Stochastic Stability of TSK Fuzzy Model

Leila Fallah Araghi, Hamid Khaloozade

Abstract— **in this paper, a theorem is derived for the existence of a common quadratic Lyapunov function for stability analysis of TSK fuzzy model.**

This paper proposed a new method based on stochastic stability. In this paper we study the stochastic stability properties of certain TSK Fuzzy system has been studied.

Index Terms— **quadratic Lyapunov function, TSK Fuzzy system stochastic stability**.

I. Introduction

For many real-world systems, a mathematical description in the form of differential/difference equations or similar conventional model is either infeasible or impracticable; due to the complexities involved, and the intrinsic nature of information incompleteness. The fuzzy modeling is generally presented to overcome these difficulties. Stability is very important property of hybrid systems, that switching system and fuzzy system are types of hybrid systems.

Recently many researchers have worked on hybrid systems stability analysis.

Begian et al., proposed a novel inference mechanism to design stable interval type-2 Takagi-Sugeno-Kang (TSK) dynamic fuzzy systems, they are derived stability conditions for type-2 TSK dynamic systems utilizing the proposed inference engine [1]. Sonbol et al., proposed a new approach for the stability analysis of continuous Sugeno Types II and III dynamic fuzzy systems [2].

Guisheng Zhai et al., analyzed stability for switched systems which are composed of both continuous-time and discrete-time subsystems. By considering a Lie algebra generated by all subsystem matrices, they showed that if all subsystems are Hurwitz/Schur stable and this Lie algebra is solvable, then there is a common quadratic Lyapunov function for all subsystems and thus the switched system is exponentially stable under arbitrary switching [3]. Lin et al., have given a brief overview of the most recent developments in the field of stability and stabilizability of switched linear systems [4]. R. N. Shorten et al., proposed necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for a finite number of stable second order linear time-invariant systems [5]. In their method for stability analysis a common positive definite matrix P_i should be found to satisfy a set of Lyapunov equations [5].

Tanaka et al., have proposed a method for stability analysis of TSK model by finding a common Lyapunov function using LMI method [6]. Percup et al. used the center manifold theory for fuzzy system stability analysis [7]. Farinawata [8] and Linder [9] separately worked on a robust stabilizing controller design. A rather good survey of many other similar results in the TSK stability analysis could be found in [10]. Suratgar and et al., proposed some theorems for stability analysis of TSK and linguistic fuzzy models [11, 12]. Benrejeb et al., deals with the impact of the choice of conjunctive operator between input variables of discrete TSK fuzzy models, t-norm, on stability domain estimation [13]. Gurvitz et al worked on stability of switched positive linear systems [14].

II. A REVIEW ON STABILITY ANALYSIS OF SWITCHING SYSTEM [7, 8, 9, 10]

In switched linear systems, the subsystems of which are continuous-time linear time-invariant (LTI) systems

$$\dot{x} = A_i x, i = \{1,...,n\}$$

(1)

Or a collection of discrete-time LTI systems

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$$x[k+1] = A_i x[k],$$
$$k \in Z^+, i \in \{1,2,...,n\}$$

(2)

Where $A_i \in R^{n \times n}$.

The existence of a common quadratic Lyapunov function (CQLF) for all its subsystems assures the quadratic stability of the switched system [15, 16]. Quadratic stability is a special class of exponential stability, which implies asymptotic stability, and has attracted a lot of research efforts due to its importance in practice [7, 8]. It is known that the conditions for the existence of a CQLF can be expressed as linear matrix inequalities (LMIs) [7, 8]. Namely, there exists a positive definite symmetric matrix $P, P \in R^{n \times n}$, such that

$$A_i^{\,T} P + PA_i \prec 0, i \in \{1,...,n\}$$

(3)

For the continuous-time case, or

$$A_i^{\,T} PA_i - P_i \prec 0$$

(4)

For the discrete-time case, hold simultaneously. However, the standard interior point methods for LMIs may become ineffective have the number of modes increases [8].

