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Three-player Hackenbush played on strings is $\mathcal{NP}$-complete

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Abstract— **Why are three-player games much more complex than two-player games? Is it much more difficult to cooperate or to compete? Three-player Hackenbush is a three-player version of Black-White Hackenbush, a classic combinatorial game. When Black-White Hackenbush is played on strings, cooperation is much more difficult than competition and, as consequence, three-player Hackenbush played on strings is $\mathcal{NP}$-complete.**

Keywords: combinatorial games, $\mathcal{NP}$-complete, three-player Hackenbush

1 Introduction

Combinatorial game theory is a branch of mathematics devoted to studying the optimal strategy in perfect-information games where typically two players are involved. To extend this theory so to allow more than two players is a challenging and fascinating problem for different reasons.

Typically, more than two parties are involved in a real-world economical, social or political conflict and a winning strategy is often the result of alliances. In two-player games there exist no coalitions because the two players are in conflict each other, but in three-player games cooperation is a key-factor because to determine the winning strategy of a player means to consider the worst scenario, i.e., assuming that both opponents are allied against him/her.

The first theories of Li [1] and Straffin [2] concerning impartial three-player combinatorial games have made various restrictive assumptions about the rationality of one's opponents and the formation and behavior of coalitions. Loeb [3] introduces the notion of a stable winning coalition in a multi-player game as a new system of classification of games. Differently, Propp [4] adopts in his work an agnostic attitude toward such issues, and seeks only to understand in what circumstances one player has a winning strategy against the combined forces of the other two.

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Cincotti [5] presents an extension of Conway's theory of partizan games [6], [7] to classify three-player partizan games. Such a theory has been applied to three-Player Hackenbush, that is to say a three-player version of Black-White Hackenbush, a classic combinatorial game. In this paper we show that, in Black-White Hackenbush played on strings, cooperation is much more difficult than competition, and it causes complexity in three-player Hackenbush.

2 Three-player Hackenbush

Black-White Hackenbush is a classic combinatorial game defined in [6], [8]. Every instance of this game is represented by $G = \{G_1, G_2, \ldots, G_n\}$ where G_i for all $1 \leq i \leq n$ is an undirected connected graph such that:

- at least one edge is connected to a certain line called the *ground*, and
- every edge is colored either black or white.

Two players, called Left and Right, move alternately. Left moves by deleting any black edge together with all the edges that are no longer connected to the ground and Right moves by deleting any white edge together with all the edges that are no longer connected to the ground. The first player unable to move because there are no edge of his/her color is the loser. An example of Black-White Hackenbush is shown in Fig. 1.

Three-player Hackenbush is the natural extension of Black-White Hackenbush where it has been introduced a third player called Center. Therefore, an instance of three-player Hackenbush can be represented by $G = \{G_1, G_2, \ldots, G_n\}$ where G_i for all $1 \leq i \leq n$ is an undirected connected graph such that:

- at least one edge is connected to a certain line called the *ground*, and
- every edge has a color (black for Left, gray for Center, and white for Right).

Players take turns making legal moves in cyclic fashion (..., Left, Center, Right, Left, Center, Right, ...) until

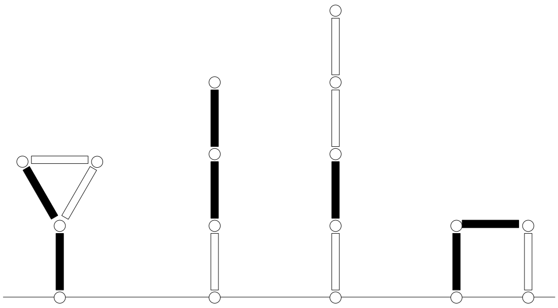

Figure 1: An example of Black-White Hackenbush.

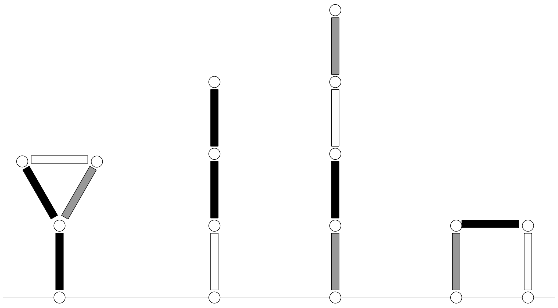

Figure 2: An example of three-player Hackenbush.

one of the players is unable to move. Then, that player leaves the game and the remaining players continue in alternation until one of them cannot move. Then that player leaves the game and the remaining player is the winner. An example of three-player Hackenbush is shown in Fig. 2.

We briefly recall the definition of *queer* game introduced by Propp [4]:

Definition 1. *A position in a three-player combinatorial game is called queer if no player can force a win.*

In the game of three-player Hackenbush is not always possible to determine the winner because of queer games as shown in Fig. 3.

In this case, when Left starts the game no player has a winning strategy because if Left removes the first string, then Right has a winning strategy but if Left removes the second string, then Center has a winning strategy.

3 Three-player Hackenbush played on strings is $\mathcal{NP}$-complete

We prove that to solve three-player Hackenbush played on strings is a $\mathcal{NP}$-complete problem.

