T_1, T_2 into some new dependent 'in system' **life times $\boldsymbol{X_1, X_2}$.**

Now we can formulate the paper's **objective** as to **determine the joint reliability function** of the lifetimes' random vector **($\boldsymbol{X_1, X_2}$).** This function, equivalent of the joint probability distribution, will serve as the **model** of the considered **system' reliability**. Other, similar stochastic models one can find in quite a large subject's literature. See, for example, [3]–[7] .

To start with the model' construction based on the 'micro-damages → probability-micro-changes' pattern we must to take into consideration that each micro-damage that occurs is **extremely small** so that there is usually no practical possibility as to detect any significant influence of this micro-damage on a given unit' reliability. This can only be realized after a long enough time period as the micro-damages cumulate their effects. Since also the time periods the micro-damages occur are extremely small, while the number of these micro-events per time unit is a "huge" one, we have chosen the **(**typical in such situations**) continuous approximation** as an **analytical description** of these "fugitive" phenomena. In other words we utilize the familiar calculus notion of the '**infinitesimal quantities'.** Thus, in the classical analytical model, an accumulation of the micro-changes in both the components' failure rates is to be expressed by the **Riemann integral** device.

II. THE MODEL'S CONSTRUCTION

In this section we find the joint probability distribution of the random vector **($\boldsymbol{X_1,X_2}$)** in terms of its **joint survival function $\boldsymbol{S(x_1,x_2) = Pr(X_1 > x_1 , X_2 > x_2)}$.** Recall, that the joint survival function **$s^*(x_1,x_2)$** of the independent 'off-system' component life-times T_1, T_2 is given by the following 'product' form**:**

$s^*(x_1,x_2) =$

$$\exp[-\int_0^{x_1} \lambda_1(t_1)\, d t_1 - \int_0^{x_2} \lambda_2(t_2)\, d t_2] \quad \textbf{(1)}$$

where $\lambda_1(t_1)$, $\lambda_2(t_2)$ are the component's 'off-system' failure rates.

When the components u_1, u_2 work **in** the system then, in accordance with the adopted (in the analytical model) assumption, in "every" infinitesimal small time interval $[\tau_k , \tau_k + d\tau_k)$ an occurrence of an infinitesimal **micro-damage** of the component, say **u_k , $k = 1, 2$; (**that is caused by '**side effects'** accompanying an activity of component u_m , where **$m = 1, 2$; $m \neq k$)** results in an **infinitesimal increment $\alpha_{k\,m}(\tau_k)\, d\tau_k$ of** u_k's failure rate. For every 'past' time instant **τ_k** that increment is given by a predetermined quantity **$\alpha_{km}(\tau_k)$** which, in a stochastic way, reflects "an amount" (or a "density") of the physical influence of the component u_m on component u_k at a given time instant τ_k**.** That quantity is chosen **(**and then must be **statistically verified** for fit to available data**)** to be a continuous function of all the past epochs **τ_k** the micro-damages occurred. 'In part' it stands for time derivative of an overall failure rate.

In the mathematical model, all the **stochastic** effects $\alpha_{k\,m}(\tau_k) d\tau_k$ **(**of the physical micro-damage's **)** "sum up" over the time. Each of their "partial accumulation" **(**created from the 'time zero', up to the epoch , say **t_k considered to be** "current"**)** is expressed as the following Riemann integral**:**

$$\boldsymbol{\varphi_k(t_k)} = \left(\int_0^{t_k} \alpha_{k\,m}(\tau_k)\, d\tau_k , \ (k \neq m) \right). \quad \textbf{(2)}$$

This integral itself is a non decreasing continuous function of the <u>current</u> time **t_k** as is taken over all time intervals $[\, 0, t_k]$, with **$0 \le t_k \le x_k$** $< \infty$, for $k = 1, 2$**.**

Both the variables **x_k** , ($k = 1, 2$) as present in the foregoing condition, are the same as the arguments of the survival function (1). The integral (2) will be thought of as a <u>measure</u> of "magnitude" of the u_k 's micro-damage's accumulation up to the current time epoch t_k**.**

At every "current" time instant **t_k** **,** the overall '**in-system' failure rate $r_k(t_k)$** of component u_k ($k = 1, 2$) is defined to be a simple arithmetic sum of the 'off-system' failure rate $\lambda_k(t_k)$ and the "additional failure rate" given by the integral (2). Thus, at every time epoch **t_k** one obtains the following formula for the 'in-system' **failure rate $r_k(t_k)$** of component **u_k :**

$$r_k(t_k) = \lambda_k(t_k) + \int_0^{t_k} \alpha_{k\,m}(\tau_k)\, d\tau_k \quad \textbf{(3)}$$

as $k, m = 1, 2$, and $k \neq m$**.**

We consider the failure rate formula (3) to be valid for each time argument **t_k** satisfying $0 \le t_k \le$ **x ,** where **x** = minimum (x_1, x_2) is considered to be the time of the first_failure in the system. From the above one obtains the following survival function**:**

