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# Existence of Periodic Motions for Dynamical Systems with Finite Delay

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***Abstract*—We give a concise existence result of periodic motions for nonlinear dynamical system**

$$\ddot{x}(t)+a\dot{x}(t)+[2+\sin(bx(t))]\,f(x(t-r))=\cos(\omega t+\varphi)$$

**by means of the analysis and computing method.**

***Index Terms*—boundedness, dynamical system, model, periodic motion.**

## I. INTRODUCTION

Consider the mathematical model

$$\begin{aligned}&\ddot{x}(t)+a\dot{x}(t)+[2+\sin(bx(t))]\,f(x(t-r))\\&=\cos(\omega t+\varphi)\end{aligned}\tag{1}$$

where $a=const.>0$, $f(x)\in C^1$, $b,\omega,\varphi=const.$, the finite delay $r=const.>0$. The nonlinear dynamical system (1) can be used to describe many practical engineering problems [1—8]. The existence problem of periodic motions of dynamical system (1) is not only the considerable significance in theory, but also of important background in application [1, 2, 5—9]. In this paper, a convenient and efficient result is given to solve the problem above.

## II. ANALYSIS AND COMPUTING

Equivalent system of the dynamical system (1) is

$$\begin{cases}\dot{x}(t)=y(t)\\ \dot{y}(t)=-ay(t)-[2+\sin(bx(t))]\,f(x(t))+\\ [2+\sin(bx(t))]\int_{-r}^{0}y(t+s)f_x'(x(t+s))\,ds\\ +\cos(\omega t+\varphi).\end{cases}\tag{2}$$

If $f(x)\operatorname{sgn}x\to+\infty$ ($|x|\to+\infty$), then we choose

$$V(x,y)=\int_0^x[2+\sin(b\xi)]\,f(\xi)\,d\xi+\frac{1}{2}y^2$$

as the V-functional of the system (2). Therefore,

$$\begin{aligned}\dot{V}_{(2)}(x(t),y(t))&=[2+\sin(bx(t))]\,f(x(t))\dot{x}(t)\\&\quad+y(t)\dot{y}(t)\\&=[2+\sin(bx(t))]\,f(x(t))y(t)\\&\quad+y(t)\{-ay(t)\\&\quad-[2+\sin(bx(t))]\,f(x(t))\\&\quad+[2+\sin(bx(t))]\times\\&\quad\int_{-r}^{0}y(t+s)f_x'(x(t+s))\,ds\\&\quad+\cos(\omega t+\varphi)\}\\&=-ay^2(t)+y(t)\cos(\omega t+\varphi)+\\&\quad[2+\sin(bx(t))]y(t)\times\\&\quad\int_{-r}^{0}y(t+s)f_x'(x(t+s))\,ds.\end{aligned}$$

If $3rc<a$, then there is a positive constant $q>1$ such that $3qrc<a$. When $|f'(x)|\le c$, $c=const.>0$, $|y(t)|>0$ and $q|y(t)|\ge|y(t+s)|$ $(-r\le s\le0)$, we have

$$\begin{aligned}\dot{V}_{(2)}(x(t),y(t))&=-ay^2(t)+y(t)\cos(\omega t+\varphi)+\\&\quad[2+\sin(bx(t))]y(t)\times\\&\quad\int_{-r}^{0}y(t+s)f_x'(x(t+s))\,ds\\&\le-ay^2(t)+y(t)\cos(\omega t+\varphi)+\\&\quad|2+\sin(bx(t))|\,|y(t)|\times\\&\quad\int_{-r}^{0}|y(t+s)|\,|f_x'(x(t+s))|\,ds\\&\le-ay^2(t)+y(t)\cos(\omega t+\varphi)\\&\quad+3\,|y(t)|\int_{-r}^{0}c\,|y(t+s)|\,ds\\&\le-ay^2(t)+y(t)\cos(\omega t+\varphi)\\&\quad+3c\,|y(t)|\int_{-r}^{0}q\,|y(t)|\,ds\\&=-ay^2(t)+y(t)\cos(\omega t+\varphi)+\\&\quad+3qrc\,y^2(t)\end{aligned}$$