We briefly recall the definition of Subset Sum Problem.

Definition 2. *Let*

$$\mathcal{U} = \{u_1, \ldots, u_n\}$$

be a set of natural numbers and K a given natural number. The problem is to determine if there exists $\mathcal{U}' \subseteq \mathcal{U}$ such that

$$\sum_{u_i \in \mathcal{U}'} u_i = K.$$

This problem is known to be $\mathcal{NP}$-complete [9].

Starting from a general instance of Subset Sum Problem it is possible to create an instance of three-player Hackenbush as shown in Fig. 4. We use U to indicate $\sum_{u_i \in \mathcal{U}} u_i$. For every $u_i \in \mathcal{U}$ we have a string containing u_i gray edges on the bottom and u_i black edges on the top. Moreover, we add two more strings: the first one containing $U + 2$ black edges and $U + 1$ gray edges (the first string on the left as shown in Fig. 4), and second one containing $2U + 2$ white edges (the last string on the right as shown in Fig. 4).

Who has a winning strategy?

Table 1 shows the number of edges for each player. We observe that Right can always make $2U + 2$ moves removing always the highest edge, but Center has just $2U + 1$ edges therefore, even in the best case, he/she will never be the winner.

The situation is a bit different for Left because even if he/she has $2U + 2$ edges, he/she can make $2U + 2$ moves only if he/she starts to play before Center because after Center has removed all his/her edges Left can make just one more move.

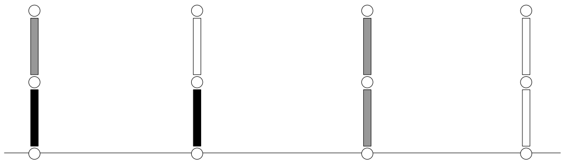

Figure 3: A queer game.

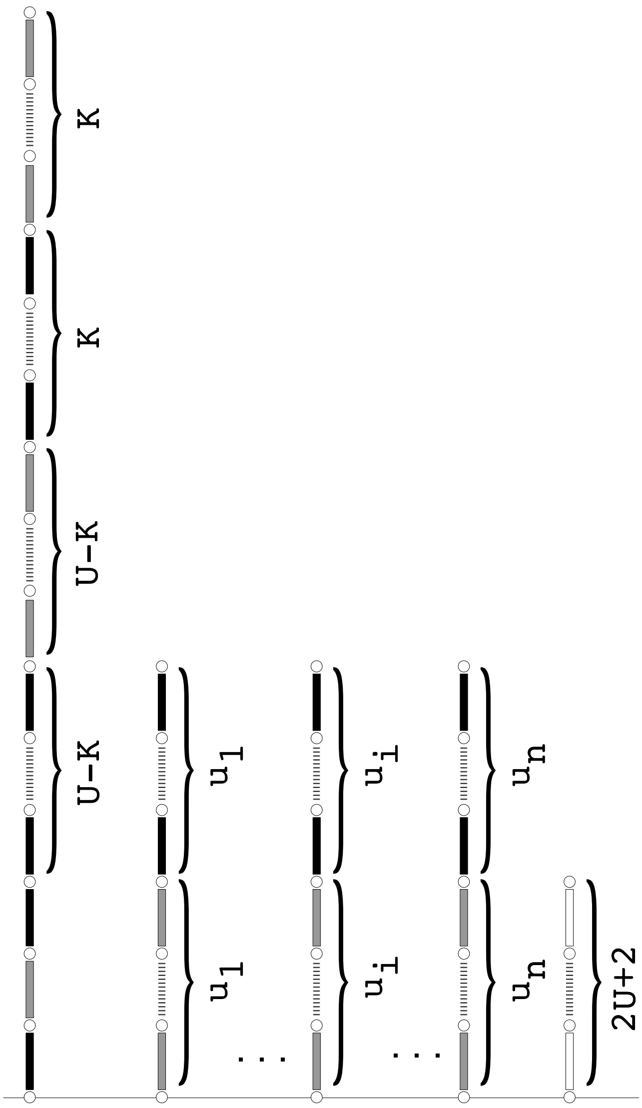

Figure 4: Subset Sum Problem is reducible to three-player Hackenbush.

Table 1: Number of edges.

Player	Number of edges
Left	$2U + 2$
Center	$2U + 1$
Right	$2U + 2$

It follows that, if Left or Center starts the game, then Right has always a winning strategy; however, when Right starts the game we have two possibilities:

- If Left, cooperating with Center, is not able to make $2U+2$ moves, then Right has still a winning strategy,
- If Left, cooperating with Center, is able to make $2U + 2$ moves then, Right does not have a winning strategy and the game is queer because Left can win the game only assuming that Center cooperates with him/her.

The problem to determine if Right has a winning strategy when he/she starts the game is strictly connected to Subset Sum Problem as shown in the following theorem.

Theorem 1. *Let $G = \{G_1 \cup G_2 \cup \dots G_n\}$ be a general instance of three-player Hackenbush where every G_i is a string for all $1 \leq i \leq n$. Then, to establish the outcome of G is a $\mathcal{NP}$-complete problem.*

Proof. The problem is clearly in $\mathcal{NP}$.

