On the Global Uniform Asymptotic Stability of Nonlinear Dynamic System

Jiemin Zhao

Abstract—**We give a concise result of global uniform asymptotic stability for nonlinear dynamical system**

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = [1+\sin^2(A\,y(t))] \int_{-r}^{0} y(t+s)\,K(x(t+s))\,ds \\ -B\,y(t) + C\,x(t)[1+\sin^2(A\,y(t))] \end{cases}$$

by means of the method of Liapunov functional.

Index Terms—**dynamical system, finite delay, model, Stability.**

I. INTRODUCTION

Consider the mathematical model

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = [1+\sin^2(A\,y(t))] \int_{-r}^{0} y(t+s)\,K(x(t+s))\,ds \\ -B\,y(t) - C\,x(t)[1+\sin^2(A\,y(t))] \end{cases} \quad (1)$$

where $A, B, C = const.$, the finite delay $r = const. > 0$, $K(x)$ is a continuous function, $B, C > 0$. The nonlinear dynamical system (1) can be used to describe many practical engineering problems [1—10]. The problem of global uniform asymptotic stability of dynamical system (1) is not only the considerable significance in theory, but also of important background in application [1, 2, 5—11]. In this paper, a convenient and efficient result is given to solve the problem above.

II. ANALYSIS AND COMPUTING

Let

$$V = Cx^2 + \int_0^y \frac{2\xi}{1+\sin^2(A\xi)} d\xi + a\int_{-r}^{0}\int_{t+s}^{t} y^2(u)\,du\,ds$$

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where a is an arbitrary constant. Thus, if

$$(x(t),\, y(t)) = (x(t, t_0, x_0),\, y(t, t_0, y_0))$$

is a solution of dynamical system (1), then the derivative $\dot{V}_{(1)}$ of V along $(x(t),\, y(t))$ satisfies

$$\dot{V}_{(1)} = 2Cx(t)\dot{x}(t) + \frac{2y(t)}{1+\sin^2(Ay(t))}\dot{y}(t) + a\int_{-r}^{0} [\,y^2(t) - y^2(t+s)\,]ds.$$

Using (1), we obtain

$$\begin{aligned} \dot{V}_{(1)} &= 2Cx(t)\dot{x}(t) + \frac{2y(t)}{1+\sin^2(Ay(t))}\dot{y}(t) + a\int_{-r}^{0} [\,y^2(t) - y^2(t+s)\,]ds. \\ &= 2Cx(t)y(t) + \frac{2y(t)}{1+\sin^2(Ay(t))}[1+ \sin^2(A\,y(t))] \int_{-r}^{0} y(t+s)\,K(x(t+s))\,ds - \\ &\quad B\,y(t) - C\,x(t)[1+\sin^2(A\,y(t))] + a\int_{-r}^{0} [\,y^2(t) - y^2(t+s)\,]ds \\ &= 2\,y(t)\int_{-r}^{0} y(t+s)\,K(x(t+s))\,ds - \frac{2B\,y^2(t)}{1+\sin^2(Ay(t))} + a\int_{-r}^{0} [\,y^2(t) - y^2(t+s)\,]ds. \end{aligned}$$

Using the inequality

$$\begin{aligned} &2\,y(t)\int_{-r}^{0} y(t+s)\,K(x(t+s))\,ds \\ &\le 2\int_{-r}^{0} |\,y(t)\,|\;|\,y(t+s)\,|\;|\,K(x(t+s))\,|ds, \end{aligned}$$

we have

$$\dot{V}_{(1)} = 2\,y(t)\int_{-r}^{0} y(t+s)\,K(x(t+s))\,ds -$$

$$\frac{2By^2(t)}{1+\sin^2(Ay(t))}+a\int_{-r}^{0}[\,y^2(t)-y^2(t+s)\,]ds.$$

$$\le 2\int_{-r}^{0}|\,y(t)|\,|\,y(t+s)|\,|\,K(x(t+s))|ds-$$

$$\frac{2By^2(t)}{1+\sin^2(Ay(t))}+a\int_{-r}^{0}[\,y^2(t)-y^2(t+s)\,]ds.$$

If there is a constant $\mu>0$ such that $\left|K(x)\right|\le\mu$, then the derivative $\dot{V}_{(1)}$ of V satisfies

$$\dot{V}_{(1)}\le 2\mu\int_{-r}^{0}|\,y(t)|\,|\,y(t+s)|ds-$$

$$\frac{2By^2(t)}{1+\sin^2(Ay(t))}+a\int_{-r}^{0}[\,y^2(t)-y^2(t+s)\,]ds.$$

Taking $a=\mu$, we have

$$\dot{V}_{(1)}\le 2\mu\int_{-r}^{0}|\,y(t)|\,|\,y(t+s)|ds-$$

$$\frac{2By^2(t)}{1+\sin^2(Ay(t))}+\mu\int_{-r}^{0}[\,y^2(t)-y^2(t+s)\,]ds.$$

Using the inequality

$$2\alpha\beta\le\alpha^2+\beta^2,$$

we have

$$\dot{V}_{(1)}\le\mu\int_{-r}^{0}[\,y^2(t)+y^2(t+s)\,]\,ds-$$

$$\frac{2By^2(t)}{1+\sin^2(Ay(t))}+\mu\int_{-r}^{0}[\,y^2(t)-y^2(t+s)\,]ds$$

$$=2\mu\int_{-r}^{0}y^2(t)ds-\frac{2By^2(t)}{1+\sin^2(Ay(t))}$$

$$=2\mu r\,y^2(t)-\frac{2By^2(t)}{1+\sin^2(Ay(t))}$$

$$\le 2\mu r\,y^2(t)-By^2(t).$$

If $2\mu r<B$, then $\dot{V}_{(1)}\le(2\mu r-B)\,y^2(t)\le 0$.

Thus,

$$\dot{V}_{(1)}\le 0 \text{ and } \dot{V}_{(1)}=0 \text{ only if } (x(t),y(t))=(0,0).$$

In fact, since $y(t)=0$ we have

$$\dot{x}(t)=y(t)=0,$$

$$\dot{y}(t)=[1+\sin^2(A\,y(t))]\int_{-r}^{0}y(t+s)\,K(x(t+s))\,ds$$

$$-By(t)-Cx(t)[1+\sin^2(A\,y(t))]=-Cx(t).$$

Thus, $-Cx(t)=0$. since $C=const.>0$ we have $x(t)=0$.

On the other hand,

$$Cx^2\to+\infty\,(\,|x|\to+\infty\,)$$

And

$$\int_0^y\frac{2\xi}{1+\sin^2(A\xi)}d\xi\to+\infty\,(\,|y|\to+\infty\,)$$

Thus, the nonlinear dynamical system (1) is globally uniformly asymptotically stable.

III. Main Result

From analysis and computing above, we have result as follow:

Suppose $K(x)$ is a continuous function. If there is a constant $\mu>0$ such that

(i) $\left|K(x)\right|\le\mu$,
(ii) $2\mu r<B$,

then the system (1) is globally uniformly asymptotically stable.

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