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A Fuzzy TOPSIS Decision Making Model with Entropy Weight under Intuitionistic Fuzzy Environment

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***Abstract*—The theory of intuitionistic fuzzy set (IFS) is well suitable to deal with vagueness and hesitancy. In this study, we propose a new fuzzy TOPSIS decision making model using entropy weight for dealing with multiple criteria decision making (MCDM) problems under intuitionistic fuzzy environment. This model allows measuring the degree of satisfiability and the degree of non-satisfiability, respectively, of each alternative evaluated across a set of criteria. To obtain the weighted fuzzy decision matrix, we employ the concept of Shannon's entropy to calculate the criteria weights. An investment example is used to illustrate the application of the proposed model.**

***Index Terms*—Entropy, Intuitionistic fuzzy set (IFS), Multiple criteria decision making (MCDM), TOPSIS**

I. Introduction

A lot of multiple criteria decision making (MCDM) approaches have been developed and applied to diverse fields, like engineering, management, economics, etc. As one of the known classical MCDM approaches, TOPSIS (technique for order performance by similarity to ideal solution) was first developed by Hwang and Yoon [6]. The primary concept of TOPSIS approach is that the most preferred alternative should not only have the shortest distance from the positive ideal solution (PIS), but also have the farthest distance from the negative ideal solution (NIS) [6, 13]. General speaking, the advantages for TOPSIS include (a) simple, rationally comprehensible concept, (b) good computational efficiency, (c) ability to measure the relative performance for each alternative in a simple mathematical form [11].

In 1965, Zadeh [12] introduced first the theory of fuzzy sets. Later on, many researchers have been working on the process of dealing with fuzzy decision making problems by applying fuzzy sets theory. Roughly speaking, Zadeh's fuzzy set only assigns a single membership value between zero and one to each element. In 1993, Gau and Buehrer [5] pointed out that this single value could not attest to its accuracy and proposed the concept of vague sets. Bustince and Burillo [2], however, pointed out that the notion of vague sets coincides with that of intuitionistic fuzzy sets (IFSs) proposed by Atanassov [1] almost ten years earlier. IFSs are proposed using two characteristic functions expressing the degree of membership and the degree of non-membership of elements of the universal set to the IFS. It can cope with the presence of vagueness and hesitancy originating from imprecise knowledge or information. In the last two decades, there exists a large amount of literature for the theory and application of IFS. Different from other studies, in this study, the criteria weights are obtained by conducting Shannon's entropy concept; after that, a fuzzy TOPSIS method is employed to order the alternatives. The proposed model fits the reality of the situation and its calculation is not difficult, so it can provide an efficient way to help the decision maker (DM) in making decisions.

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II. Preliminaries

2. 1. Intuitionistic fuzzy sets

Definition 1 [1]. An IFS A in the universe of discourse X is defined with the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\},$$

where

$$\mu_A: X \to [0, 1],\ \nu_A: X \to [0, 1]$$

with the condition

$$0 \le \mu_A(x) + \nu_A(x) \le 1,\ \forall x \in X.$$

The numbers $\mu_A(x)$ and $\nu_A(x)$ denote the membership degree and the non-membership degree of x to A, respectively.

Obviously, each ordinary fuzzy set may be written as

$$\{\langle x, \mu_A(x), 1-\mu_A(x)\rangle \mid x \in X\}.$$

That is to say, fuzzy sets may be reviewed as the particular cases of IFSs.

Noted that A is a crisp set if and only if for $\forall x \in X$, either $\mu_A(x) = 0,\ \nu_A(x) = 1$ or $\mu_A(x) = 1,\ \nu_A(x) = 0$.

For each IFS A in X, we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

the intuitionistic index of x in A. It is a measure of hesitancy degree of x to A [1]. It is obvious that $0 \le \pi_A(x) \le 1$ for each $x \in X$.

For convenience of notation, IFSs(X) is denoted as the set of all IFSs in X.

Definition 2 [4]. For every $A \in$ IFSs(X), the IFS λA for any positive real number λ is defined as follows:

$$\lambda A = \left\{ \langle x,\, 1-(1-\mu_A(x))^{\lambda},\, (\nu_A(x))^{\lambda} \rangle \mid x \in X \right\}. \tag{1}$$

2.2. Entropy of IFS

In 1948, Shannon [7] proposed the entropy function, $H(p_1, p_2, ..., p_n) = -\sum_{i=1}^{n} p_i \log(p_i)$, as a measure of uncertainty in a discrete distribution based on the Boltzmann entropy of classical statistical mechanics, where $p_i (i = 1, 2, ..., n)$ are the probabilities of random variable computed from a probability mass function P. Later, De Luca and Termini [3] defined a non-probabilistic entropy formula of a fuzzy set based on Shannon's function on a finite universal set $X = \{x_1, x_2, ..., x_n\}$ as Eq. (2):

$$E_{\mathrm{LT}}(A) = -k \sum_{i=1}^{n} \left[\mu_A(x_i) \ln \mu_A(x_i) + (1-\mu_A(x_i)) \ln(1-\mu_A(x_i)) \right],\ k > 0. \tag{2}$$

