An ε-Uniform Initial Value Technique For Convection-Diffusion Singularly Perturbed Problems

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Abstract—**In this paper, we have proposed an ε-uniform initial value technique for singularly perturbed convection-diffusion problems in which an asymptotic expansion approximation of the solution of boundary value problem is constructed using the basic idea of WKB method. In this computational technique, the original problem reduces to combination of an initial value problem and a terminal value problem. The initial value problem happened to be singularly perturbed problem, which is then solved by using a hybrid scheme on an appropriate piecewise uniform Shishkin mesh, whereas trapezoidal scheme is applied to terminal boundary value problems. Necessary error estimates are derived for the method. Computational efficiency and accuracy are verified through numerical examples.**

Key words: Asymptotic expansion approximation, Implicit Euler method, Piece-wise Uniform Shishkin mesh, Singular perturbation problem, Trapezoidal method

1 Introduction

Singularly perturbation problems(SPP's) arise in several branches of computational science which include fluid dynamics, quantum mechanics, elasticity, chemical reactor theory, gas porous electrodes theory, etc. The presence of small parameter(s) in these problems prevents us from obtaining satisfactory numerical solutions. It is a well known fact that the solutions of the SPP's have a multi-scale character. That is, there are thin transition layers where the solutions can jump abruptly, while away from the layers the solution behave regularly and varies slowly.

It is well known that classical methods fail to provide reliable numerical results for such problems (in the sense that the parameter ε and the mesh size h can not vary independently). To solve these type of problems, mainly there are two approaches namely, fitted operator and fitted mesh methods. The first one has advantage that it does not require the knowledge of location and width of the boundary layer, however, they are difficult to extend for higher dimensional problems. Whereas, the disadvantage of second approach is requirement of knowledge of location and width of boundary layer, but is gaining popularity because simple piecewise uniform meshes like Shishkin meshes are sufficient to give satisfactory results and extension to higher dimensional problems is comparatively easy. For solving various types of singular perturbation problems, many techniques are available in the literature, more details can be found in books of Farrell et.al.[2] and Roos et.al.[3].

In this article we consider the following convection-diffusion singularly perturbed boundary value problems
(SPBVP's):

$$Lu(x) \equiv \varepsilon u''(x) + a(x)u'(x) = f(x), \qquad (1.1)$$

$$x \in \Omega = (0,1) \qquad (1.2)$$

$$u(0) = p, \quad u(1) = q \qquad (1.3)$$

where $0 < \varepsilon \leq 1$ is a small positive parameter, $a(x)$ and $f(x)$ are sufficiently smooth functions, such that $a(x) \geq \beta > 0$ on $\overline{\Omega} = [0,1]$. Under these assumptions, the (SPBVP)(1.1-1.3) possesses a unique solution $u(x) \in C^2(\overline{\Omega})$ with a boundary layer of width $O(\varepsilon)$ at x=0.

In recent past, some researchers employed techniques for solving (SPBVP's), which are based on the idea of replacing a two point boundary value problem by two suitable initial-value problems. For example Gasparo and Macconi[7] considered a semi-linear ordinary differential equation which was integrated to obtain a first-order ordinary differential equation and considered both the inner and outer solutions. A similar matching combing the reduced problem and a WKB approximation for the full problem has also been employed by Gasparo and Macconi[8] for linear and semi-linear (SPBVP's). These matching ideas are based on on the work of Robert[9]. Robert's idea has been extended by Valanarasu and Ramanujam[10]

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to BVP of singularly perturbed ode's, where the authors used the combination of Euler's method and exponentially fitted method. The main disadvantage of this method is that, it works only for those values of N(number of mesh points) which are of same order as of ε(perturbation parameter). Also, its theoretical order of convergence is only one. To overcome these drawbacks, we have used the method proposed in[6], which is combination of Euler and Trapezoidal method on well known Shishkin mesh in such a way that, the order of convergence is increased to two and it works for different values of N and ε,even for h(mesh size)<< ε.

