BEM with Linear Boundary Elements for Solving the Problem of the 3D Compressible Fluid Flow around Obstacles

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***Abstract*—This paper presents a solution of the singular boundary integral equation of the 3D compressible fluid flow around an obstacle, which uses isoparametric linear boundary elements of Lagrangean type. The singular boundary integral equation formulated in velocity vector terms is deduced by applying the indirect technique of the BEM with sources distribution. The problem is reduced to a linear system of equations and for evaluating the coefficients arising from integrals of singular kernels a suitable parametric representation is used and the finite part of the integrals involved is considered. Based on the method exposed a computer code in MATHCAD is made. We test the method solving the problem in a particular case, in which an exact solution is known. A comparison between the exact solution and the numerical one shows a high degree of accuracy.**

***Index Terms*—boundary element method, compressible fluid flow, linear boundary elements, singular boundary integral equation, singular kernels.**

I. INTRODUCTION

The Boundary Element Method (BEM) is an important numerical technique, a method of great efficiency, used to solve boundary value problems for systems of partial differential equations.

The principal advantage of the BEM over other numerical methods is the ability to reduce the problem dimension by one. This property is advantageous as it reduces the size of the system the problem is reduced at, leading to improved computational efficiency.

To achieve this reduction of dimension it is necessary to obtain an equivalent boundary integral formulation for the governing equations. Usually a singular boundary integral equation is obtained.

In order to solve the integral equation different types of boundary elements can be used. As shown in [1] the type of boundary elements plays an important role in applying BEM, because the accuracy of the numerical solution is affected by the approximation models brought into solving through them.

If the body is three-dimensional, the boundary elements are usually of two types: quadrilateral and triangular elements.

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In this paper we use triangular linear isoparametric elements of Lagrangean type for solving the singular integral equation resulting as an application of the indirect boundary element method with sources distribution to the three-dimensional problem of a compressible fluid flow past an obstacle.

II. THE BOUNDARY INTEGRAL EQUATION

We think that it is necessary to make a short presentation of the problem we want to solve. We consider a 3D uniform, steady, potential motion of an ideal compressible fluid of subsonic velocity $U_\infty \bar{i}$, pressure p_∞ and density ρ_∞ perturbed by the presence of a fixed obstacle of a known boundary, noted Σ , assumed to be smooth and closed, which equation is: $F(X,Y,Z)=0$. We want to find out the perturbation, and the fluid action on the body.

The problem was studied by other authors too but with other numerical techniques, and even when BEM was applied the boundary integral formulations were obtained in terms of potential functions, or stream function, not in terms of velocity field like the approach considered in this paper.

Using dimensionless variables, we have, for the velocity and pressure fields, the following relations:

$$\bar{V}_1 = U_\infty\left(\bar{i}+\bar{V}\right),\ p_1 = p_\infty + \rho_\infty U_\infty^2 P$$

After some changes of coordinates the mathematical model in dimensionless variables for the perturbed motion is:

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0 \\ \dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}=0 \\ \dfrac{\partial w}{\partial x}-\dfrac{\partial u}{\partial z}=0 \end{cases} \quad (1)$$

with boundary condition:

$$un_x+\beta^2\left(vn_y+wn_z\right)=-\beta n_x \ on\ \Sigma \ , \quad (2)$$

where $\bar{v}$ represents the perturbation velocity and

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$$\overline{n} = \frac{gradF}{|gradF|}.$$

It is also required that the perturbation velocity vanishes at infinity: $\lim_{\infty}(u, v, w) = 0$.

The first equation ensures the existence of the potential function $\varphi(x, y, z)$, so as:

$$u = \frac{\partial \varphi}{\partial x}, \quad v = \frac{\partial \varphi}{\partial y}, \; w = \frac{\partial \varphi}{\partial z},$$

and $\Delta \varphi = 0$.