Consider a dynamical system as

$$\dot{x} = A_i x, i = \{1,...,n\}$$

(5)

Where the matrices, A_i belonging to set { $A_1,....,A_n$ } and A_i are constant matrices in $R^{n \times n}$. This system will be referred as the switching system.

Matrices A_i are asymptotically stable if the Eigen values of each A_i matrix lies in the open left half of the complex plan. So that matrices A_i are assumed to be Hurwitz.

An important problem is to determine necessary and sufficient condition for the existence of a quadratic Lyapunov function V(x) = x^T P x, P = P^T > 0, P belong to $R^{n \times n}$, such that $\frac{dv}{dt}$ along any trajectory of the system (1) is negative definite, or alternatively that

$$A_i^{\,T} P + PA_i = -Q_i \qquad (6)$$

Where Q_i positive definite and P are negative definite. The function V(x) is a common quadratic Lyapunov function (CQLF) for the switching linear time-invariant (LTI) dynamic systems,

$$\sum A_i : \dot{x} = A_i x, i = \{1,...,n\}$$

(7)

Where A_i belong to $R^{n \times n}$.

The existence of such a Lyapunov function is sufficient to guarantee the uniform asymptotic (exponential) stability of switching system (1) [9].

Theory 1: If we consider switching linear time-invariant (LTI) dynamic systems,

$$\sum A_i : \dot{z} == A_i z, i = \{1,...,n\}$$

(8)

The system described by Equation (7) with initial conditions $z(t_0) = z_0$ has the following response [10]:

$$z(t) = \exp(A_i(t - t_k))\left(\prod_{j=1}^{k} M(j)\right) z_0$$

(9)

And

$$M(j) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \exp(A_i h(j)) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

Definition #1 [10]

The equilibrium $z = 0$ of a system described by $\dot{z} = f(z,t)$ with initial condition $z(t_0) = z_0$ is almost sure (or with probability-1) asymptotically stable at large (or globally) if for any $\beta \succ 0$ and $\varepsilon \succ 0$ the solution of $\dot{z} = f(z,t)$ satisfies

$$\lim_{\partial \to \infty} \left\{ P(\sup_{t \geq \partial} \|z(t, z_0, t_0)\| \succ \varepsilon \right\} = 0$$ Where $\| z_0 \| \prec \beta$.

Corollary #1[10]

The system described by Equation (2), with update times $h(j)$ that are independent identically distributed random variable with probability distribution $F(h)$ is globally almost sure (or with probability-1) asymptotically stable around the solution $z = 0$ if $T = E[\exp(\sigma(A_i)h)] \prec \infty$ and the expected value of the maximum singular value of the test matrix :

M, $E[\|M\|] = E\left[\bar{\sigma}_M\right]$, is strictly less than one,

where $M(j) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \exp(A_i h(j)) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

III. A REVIEW ON TSK FUZZY SYSTEM [11]

In these articles, classical approaches for stability analysis are used for fuzzy systems stability analysis

Theorem 4:

Consider a TSK Fuzzy model as uniform:

$$Rule_{ith} if \ x_1(t) is \ B_1^i and ... and \ x_N(t) is \ B_n^i then:$$

$$\left[\dot{x}_1(t).....\dot{x}_N(t)\right]^T = A_i\left[x_1(t).....x_N(t)\right]^T$$

(10)

This system is asymptotic stable if and only if the $A_i \in \{A_1,...,A_n\}$, $A_i \in R^{n\times n}$ be Metzler-Hurwitz.