$S_1(x) = Pr (\min(X_1 , X_2) > x)$, with x = **$\min(x_1, x_2)$),** for the first order statistics **X** of the set of random variables: $\{ X_1, X_2 \}$**.** That is**:**

$$\mathbf{S_1(x) = \exp[-\int_0^{x} r_1(t_1)\, d t_1 - \int_0^{x} r_2(t_2)\, d t_2]} \quad \textbf{(4)}$$

where $\mathbf{r_1(t_1)}$ and $\mathbf{r_2(t_2)}$ are given by (1) for $k = 1, 2$.
These two functions represent **the 'in system' failure rates** of the components u_1 and u_2 respectively, at the time instances t_1, t_2, both prior to the time, say x, after which the first failure in the system occurs. Consequently, the given by (4) function $\mathbf{S_1(x)}$ is the (whole) system' reliability function, **if** the system **reliability structure was series.**
At next, consider the (parallel) system's residual life -time's failure rate, say $\mathbf{r_k(t)}$ i.e., the failure rate of either surviving component $\mathbf{c_k}$, at any time $\mathbf{t}$ satisfying $x \le \mathbf{t} \le y$, **(** where $\mathbf{x, y}$ are the time epochs of the first and the second failure in the system respectively **).** For that period of the time we have chosen the following failure pattern.
Namely, we define the failure rate $\mathbf{r_k(t)}$ in the time interval $\mathbf{[x, y]}$ as the following arithmetic sum**:**

$$\mathbf{r_k(t) = \lambda_k(t) + \int_0^x \alpha_{k\,m}(\tau)\, d\tau .} \quad \textbf{(5)}$$

In this context, **the integral** $\int_0^x \alpha_{k\,m}(\tau)\, d\tau$ **is constant** over time past x **(** $k, m = 1, 2$, and $k \neq m$ **).**
The reason for its constancy is based on a simple observation that in the time interval $[x, y]$ only (one) component $\mathbf{c_k}$ is working in the system, and thus the process of micro -damages accumulation is terminated since the time x passed. This integral **(**present as a part in (6), (6a) that follow**)** is an additional part of the overall failure rate $r_k(t)$ of component c_k, and may be understood as a **measure of "memory"** of the micro-incentives the c_k received before the other component c_m stopped its activity at time x**.**

The Final Formula for the **joint survival function** $\mathbf{S(x_1, x_2) = Pr(X_1 > x_1, X_2 > x_2)}$ of the in-system component life-times $\mathbf{X_1}$, $\mathbf{X_2}$, is given below as:

$$\mathbf{Pr(X_1 > x_1, X_2 > x_2) = \exp\Big[-\int_0^{x_1} \{ \lambda_1(t_1) + \int_0^{t_1} \alpha_{1,2}(\tau_1)\, d\tau_1 \} dt_1 - \int_0^{x_1} \{ \lambda_2(t_2) + \int_0^{t_2} \alpha_{2,1}(\tau_2)\, d\tau_2 \}\, dt_2 \Big]}$$

$$\mathbf{\exp\Big[-\int_{x_1}^{x_2} \{ \lambda_2(t_2)\, dt_2 \} - (x_2 - x_1) \int_0^{x_1} \alpha_{2,1}(\tau_1)\, d\tau_1 \Big];} \quad \textbf{(6)}$$

when $\mathbf{x_1 \le x_2}$ **, and**

$$\mathbf{Pr(X_1 > x_1, X_2 > x_2) = \exp\Big[-\int_0^{x_2} \{ \lambda_2(t_2) + \int_0^{t_2} \alpha_{2,1}(\tau_2)\, d\tau_2 \} dt_2 - \int_0^{x_2} \{ \lambda_1(t_1) + \int_0^{t_1} \alpha_{1,2}(\tau_1)\, d\tau_1 \}\, dt_1 \Big]}$$

$$\mathbf{\exp\Big[-\int_{x_2}^{x_1} \{ \lambda_1(t_1)\, dt_1 \} - (x_1 - x_2) \int_0 \alpha_{1,2}(\tau_1)\, d\tau_1 \Big],} \quad \textbf{(6a)}$$

when $\mathbf{x_1 > x_2}$ **.**

If in both formulas (6) and (6a) one sets $x_2 = 0$, then one obtains the **marginal** probability distribution **of** $\mathbf{X_1}$ to be the same as the original probability distribution $\mathbf{F_1(x_1)}$ of the off-system life-time T_1, **related to** the given in advance original failure rate $\mathbf{\lambda_1(t_1)}$**.** The similar result one obtains when imposing in (6), (6a) the condition $x_1 = 0$.

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The latter condition yields the marginal distribution of the $\mathbf{X_2}$ to be equal $\mathbf{F_2(x_2)}$. As the conclusion one derives the following surprising Property, shared by all the models that obey the pattern expressed by (6), (6a).