As it is known (see for example[2]), the fundamental solution of this equation is :

$$\varphi(\overline{x}) = -\frac{1}{4\pi} \frac{1}{|\overline{x} - \overline{\xi}|}, \tag{3}$$

where $|\overline{x} - \overline{\xi}| = \sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2}$,

and $\varphi(x, y, z)$ represents the potential of he motion produced by an unitary source situated in point $\overline{\xi}$ (position vector). The velocity field is given by:

$$\overline{v} = grad\varphi(\overline{x}) = \frac{1}{4\pi} \frac{\overline{x} - \overline{\xi}}{|\overline{x} - \overline{\xi}|^3}. \tag{4}$$

Assimilating the body with a continuous distribution of sources on the boundary, so on Σ, having an unknown intensity $m(\overline{x})$ (presumed to satisfy hölder condition on Σ), we have for the perturbation velocity, $\overline{v}$, the integral representation:

$$\overline{v}(\overline{x}) = -\frac{1}{4\pi} \iint_{\Sigma} m(\overline{\xi}) \frac{\overline{\xi} - \overline{x}}{|\overline{x} - \overline{\xi}|^3} da. \tag{5}$$

For $\overline{\xi} \to \overline{x}_0 \in \Sigma$ we get the perturbation velocity in any point of the boundary:

$$\overline{v}(\overline{x}_0) = -\frac{1}{2} m(\overline{x}_0)\overline{n}_0 - \frac{1}{4\pi} {\iint_{\Sigma}}' m(\overline{x}) \frac{\overline{x} - \overline{x}_0}{|\overline{x} - \overline{x}_0|^3} da. \tag{6}$$

where $\overline{n}_0 = \overline{n}(\overline{x}_0)$.

Using the boundary condition (2) a singular integral equation for the unknown m is obtained in [3]:

$$\left\{ (n_x^0)^2 + \beta^2 \left[(n_y^0)^2 + (n_z^0)^2 \right] \right\} m(\overline{x}_0) + \frac{1}{2\pi} {\iint_{\Sigma}}' m(\overline{x}) \frac{(x - x_0)n_x^0 + \beta^2 \left[(y - y_0)n_y^0 + (z - z_0)n_z^0 \right]}{|\overline{x} - \overline{x}_0|^3} da = 2\beta n_x^0 \tag{7}$$

where the sign "′ " denotes the principal value in Cauchy sense of the integral.

For $\beta = 1$ we obtain the boundary integral equation for the incompressible fluid flow.

III. Solving the Singular Boundary Integral Equation

A collocation method is used for example in [4] for solving integral equation (7).

In the boundary element approach used herein, for solving the integral equation (7) we use linear isoparametric boundary elements of Lagrangean type. The body surface, Σ, is divided into M plane triangles, noted T_j, $j = 1, M$, the extremes of the panels, noted $\overline{x}_i$, $i = \overline{1, N}$, being situated on Σ. Introducing this geometric approximation in (7) we obtain the following boundary integral equation:

$$\left\{ (n_x^0)^2 + \beta^2 \left[(n_y^0)^2 + (n_z^0)^2 \right] \right\} m(\overline{x}_0) + \frac{1}{2\pi} {\sum_{j=1}^{M}}' \iint_{T_j} m(\overline{x}) \frac{(x - x_0)n_x^0 + \beta^2 \left[(y - y_0)n_y^0 + (z - z_0)n_z^0 \right]}{|\overline{x} - \overline{x}_0|^3} da = 2\beta n_x^0 \tag{8}$$

Considering $\overline{x}_0 = \overline{x}_i$, $i \in \{1, 2, ..., N\}$ we have to calculate two types of integrals on T_j, with (if $\overline{x}_i$ is one of the triangle T_j vertices) and without singularities. Thus, we have, for a fixed *i*, the integral equation:

$$\left\{ (n_x^i)^2 + \beta^2 \left[(n_y^i)^2 + (n_z^i)^2 \right] \right\} m(\overline{x}_i) + \frac{1}{2\pi} \sum_{j \in A_1} \iint_{T_j} m(\overline{x}) \frac{(x - x_i)n_x^i + \beta^2 \left[(y - y_i)n_y^i + (z - z_i)n_z^i \right]}{|\overline{x} - \overline{x}_i|^3} da + \frac{1}{2\pi} {\sum_{j \in A_2}}' \iint_{T_j} m(\overline{x}) \frac{(x - x_i)n_x^i + \beta^2 \left[(y - y_i)n_y^i + (z - z_i)n_z^i \right]}{|\overline{x} - \overline{x}_i|^3} da = 2\beta n_x^i \tag{9}$$

where A_1 and A_2 represent the sets of triangles that don't have, respective have, an extreme in $\overline{x}_i$.