In this paper, we are interested in obtaining numerical solutions of (1.1-1.3) for small values of positive parameter ε. We suggest an initial-value method, in line of [8] for SPBVP (1.1-1.3). In [8], the BVP is replaced with one suitable initial value problems(IVP) and a terminal value problem(TVP). The integration of these problems goes in opposite directions, but each problem can be solved independently of the other. The IVP is of singularly perturbed type, whereas the TVP does not contain any small parameter. Hence, we solve TVP by Trapezoidal method and IVP by hybrid scheme proposed by Bawa and Kumar [6]. The hybrid scheme, is the combination of Trapezoidal method and Implicit Euler method and is applied on modified Shishkin mesh. The scheme retains the oscillation free behavior of Euler's method and higher order convergence of Trapezoidal method.

The outline of this paper is as follows: we present some results in the form of theorems and asymptotic expansion approximation in section 2. Shishkin mesh and initial-value technique are given Section 3. Finally, some numerical examples are presented in Section 4. The paper ends with conclusion.

2 Preliminaries

2.1 Asymptotic Expansion Approximation

It is well known that, by using the fundamental idea of WKB [1], an asymptotic expansion approximation for the solution of SPBVP (1.1-1.3)is given by:

$$u_{as}(x) = u_R(x) + (p - u_R(0))u_1(x) + O(\varepsilon) \quad (2.1)$$

where $u_R(x)$ is the solution of reduced problem

$$u'_R(x) = \tfrac{f(x)}{a(x)}, \quad x \in \Omega \quad (2.2)$$

$$u_R(1) = q \quad (2.3)$$

and $u_1(x)$ is defined on $\overline{\Omega}$ by

$$u_1(x) = exp\{-\int_0^x \frac{a(s)}{\varepsilon} ds\} \quad (2.4)$$

Also,$u_1(x)$ satisfies the following initial value problem

$$\varepsilon u'_1(x) + a(x)u_1(x) = 0, \quad x \in \overline{\Omega} \quad (2.5)$$

$$u_1(0) = 1 \quad (2.6)$$

Theorem 2.1 *The zero order asymptotic expansion approximation* u_{as}*satisfies the inequality*

$$|(u - u_{as})(x)| \leq C\varepsilon, \quad x \in \overline{\Omega}$$

where u(x) is he solution of BVP (1.1-1.3).

Proof. See Ref.[4].

Theorem 2.2 *u(x) is he solution of BVP (1.1-1.3).Then:*

$$|u^{(k)}(x)| \leq C[1 + \varepsilon^{-k} e^{\frac{-\beta x}{\varepsilon}}], \quad x \in \overline{\Omega}, k = 1, 2....$$

Proof. See Ref.[14].

3 Discretization and Mesh for Initial-Value Problem

The singularly perturbed initial value problem(2.5-2.6) is solved by following hybrid scheme[6].

Let the mesh points of $\Omega = [0, 1]$ be

$$x_0, x_i = \overline{\Omega}_{k=0}^{i-1} h_k, h_k = x_{k+1} - x_k, x_N = 1, i = 1, 2, ..., N-1.$$

We define the scheme as

$$L^N U_i \equiv \begin{cases} \varepsilon D^- U_i + \frac{a_{i-1}U_{i-1} + a_i U_i}{2} = \frac{f_{i-1} + f_i}{2} & 0 < i \leq N/2, \\ \varepsilon D^- U_i + a_i U_i = f_i, \quad N/2 < i \leq N \end{cases} \quad (3.1)$$

where $D^- U_i = \frac{U_i - U_{i-1}}{h_i}$ and $a_i = a(x_i), f_i = f(x_i)$.

3.1 Piece-wise uniform Shishkin mesh

It is known that on an equidistant mesh no scheme can attain convergence at all mesh points uniformly in ε, unless its coefficients have an exponential property. In order to be $\varepsilon-$uniform convergent, we will use the Shishkin mesh on the difference scheme defined above. Shishkin mesh is attractive because of its simplicity and adequate for handling a vide variety of singularly perturbed problems[14]. The basic idea behind this mesh is to divide $\overline{\Omega}$ into $[0, \sigma]$ and $[\sigma, 1]$, where σ is a transition point(a function of N and ε)and place $N/2$ points of the mesh in the region $[0, \sigma]$ known as "inner region " where the solution varies fast and place remaining $N/2$ mesh points in the region $[\sigma, 1]$ called "outer region" where the solution varies slowly. In this hybrid scheme, we use Trapezoidal scheme in the inner region and Euler's scheme in the outer region. The transition point σ,which separates the fine and coarse portions of the mesh is obtained by:

$$\sigma = min\{\frac{1}{2}, \sigma_0 \varepsilon \ln N\}$$

and $\sigma_0 \geq \frac{2}{\sqrt{\beta}}$.