Property 1. For any joint probability distribution given by the joint reliability function $\mathbf{S(x_1, x_2)}$ defined by (6), (6a), any, given in advance, original probability distributions $\mathbf{F_1(x_1), F_2(x_2)}$ of the 'off-system' life-times T_1, T_2 **(**with the assumed notation: $x_1 = t_1$, $x_2 = t_2$ **) are preserved !** as **the marginal distributions** of the joint probability distribution of the 'in-system' life-times X_1, X_2 of the considered units u_1, u_2. □

From Property 1, the conclusion can be derived as the following:

Corollary. Suppose we are given a pair of probability distributions $G_1(x_1)$, $G_2(x_2)$ that belong to any class of probability distribution functions, whose all members posses continuous failure (hazard) rates, say $\lambda_1(t_1)$, $\lambda_2(t_2)$. If one puts any arbitrary single pair of such distributions into the scheme defined by (7), (7a) then, as a result, one can generate a wide class of the bivariate survival functions $S(x_1, x_2)$, whose marginals remain to be the $G_1(x_1)$, $G_2(x_2)$. The class of the so obtained bivariate probability distributions, "given the (fixed) marginals $G_1(x_1)$, $G_2(x_2)$", is determined by the family of all the continuous functions $\alpha_{i,j}(\tau_i)$, (i, j = 1, 2, with $i \neq j$) that produce all the integrals in (7), (7a) finite. □

So, in this particular sense one can consider **the "bivariate Weibull, gamma (in particular, exponential), the extreme value"** and other joint probability distributions.

Realize, however that the marginal distributions $\mathbf{G_1(x_1)}$, $\mathbf{G_2(x_2)}$, in the Corollary also may represent two **distinct distribution classes.** The last possibility may be utilized in modeling reliability of two stochastically dependent units (such as system components) each one subjected to a different failure mechanism . Apparently such cases often are realistic.

III. EXAMPLE

As a particular class of the bivariate survival functions $S(x_1, x_2)$, satisfying the pattern given by (6), (6a), we now choose a class of bivariate exponential distributions, given by two arbitrary constant failure rates λ_1, λ_2 for the marginals. We also restrict the dependence structure, by assuming it is only determined by two **constant** functions $\alpha_{1,2}(\ )$, $\alpha_{2,1}(\ )$. Recall, they represent the rates of increment in the failure rate of the unit u_1 caused by u_2 and the failure rate increment of u_2, caused by u_1, respectively. The resulting class of the **joint exponential survival functions** can be expressed by the following specification of the previous, more general, patterns (6), (6a) **:**

$$\Pr(X_1 > x_1, X_2 > x_2) =$$

$$\exp\Big[-\int_0^{x_1} \{ \lambda_1 + \int_0^{t_1} \alpha_{1,2}\, d\tau_1 \}\, dt_1 - \int_0^{x_1} \{ \lambda_2 + \int_0^{t_2} \alpha_{2,1}\, d\tau_2 \}\, dt_2 \Big] \qquad (7)$$

$$\exp\Big[-\int_{x_1}^{x_2} \{ \lambda_2\, dt_2 \} - (\alpha_{2,1}\, x_1)(x_2 - x_1) \Big]$$

for $\mathbf{x_1 \leq x_2}$ **,** and

$$\Pr(X_1 > x_1, X_2 > x_2) =$$

$$\exp\left[-\int_0^{x_2} \{ \lambda_2 + \int_0^{t_2} \alpha_{2,1}\, d\tau_2 \}\, dt_2 - \int_0^{x_2} \{ \lambda_1 + \int_0^{t_1} \alpha_{1,2}\, d\tau_1 \}\, dt_1 \right] \quad \textbf{(7a)}$$

$$\exp\left[-\int_{x_2}^{x_1} \{ \lambda_1\, dt_1 \} - (\alpha_{1,2}\, x_2)(x_1 - x_2) \right],$$

for $x_1 > x_2$ **.**

Upon additionally simplifying the assumption stating that $\alpha_{1,2}(\ ) = \alpha_{2,1}(\ )$ **=** α **= constant,** both formulas (7) and (7a) reduce to the following single one**:**

$$\Pr(X_1 > x_1, X_2 > x_2) =$$

$$\exp[-\lambda_1 x_1 - \lambda_2 x_2 - \alpha x_1 x_2] \quad \textbf{(8)}$$

Thus, as a special case of the model one obtains the first bivariate exponential **Gumbel** probability distribution **as** system **reliability model.** □

IV. REMARK

There is an interesting relationship between the models defined in this paper (in particular the Gumbel model) and the model given by **Freund** in [5]. In Freund model the **two components work independently** (no interactions) **until the first failure** and then the remaining component is affected 'by lack' of the other component. Unlike in Freund's case we have exactly opposite , say **complementary** failure mechanism. The components interact as long as they work together while after the first failure, the remaining component enjoys the off system (normal) situation.

Also, the constructions presented here **differ** significantly from the models considered in [1] or in similar papers cited in [1].

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