Szmidt and Kacprzyk [9] extended De Luca and Termini axioms presenting the four definitions with regard to entropy measure on IFSs(X). Recently, Vlachos et al. [10] presented Eq. (3) as the measure of intuitionistic fuzzy entropy which was proved to satisfy the four axiomatic requirements.

$$E_{\mathrm{LT}}^{\mathrm{IFS}}(A) = -\frac{1}{n \ln 2} \sum_{i=1}^{n} [\mu_A(x_i) \ln \mu_A(x_i) + \nu_A(x_i) \ln \nu_A(x_i) - (1-\pi_A(x_i)) \ln(1-\pi_A(x_i)) - \pi_A(x_i) \ln 2]. \tag{3}$$

It is noted that that $E_{\mathrm{LT}}^{\mathrm{IFS}}(A)$ is composed of the hesitancy degree and the fuzziness degree of the IFS A.

III. PROPOSED FUZZY TOPSIS DECISION MAKING MODEL

The procedures of calculation for this proposed model can be described as follows:

Step 1. Construct an intuitionistic fuzzy decision matrix.

A MCDM problem can be concisely expressed in matrix format as

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \!\!\begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} \end{array} \tag{4}$$

$$W = (w_1, w_2, ..., w_n)$$

Let $A = \{A_1, A_2, ..., A_m\}$ be a set of alternatives which consists of m non-inferior decision-making alternatives. Each alternative is assessed on n criteria, and the set of all criteria is denoted $C = \{C_1, C_2, ..., C_n\}$. Let $W = (w_1, w_2, ..., w_n)$ be the weighting vector of criteria, where $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$.

In this study, the characteristics of the alternatives A_i are represented by the IFS as:

$$A_i = \left\{ \langle C_j, \mu_{A_i}(C_j), \nu_{A_i}(C_j) \rangle \mid C_j \in C \right\},\quad i = 1, 2, ..., m, \tag{5}$$

where $\mu_{A_i}(C_j)$ and $\nu_{A_i}(C_j)$ indicate the degrees that the alternative A_i satisfies and does not satisfy the criterion C_j, respectively, and $\mu_{A_i}(C_j) \in [0, 1]$, $\nu_{A_i}(C_j) \in [0, 1]$, $\mu_{A_i}(C_j) + \nu_{A_i}(C_j) \in [0, 1]$. The intuitionistic index $\pi_{A_i}(C_j) = 1 - \mu_{A_i}(C_j) - \nu_{A_i}(C_j)$ is such that the larger $\pi_{A_i}(C_j)$ the higher a hesitation margin of the DM about the alternative A_i with respect to the criterion C_j.

Step 2. Determine the criteria weights using the entropy-based method.

The well-known entropy method [6, 13] can obtain the objective weights, i.e. called entropy weights. The smaller entropy values to which all alternatives A_i $(i = 1, 2, ..., m)$ with littler similar criteria values with respect to a set of criteria can be obtained. According to the idea mentioned as above, for the decision matrix, $\tilde{D} = [\tilde{x}_{ij}]_{m \times n}$, $i = 1, ..., m$, $j = 1, ..., n$, under intuitionistic fuzzy environment, the expected information content emitted from each criterion C_j can be measured by the entropy value, denoted as $E_{\mathrm{LT}}^{\mathrm{IFS}}(C_j)$, as

$$E_{\mathrm{LT}}^{\mathrm{IFS}}(C_j) = -\frac{1}{m \ln 2} \sum_{i=1}^{m} \left[\mu_{ij}(C_j) \ln \mu_{ij}(C_j) + \nu_{ij}(C_j) \ln \nu_{ij}(C_j) - (1-\pi_{ij}(C_j)) \ln(1-\pi_{ij}(C_j)) - \pi_{ij}(C_j) \ln 2 \right], \tag{6}$$

where $j = 1, 2, ..., n$ and $1/(m \ln 2)$ is a constant which assures $0 \leq E_{\mathrm{LT}}^{\mathrm{IFS}}(C_j) \leq 1$.

Therefore, the degree of divergence (d_j) of the average intrinsic information provided by the corresponding performance ratings on criterion C_j can be defined as

$$d_j = 1 - E_{\mathrm{LT}}^{\mathrm{IFS}}(C_j),\quad j = 1, 2, ..., n. \tag{7}$$

The value of d_j represents the inherent contrast intensity of criterion C_j, then the entropy weight of the jth criterion is

$$w_j = d_j \Big/ \sum_{j=1}^{n} d_j . \quad (8)$$

Step 3. Construct the weighted intuitionistic fuzzy decision matrix.