Further, we denote the mesh size in the region $[0, \sigma]$ by $h = 2\frac{\sigma}{N}$ and in $(\sigma, 1]$ by $H = 2\frac{(1-\sigma)}{N}$.

Proposition 3.1 *Let $u(x)$ and U_i be respectively the solutions of (2.5-2.6) and (3.1). Then, the local truncation error satisfies the following bounds:*

$$|L_\varepsilon^N(U_i - u_i)| \leq CN^{-2}\sigma_0^2 \ln^2 N, for\, 0 < i \leq N/2,$$

$$|L_\varepsilon^N(U_i - u_i)| \leq C(N^{-1}\varepsilon + N^{-\beta\sigma_0}), for\, N/2 < i \leq N \quad and \quad H \leq \varepsilon,$$

$$|L_\varepsilon^N(U_i - u_i)| \leq C(N^{-2} + N^{-\beta\sigma_0}), for\, N/2 < i \leq N \quad and \quad H > \varepsilon.$$

Proof. See Ref.[6]

3.2 Description of Method

In this sub-section, we describe the initial value technique [8] in brief to solve the (SPBVP) (1.1-1.3)

- **Step 1.** Solve the TVP (2.2-2.3) by using Trapezoidal Method.
  Let $U_0(x_i)$ be its solution.
- **Step 2.** Solve the IVP (2.5-2.6) by using hybrid scheme 3.1 on described Shishkin mesh in section (3.1).
  Let $V_0(x_i)$ be its solution.
- **Step 3.** Compute solution of(1.1-1.3) as

$$U_i = U_0(x_i) + (p - u_R(0))V_0(x_i), \quad x \in \overline{\Omega}$$

4 Numerical Experiments and Discussions

To illustrate the present technique, two examples are provided here. Computational results are given in the form of tables. The results are presented with maximum point wise error's for various values of ε and N. We have also computed the computational order of convergence and shown in the same table along with maximum errors. In all the cases, we take the constant σ_0=2. From, Table 1 and 2, we clearly deduce the second order of convergence(except by a logarithmic factor). Maximum point wise error are calculated as:

$$E_\varepsilon^N = \max_{x_i \varepsilon \overline{\Omega}^N} \{|u(x_i) - u_i^N|\}$$

Where $u(x_i)$ is the exact solution and u_i^N is the numerical solution obtained by using N mesh intervals in the domain $\overline{\Omega}^N$. The rates of convergence are calculated as:

$$p^N = \frac{lnE_\varepsilon^N - lnE_\varepsilon^{2N}}{ln2}$$

Example 4.1 *Consider the BVP:*

$$\varepsilon u''(x) + u'(x) = 0, \quad x \in (0,1)$$
$$u(0) = 1, \quad u(1) = 1$$

The exact solution of this problem is

$$u(x) = \frac{1 - e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}}.$$

Results are given in Table 1, for various values of N and ε .

Example 4.2 *Consider the non-homogeneous BVP:*

$$\varepsilon u''(x) + u'(x) = 1 + 2x, \quad x \in (0,1)$$
$$u(0) = 0, \quad g(x) = u(1) = 1.$$

The exact solution of this problem is

$$u(x) = x(x + 1 - 2\varepsilon) + (2\varepsilon - 1)\frac{1 - e^{-x/\varepsilon}}{1 - e^{-1/\varepsilon}}.$$

The numerical results are given in Table 2.

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Table 1: *Maximum point wise errors and rates of convergence for Example 4.1.*