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$$\leq [\,-a+\frac{|\cos(\omega t+\varphi)|}{|\,y(t)\,|}+3\,q\,r\,c\,]\,y^2(t)$$
$$=-[\,a-3\,q\,r\,c-\frac{|\cos(\omega t+\varphi)|}{|\,y(t)\,|}\,]\,y^2(t)$$
$$\leq -[\,a-3\,q\,r\,c-\frac{1}{|\,y(t)\,|}\,]\,y^2(t).$$

Therefore, there is a positive constant $M$ and a positive constant $\mu$ such that

$$[\,a-q\,r\,c-\frac{1}{|\,y(t)\,|}\,]\geq \mu$$

for $|\,y(t)\,|\geq M$ *and* $q\,|\,y(t)\,|\geq |\,y(t+s)\,|$. Thus, we have

$$\dot V_{(2)}(\,x(t),\,y(t)\,)\leq -\mu\,y^2(t)$$

for $|\,y(t)\,|\geq M$ *and* $q\,|\,y(t)\,|\geq |\,y(t+s)\,|$. Hence, the $y$ coordinate of the solutions of system (1) is uniformly ultimately bounded for a positive constant $D$.

If $|\,y\,|\leq D$ and $\tilde V(\,x\,,\,y\,)=V(\,x\,,\,y\,)+y$, then there is a positive constant $L$ such that

$$\begin{aligned}\dot{\tilde V}_{(2)}(\,x(t),\,y(t)\,)&=\dot V_{(2)}(\,x(t),\,y(t)\,)+\dot y\\
&=-a\,y^2(t)+y(t)\cos(\omega t+\varphi)+\\
&\quad[\,2+\sin(bx(t))\,]\,y(t)\times\\
&\quad\int_{-r}^{0}y(t+s)f_x'(x(t+s))\,ds\\
&\quad-[\,2+\sin(bx(t))\,]\,f(x(t))+\\
&\quad[\,2+\sin(bx(t))\,]\times\\
&\quad\int_{-r}^{0}y(t+s)f_x'(x(t+s))\,ds\\
&\quad+\cos(\omega t+\varphi)-a\,y(t)\\
&\leq -a\,y^2(t)+|\,y(t)\,|+\\
&\quad 3\,|\,y(t)\,|\int_{-r}^{0}|\,y(t+s)\,||\,f_x'(x(t+s))\,|\,ds\\
&\quad-[\,2+\sin(bx(t))\,]\,f(x(t))+\\
&\quad 3\int_{-r}^{0}|\,y(t+s)\,||\,f_x'(x(t+s))\,|\,ds\\
&\quad-a\,y(t)+1\\
&\leq -[\,2+\sin(bx(t))\,]\,f(x(t))+L\,.\end{aligned}$$

Since $f(x)\,\mathrm{sgn}\,x\to+\infty\,(\,|\,x\,|\to+\infty\,)$, we can choose a constant $\tilde L>0$ such that

$$\dot{\tilde V}_{(2)}(\,x(t),\,y(t)\,)<-0.5\quad \text{if}\; x(t)\geq \tilde L\,.$$

Therefore, there is a positive constant $\beta$ such that the $x$ coordinate of the solutions of system (1) satisfies $x(t)\leq\beta$ for $|\,y\,|\leq D$. Similarly, we can obtain the $x$ coordinate of the solutions of system (1) satisfies $x(t)\geq-\beta$ for $|\,y\,|\leq D$. It is shown that solutions of system (1) are uniform ultimately bounded [1, 2, 5].

If $f(x)\in C^1$, then the system (1) generates a $2\pi/\omega-$ periodic process $\tilde U$ on the Banach space $\tilde C(\,[\,-r,\;0\,],\;R^2\,)$. Since the solutions of system (1) are uniform ultimately bounded, there is a bounded set $\tilde D\subseteq\tilde C(\,[-r,\;0],\;R^2)$ such that $R\times\tilde D$ attracts all points in $\tilde C([-b,\;0],\;R^2)$. Hence, the system (1) is $2\pi/\omega-$periodic [1, 2, 5].

## III. Main Result

From analysis and computing above, we have result as follow:

Suppose $f'(x)$ is a continuous function. If there is a constant $c>0$ such that

( i ) $|\,f'(x)\,|\leq c$,

( ii ) $3cr<a$,

(iii) $f(x)\,\mathrm{sgn}\,x\to+\infty\,(\,|\,x\,|\to+\infty\,)$,

then there is a $2\pi/\omega-$ periodic motion of the dynamical system (1) .

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