A weighted intuitionistic fuzzy decision matrix $\tilde{Z}$ can be obtained by aggregating the weight vector W and the intuitionistic fuzzy decision matrix $\tilde{D}$ as:

$$\tilde{Z} = W^{\mathrm{T}} \otimes \tilde{D} = W^{\mathrm{T}} \otimes \left[\tilde{x}_{ij} \right]_{m\times n} = \left[\hat{x}_{ij} \right]_{m\times n} , \quad (9)$$

where

$$W = \left(w_1, w_2, ..., w_j, ..., w_n \right);$$

$$\hat{x}_{ij} = \left\langle \hat{\mu}_{ij}, \hat{\nu}_{ij} \right\rangle = \left\langle 1-(1-\mu_{ij})^{w_j}, \nu_{ij}^{w_j} \right\rangle , \quad w_j > 0.$$

Step 4. Determine intuitionistic fuzzy positive-ideal solution (IFPIS, A^+) and intuitionistic fuzzy negative-ideal solution (IFNIS, A^-).

In general, the evaluation criteria can be categorized into two kinds, benefit and cost. Let G be a collection of benefit criteria and B be a collection of cost criteria. According to IFS theory and the principle of classical TOPSIS method, IFPIS and IFNIS can be defined as:

$$A^+ = \Big\{ \Big\langle C_j, \Big((\max_i \hat{\mu}_{ij}(C_j) \mid j \in G), (\min_i \hat{\mu}_{ij}(C_j) \mid j \in B) \Big), \Big((\min_i \hat{\nu}_{ij}(C_j) \mid j \in G), (\max_i \hat{\nu}_{ij}(C_j) \mid j \in B) \Big) \Big\rangle \Big| i \in m \Big\}. \quad (10a)$$

$$A^- = \Big\{ \Big\langle C_j, \Big((\min_i \hat{\mu}_{ij}(C_j) \mid j \in G), (\max_i \hat{\mu}_{ij}(C_j) \mid j \in B) \Big), \Big((\max_i \hat{\nu}_{ij}(C_j) \mid j \in G), (\min_i \hat{\nu}_{ij}(C_j) \mid j \in B) \Big) \Big\rangle \Big| i \in m \Big\}. \quad (10b)$$

Step 5. Calculate the distance measures of each alternative A_i from IFPIS and IFNIS.

We use the measure of intuitionistic Euclidean distance (refer to Szmidt and Kacprzyk [8]) to help determining the ranking of all alternatives.

$$d_{\mathrm{IFS}}(A_i, A^+) = \sqrt{\sum_{j=1}^{n} \Big[\left(\mu_{A_i}(C_j) - \mu_{A^+}(C_j) \right)^2 + \left(\nu_{A_i}(C_j) - \nu_{A^+}(C_j) \right)^2 + \left(\pi_{A_i}(C_j) - \pi_{A^+}(C_j) \right)^2 \Big]} \quad (11a)$$

$$d_{\mathrm{IFS}}(A_i, A^-) = \sqrt{\sum_{j=1}^{n} \Big[\left(\mu_{A_i}(C_j) - \mu_{A^-}(C_j) \right)^2 + \left(\nu_{A_i}(C_j) - \nu_{A^-}(C_j) \right)^2 + \left(\pi_{A_i}(C_j) - \pi_{A^-}(C_j) \right)^2 \Big]} \quad (11b)$$

Step 6. Calculate the relative closeness coefficient (CC) of each alternative and rank the preference order of all alternatives.

The relative closeness coefficient (CC) of each alternative with respect to the intuitionistic fuzzy ideal solutions is calculated as:

$$CC_i = d_{\mathrm{IFS}}(A_i, A^-) \Big/ \Big(d_{\mathrm{IFS}}(A_i, A^+) + d_{\mathrm{IFS}}(A_i, A^-) \Big), \quad (12)$$

where $0 \le CC_i \le 1,\ i = 1, 2, ..., m.$

The larger value of CC indicates that an alternative is closer to IFPIS and farther from IFNIS simultaneously. Therefore, the ranking order of all the alternatives can be determined according to the descending order of CC values. The most preferred alternative is the one with the highest CC value.

IV. ILLUSTRATIVE EXAMPLE

In this section, in order to demonstrate the calculation process of the proposed approach, an example is provided [14]. An investment company wants to invest a sum of money in the best choice. There are five possible companies A_i $(i = 1, 2, ..., 5)$ in which to invest the money: (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company; and (5) A_5 is a TV company. Each possible company will be evaluated across three criteria which are: (1) C_1 is economical benefit; (2) C_2 is social benefit; (3) C_3 is environmental pollution, where C_1 and C_2 are benefit criteria, and C_3 is cost criterion.