ε	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-8}	1.474883E-02	5.822688E-03	2.084745E-03	7.052506E-04	2.301492E-04	7.281707E-05	2.247652E-05
	1.340843	1.481815	1.563663	1.615567	1.660221	1.695858	
2^{-10}	1.474883E-02	5.822688E-03	2.084745E-03	7.052506E-04	2.301492E-04	7.281707E-05	2.247652E-05
	1.340843	1.481815	1.563663	1.615567	1.660221	1.695858	
2^{-12}	1.474883E-02	5.822688E-03	2.084745E-03	7.052506E-04	2.301492E-04	7.281707E-05	2.247652E-05
	1.340843	1.481815	1.563663	1.615567	1.660221	1.695858	
2^{-14}	1.474883E-02	5.822688E-03	2.084745E-03	7.052506E-04	2.301492E-04	7.281707E-05	2.247652E-05
	1.340843	1.481815	1.563663	1.615567	1.660221	1.695858	
2^{-16}	1.474883E-02	5.822688E-03	2.084745E-03	7.052506E-04	2.301492E-04	7.281707E-05	2.247652E-05
	1.340843	1.481815	1.563663	1.615567	1.660221	1.695858	
2^{-18}	1.474883E-02	5.822688E-03	2.084745E-03	7.052506E-04	2.301492E-04	7.281707E-05	2.247652E-05
	1.340843	1.481815	1.563663	1.615567	1.660221	1.695858	
2^{-20}	1.474883E-02	5.822688E-03	2.084745E-03	7.052506E-04	2.301492E-04	7.281707E-05	2.247652E-05
	1.340843	1.481815	1.563663	1.615567	1.660221	1.695858	
2^{-24}	1.474883E-02	5.822688E-03	2.084745E-03	7.052506E-04	2.301492E-04	7.281707E-05	2.247652E-05
	1.340843	1.481815	1.563663	1.615567	1.660221	1.695858	
2^{-28}	1.474883E-02	5.822688E-03	2.084745E-03	7.052506E-04	2.301492E-04	7.281707E-05	2.247652E-05
	1.340843	1.481815	1.563663	1.615567	1.660221	1.695858	
2^{-32}	1.474883E-02	5.822688E-03	2.084745E-03	7.052506E-04	2.301492E-04	7.281707E-05	2.247652E-05
	1.340843	1.481815	1.563663	1.615567	1.660221	1.695858	
2^{-36}	1.474883E-02	5.822688E-03	2.084745E-03	7.052506E-04	2.301492E-04	7.281707E-05	2.247652E-05
	1.340843	1.481815	1.563663	1.615567	1.660221	1.695858	
2^{-39}	1.474883E-02	5.822688E-03	2.084745E-03	7.052506E-04	2.301492E-04	7.281707E-05	2.247652E-05
	1.340843	1.481815	1.563663	1.615567	1.660221	1.695858	

Table 2: *Maximum point wise errors and rates of convergence for Example 4.2.*

ε	Number of mesh points						
	16	32	64	128	256	512	1024
2^{-8}	2.037855E-02	1.131160E-02	8.282743E-03	7.700625E-03	7.636503E-03	7.622467E-03	7.617530E-03
	0.849248	0.449623	0.105133	0.012063	0.002654	0.000935	
2^{-10}	1.599539E-02	7.063400E-03	3.427317E-03	2.205096E-03	1.956126E-03	1.940117E-03	1.938630E-03
	1.179221	1.043283	0.636238	0.172843	0.011855	0.001106	
2^{-12}	1.499298E-02	6.105652E-03	2.400336E-03	1.035524E-03	5.935718E-04	4.940730E-04	4.874028E-04
	1.29607	1.346909	1.212876	0.802866	0.264699	0.01961	
2^{-14}	1.480987E-02	5.893433E-03	2.163653E-03	7.850824E-04	3.115463E-04	1.613726E-04	1.250964E-04
	1.329378	1.445639	1.462553	1.333397	0.949051	0.367356	
2^{-16}	1.476409E-02	5.840375E-03	2.104472E-03	7.252094E-04	2.498768E-04	9.294108E-05	4.415560E-05
	1.337959	1.472602	1.536989	1.537181	1.426829	1.07372	
2^{-18}	1.475265E-02	5.827110E-03	2.089677E-03	7.102403E-04	2.350812E-04	7.770487E-05	2.746580E-05
	1.340121	1.479501	1.5569	1.595148	1.597082	1.500368	
2^{-20}	1.474978E-02	5.823794E-03	2.085978E-03	7.064980E-04	2.313822E-04	7.403903E-05	2.369299E-05
	1.340663	1.481236	1.561966	1.610408	1.64392	1.643826	
2^{-24}	1.474889E-02	5.822758E-03	2.084822E-03	7.053285E-04	2.302263E-04	7.289344E-05	2.255195E-05
	1.340832	1.481779	1.563557	1.615243	1.659191	1.692537	
2^{-28}	1.474883E-02	5.822693E-03	2.084749E-03	7.052554E-04	2.301540E-04	7.282185E-05	2.248123E-05
	1.340843	1.481813	1.563656	1.615546	1.660156	1.69565	
2^{-32}	1.474883E-02	5.822689E-03	2.084745E-03	7.052509E-04	2.301495E-04	7.281737E-05	2.247681E-05
	1.340843	1.481815	1.563662	1.615565	1.660217	1.695845	
2^{-36}	1.474883E-02	5.822688E-03	2.084745E-03	7.052506E-04	2.301492E-04	7.281739E-05	2.247654E-05
	1.340843	1.481815	1.563663	1.615566	1.66022	1.695857	
2^{-39}	1.474883E-02	5.822688E-03	2.084745E-03	7.052506E-04	2.301492E-04	7.281738E-05	2.247652E-05
	1.340843	1.481815	1.563663	1.615566	1.660221	1.695858	