We show that it is possible to reduce every instance of Subset Sum Problem to G. Previously we have described how to construct the instance of three-player Hackenbush, therefore we have just to prove that Subset Sum Problem is solvable if and only if Left, cooperating with Center, can win the game, i.e., Right does not have a winning strategy.

Let's introduce some useful notations:

- $\mathcal{U}'$ indicates a solution of Subset Sum Problem such that
$$\sum_{u_i \in \mathcal{U}'} u_i = K.$$
- $\mathcal{S}$ indicates the set of all the strings containing u_i gray edges on the bottom and u_i black edges on the top for $1 \leq i \leq n$.
- $\mathcal{S}' \subseteq \mathcal{S}$ indicates a subset of strings corresponding to $\mathcal{U}'$.

If $\mathcal{U}'$ is a solution of Subset Sum Problem, then Left, cooperating with Center, has a trivial way to win the game. During the first K moves, Left removes K edges in $\mathcal{S}'$ and Center removes K edges from the top of the first string. During the next K moves, Left removes K edges from the top of first string and Center removes K edges in $\mathcal{S}'$.

At this point there are no more edges in $\mathcal{S}'$ and Left can removes $U - K$ edges in $\mathcal{S} - \mathcal{S}'$ while Center removes $U - K$ edges from the top of first string. Successively, Left removes $U - K$ edges from the top of first string and Center removes the last $U - K$ edges in $\mathcal{S} - \mathcal{S}'$. Now, the first string contains a black edge on the bottom, a gray edge in the middle and another black edge on the top therefore Left can make two more moves and Center one more move.

Summarizing, Center makes
$$K + K + U - K + U - K + 1 = 2U + 1$$
moves and Left makes
$$K + K + U - K + U - K + 2 = 2U + 2$$
moves therefore Right does not have a winning strategy.

Conversely, let us suppose that when Right starts the game, he/she does not have a winning strategy, i.e., Left, cooperating with Center, is able to to make $2U+2$ moves. We recall that Left can remove only one more edge after that Center has removed all his/her edges, therefore Center has to make exactly $2U + 1$ moves.

Let's consider the first time, during the course of the game, when the substring of K gray edges and K black edges on the top of the first string have been just removed and Left has to move. In this moment, S must contain at least $U - K$ black edges and $U - K$ gray edges to remove all the $U - K$ gray edge and all the $U - K$ black edges in the first string. Moreover, at least K black edges and K gray edges have been removed from $\mathcal{S}$, therefore there are exactly $U - K$ black edges and $U - K$ gray edges in $\mathcal{S}$.

For every string in $\mathcal{S}$ the number of black edges is always less than or equal to the number of gray edges, therefore in our case every string must contain the same number of black and gray edges, i.e., there exists a subset of strings containing $U - K$ gray edges on the bottom with $U - K$ black edges on the top.

It follows that in the original instance there exists a subset of strings containing K gray edges on the bottom and K black edges on the top. To this subset of strings there corresponds a subset $\mathcal{U}'$ such that
$$\sum_{u_i \in \mathcal{U}'} u_i = K.$$

Therefore, the problem to establish the outcome of G is $\mathcal{NP}$-hard and $\mathcal{NP}$-complete. □

It is remarkable that the same instance, represented by all the strings with black and gray edges and easily solvable when Left and Center are in competition using Berlekamp's rule [10], becomes $\mathcal{NP}$-hard when players cooperate each other.

It is interesting to observe that the strategy of Right does not affect the strategies of Left and Center because the white edges are not connected to the black and gray edges.

Moreover, strings are the simplest structure in Black-White Hackenbush, therefore the instances represented by strings are the simplest hard-instances in the game of three-player Hackenbush.

References

[1] S. Y. R. Li, "N-person Nim and N-person Moore's games," *Internat. J. Game Theory*, 7(1), pp. 31-36.

[2] P. D. Straffin Jr., "Three-person winner-take-all games with Mc-Carthy's revenge rule," *College J. Math.*, 16(5), pp. 386-394.

[3] D. E. Loeb, "Stable Winning Coalitions," in *Games of No Chance*, R.J. Nowakowski, Ed. Cambridge University Press, 1996, pp. 451-471.

[4] J. G. Propp, "Three-player impartial games," *Theoretical Computer Science*, 233(1-2), pp. 263-278.

[5] A. Cincotti, "Three-player partizan games," *Theoretical Computer Science*, 332(1-3), pp. 367-389.

[6] J. H. Conway, *On numbers and games*, 2nd ed., A K Peters, 2001.

[7] D. Knuth, *Surreal Numbers*, Addison-Wesley, 1974.

[8] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning way for your mathematical plays*, 2nd ed., A K Peters, 2001.

[9] M. R. Garey and D. S. Johnson, *Computers and Intractability*, Freeman, 1979.

[10] E. R. Berlekamp, "The Hackenbush number system for compression of numerical data," *Information and Control*, 26(2), pp. 134-140.