The proposed fuzzy TOPSIS decision making model is applied to solve this problem, and the computational procedure is described step by step as below:

Step 1. The ratings for five possible companies with respect to three criteria are represented by IFSs. The intuitionistic fuzzy decision matrix $\tilde{D}$ is constructed by the investment company can be expressed as Table 1.

Table 1. Intuitionistic fuzzy decision matrix $\tilde{D}$

	C_1	C_2	C_3
A_1	⟨0.70, 0.20⟩	⟨0.85, 0.10⟩	⟨0.30, 0.50⟩
A_2	⟨0.90, 0.05⟩	⟨0.70, 0.25⟩	⟨0.40, 0.50⟩
A_3	⟨0.80, 0.10⟩	⟨0.85, 0.10⟩	⟨0.30, 0.60⟩
A_4	⟨0.90, 0.00⟩	⟨0.80, 0.10⟩	⟨0.20, 0.70⟩
A_5	⟨0.80, 0.15⟩	⟨0.75, 0.20⟩	⟨0.50, 0.40⟩

Step 2. Determine the criteria weights. Using Eq. (6), the entropy values for criteria C_1, C_2 and C_3, respectively, are: 0.4842, 0.6341, and 0.9323. The degree of divergence d_j on each criterion C_j $(j = 1, 2, 3)$ may be obtained by Eq. (7) as 0.5158, 0.3659, and 0.0677, respectively. Therefore, the criteria weighting vector can be expressed as $W = (0.543, 0.385, 0.071)$ by applying Eq. (8).

Step 3. After determining criteria weighting vector, using Eq. (9), the weighted intuitionistic fuzzy decision matrix $\tilde{Z}$ is then obtained as Table 2.

Table 2. Weighted intuitionistic fuzzy decision matrix $\tilde{Z}$

	C_1	C_2	C_3
A_1	$\langle 0.4799, 0.4173 \rangle$	$\langle 0.5183, 0.4121 \rangle$	$\langle 0.0250, 0.9520 \rangle$
A_2	$\langle 0.7136, 0.1966 \rangle$	$\langle 0.3709, 0.5864 \rangle$	$\langle 0.0356, 0.9520 \rangle$
A_3	$\langle 0.5827, 0.2864 \rangle$	$\langle 0.5183, 0.4121 \rangle$	$\langle 0.0250, 0.9644 \rangle$
A_4	$\langle 0.7136, 0.0000 \rangle$	$\langle 0.4619, 0.4121 \rangle$	$\langle 0.0157, 0.9750 \rangle$
A_5	$\langle 0.5827, 0.3570 \rangle$	$\langle 0.4136, 0.5381 \rangle$	$\langle 0.0480, 0.9370 \rangle$

Step 4. In this case, criteria C_1 and C_2 belong to benefit criteria, and criterion C_3 belong to cost criterion. Using Eqs. (10a) and (10b), each alternative's IFPIS (A^+) and IFNIS (A^-) with respect to criteria can be determined as

$$A^+ = \left(\langle 0.7136, 0.0000 \rangle \ \langle 0.5183, 0.4121 \rangle \ \langle 0.0157, 0.9705 \rangle \right)$$
$$A^- = \left(\langle 0.4799, 0.4173 \rangle \ \langle 0.3709, 0.5864 \rangle \ \langle 0.0480, 0.9370 \rangle \right)$$

Step 5. Calculate the distance between alternatives and intuitionistic fuzzy ideal solutions (IFPIS and IFNIS) using Eqs. (11a) and (11b).

Step 6. Using Eq. (12), the relative closeness coefficient (CC) can be obtained.

The distance, relative closeness coefficient, and corresponding ranking of five possible companies are tabulated in Table 3. Therefore, we can see that the order of rating among five alternatives is $A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_5$, where "$\succ$" indicates the relation "preferred to". Therefore, the best choice would be A_4 (arms company). From the process of calculation, we can see that the proposed approach is suitable for dealing with fuzzy MCDM problems evaluated by IFSs.

Table 3. The distance measure, relative closeness coefficient and ranking

Alternatives	$d_{\text{IFS}}(A_i, A^+)$	$d_{\text{IFS}}(A_i, A^-)$	CC_i	Rank
A_1	0.5350	0.2585	0.3257	4
A_2	0.5351	0.3414	0.3895	3
A_3	0.3628	0.4346	0.5450	2
A_4	0.0862	0.7760	0.9000	1
A_5	0.6541	0.1912	0.2262	5

V. CONCLUSION

In this present work, we propose an entropy-based MCDM model, in which the characteristics of the alternatives are represented by IFSs. In information theory, the entropy is related with the average information quantity of a source. Based on the principle, the optimal criteria weights can be obtained by the proposed entropy-based model. The main difference of this method from classical TOPSIS consists in the introduction of objective entropy weight under intuitionistic fuzzy environment.

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