5 Conclusions

In the recent past, Many initial-value technique have been applied to solve various(SPBVPs), But either they work under very severe restriction on mesh size or they converge very slow.In the present paper, we have proposed a computational technique for obtaining numerical solutions of BVP (1.1-1.3). This paper demonstrates, the effectiveness of the Shishkin mesh by modifying the initial value technique[8] in a very simple way so that higher order, almost second order of convergence can be achieved with no restrictions on values of h and ε. Our method is easy for computer implementation and more effective in the sense of solution error's.

References

[1] Nayfeh A.H., *Introduction to Perturbation Methods.* Wiley, New York,1981.

[2] P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan, and G.I. Shishkin, *Robust Computational Techniques for Boundary Layers.* Chapman & Hall/CRC Press, 2000.

[3] H.-G. Roos, M. Stynes, and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations.* Springer, Berlin, 1996.

[4] E.P. Doolan, J.J.H. Miller, W.H.A. Schilders, *Uniform Numerical Methods For Problems With Initial And Boundary Layers.* Boole Press, Dublin, 1980.

[5] J.J.H. Miller, Optimal uniform difference schemes for linear initial-value problems. *Comput.Math.Appl.*, **12B**(1986),pp.1209-1215 .

[6] R.K.Bawa,Vinod Kumar *Higher Order Robust Computational Technique For Singularly Perturbed Initial Value Problems.* (submitted for publication).

[7] M.G.Gasparo, M.Macconi, New initial-value method for singularly perturbed boundary-value problems. *J.Optim.Theory Applic.(JOTA)*,**63**(1989),pp.213-224 .

[8] M.G.Gasparo, M.Macconi, Initial-value methods for second-order singularly perturbed boundary-value problems. *J.Optim.Theory Applic.(JOTA)*,**66**(1990),pp.197-210 .

[9] S.M.Roberts, A boundary value technique for singular perturbation problems. *J.Math.Anal.Applic.*,**87**(1982),pp.489-508, .

[10] T.Valanarasu, N.Ramanujam, Asymptotic intial value method for singularly perturbedboundary value problems for second order ordinary differential equations. *Appl.Math.Comput*,**116**(2003),pp.167-182 .

[11] T.Valanarasu, N.Ramanujam, An Asymptotic intial value method for second order singular perturbation problems of convection-diffusion type with a discontinuous source term. *J.Optim.Theory Applic.(JOTA)*,**23**(2007),pp.141-152 .

[12] J.Carroll, Exponentially fitted one-step methods for the numerical solution of the scalar Riccati equation. *Comput.Appl*, **16**(1986),pp.9-25.

[13] J.Jayakumar, Improvement of numerical solution by boundary value techinque for singularly perturbed one dimensional reaction diffusion problem. *Appl.Math.Coput.*, **142**(1993),pp.417-447 .

[14] J.J.H. Miller, E. O'Riordan, and G.I. Shishkin, *Fitted numerical methods for singular perturbation problems.* World Scientific,1996.

[15] K.Salvekumar, Optimal uniform finite difference schemes of order two for stiff differential equations. *Commun.Numer.Methods Eng.*, **10**(1994),pp.611-622 .

[16] M.J.O'Reilly, A uniform scheme for the singularly perturbed Riccati equation. *Numer.Math.*, **50**(1987),pp.483-501 .