Proof:

Consider a TSK fuzzy model:

$$Rule_{ith} if \ x_1(t) is \ B_1^i and ... and \ x_N(t) is \ B_n^i then:$$

$$\left[\dot{x}_1(t)......\dot{x}_N(t)\right]^T = A_i\left[x_1(t)......x_N(t)\right]^T$$

(11)

Where $x(t) = \left[x_1(t)......x_N(t)\right]^T$ is state vector and B_j^i is the fuzzy set that belong to i-th rule and j-th state and $A_i \in R^{2\times 2}$. With COA defuzzification, crisp dynamic equation of state of fuzzy system is obtained as:

$$\dot{X}(T) = \frac{\sum_{i=1}^{M}(Dof_p)_i A_i X(t)}{\sum_{i=1}^{M}(Dof_p)_i} = \frac{\sum_{i=1}^{M}(Dof_p)_i A_i}{\sum_{i=1}^{M}(Dof_p)_i} X(t) \quad (12)$$

Where M is total number of rules. $(Dof_p)_i$ is degree of firing of i-th rule. $(Dof_p)_i$ is defined as:

$$(Dof_p)_i = \mu_{B_1^l}(x_1^p(t)) \times \mu_{B_2^l}(x_2^p(t)) \times ..... \times \mu_{B_N^l}(x_N^p(t))$$

(13)

Where $\mu_{B_k^l}(x_k^p(t))$ the value of membership function of is fuzzy set B_k^i for the k-th element of state for p-th pair of data and for i-th rule Z_P And W_i are defined as follows

$$W_i = \frac{(Dof_p)_i}{\sum_{i=1}^{M}(Dof_p)_i}$$

(14)

$$Z_P = \sum_{i=1}^{M} W_i A_i$$

(15)

It is obvious that the W_i is in the [0, 1], $\sum_i w_i = 1$.

In Fuzzy TSK system inference for each linguistic value of x (t) the dynamical equation is as bellow

$$x^{\cdot}(t) = Z_p x(t), p = \{1,...,n\}$$

(16)

Where Z_p for different linguistic values of X (t) switches between $A_i \in R^{n\times n}$.

IV. STOCHASTIC STABILITY OF TSK FUZZY SYSTEM

The following statement can be say about stability of TSK inference system:

In Fuzzy TSK inference system for each linguistic value of x (t) (11), using Corollary #1 If $A_i \in R^{n\times n}$ is independent matrices then we can say that Z_p is stable.

Then we can say that $x^{\cdot}(t) = Z_p x(t), p = \{1,...,n\}$ that demonstrates dynamical equation of fuzzy system, is exponentially stable system.

For proof details see Appendix I.

V. CONCLUSION

This paper proposed a new method for stability Analysis of TSK fuzzy model. This is based on stochastic stability.

Appendix 1

If we consider $T = E\left[\exp(\sigma(A_i)h)\right] \prec \infty$ then we can say that $T = E\left[\exp(\sigma(Z_P)h)\right] \prec \infty$.

And if we consider

$$M(j)=\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\exp(Z_P h(j))\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} I0 \\ 00 \end{bmatrix}\exp((W_1 A_1)h(j)).$$

$$\exp((W_2 A_2)h(j))\ldots\exp(\sigma(W_i A_i)h(j))\begin{bmatrix} I0 \\ 00 \end{bmatrix}$$

If we consider $E\left[\|M\|\right]= E\left[\bar{\sigma}_M\right]\prec 1$ and because W_i is in the [0, 1], $\sum_i w_i = 1$. Then we can say that $T = E\left[\exp(\sigma(Z_P)h)\right]\prec\infty$.

And if we consider

$$M(j)=\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\exp(Z_P h(j))\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\exp((W_1 A_1)h(j)).$$

$$\exp((W_2 A_2)h(j))\ldots\exp(\sigma(W_i A_i)h(j))\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

If we consider $E\left[\|M\|\right]= E\left[\bar{\sigma}_M\right]\prec 1$

And because W_i is in the [0, 1], $\sum_i w_i = 1$.

Then we can say

$$M(j)=\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\exp(Z_P h(j))\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\prec 1.$$

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