For describing the local geometry and the local behavior of the unknown m, so on a boundary element, we use linear isoparametric boundary elements. They use the same basic functions to model the geometry and the unknown function, and the approximation function is continuous on the boundary.

An important and also a difficult step in solving problems with BEM is the evaluation of the coefficients of the system the problem is reduced at, specially as regarding the evaluation of the singular ones. In papers [5], [6] there are presented some methods to treat the integrals of singular kernels in both cases: bi and three-dimensional problems. An efficient method that can be applied to surpass this difficulty in the 3D case consists in using suitable geometrical transformations of coordinates in order to eliminate the singularities.

In this approach we calculate the integrals using a local system of coordinates, the intrinsic system. Denoting by $\overline{x}_1, \overline{x}_2, \overline{x}_3$ the vertices (nodes) of a triangle, and by

$\lambda_1, \lambda_2, \lambda_3$ the intrinsic triangular coordinates ([7], [8]), we have for an interior point of the triangle the relation:

$$\bar{x} = \bar{x}_1 + (\bar{x}_2 - \bar{x}_1)\lambda_2 + (\bar{x}_3 - \bar{x}_1)\lambda_3 .$$

Using the parametric representation, so a transformation that strings together the current triangle and the basic one, given by:

$$\lambda_2 = r\cos\theta, \lambda_3 = r\sin\theta \,, \theta \in \left[0, \frac{\pi}{2}\right], r \in [0, \rho],$$

with ρ and θ satisfying relation:

$$\rho(\cos\theta + \sin\theta) = 1 \quad , \tag{10}$$

we obtain the new domain of integration, Fig.1,

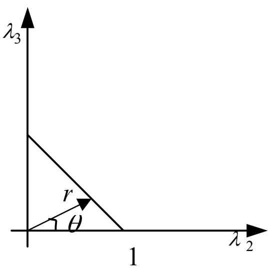

Fig. 1. New domain of integration

and further we get:

$$\bar{x} = \bar{x}_1 + (\bar{x}_2 - \bar{x}_1) r\cos\theta + (\bar{x}_3 - \bar{x}_1) r\sin\theta .$$

Evaluating the Jacobian of the transformation and noting with S the area of the initial triangle we have: $da = 2Srdrd\theta$.

First we consider that T_j has all nodes different from $\bar{x}_i$. Naming by $\bar{x}_1^j, \bar{x}_2^j, \bar{x}_3^j$ the vertices of panel T_j, and by m_j^1, m_j^2, m_j^3 the values of the unknown function in these nodes, and using the formulas below we have:

$$\bar{x} = \bar{x}_j^1 + (\bar{x}_j^2 - \bar{x}_j^1) r\cos\theta + (\bar{x}_j^3 - \bar{x}_j^1) r\sin\theta$$
$$m = m_j^1 + (m_j^2 - m_j^1) r\cos\theta + (m_j^3 - m_j^1) r\sin\theta \tag{11}$$

We can write that

$$\frac{1}{2\pi}\iint_{T_j} m(\bar{x}) \frac{(x - x_i)n_x^i + \beta^2\left((y - y_i)n_y^i + (z - z_i)n_z^i\right)}{\|\bar{x} - \bar{x}_i\|^3} da =$$
$$= \frac{S_j}{\pi}\left[m_j^1 A_{ij}^1 + (m_j^2 - f_j^1) B_{ij}^1 + (m_j^3 - f_j^1) C_{ij}^1\right]$$

where

$$A_{ij}^1 = n_x^i \left[(x_j^1 - x_i) \int_0^{\frac{\pi}{2}} I_1(\theta) d\theta + \int_0^{\frac{\pi}{2}} e_j^1\{\theta\} I_2(\theta) d\theta \right] +$$
$$+ \beta^2 n_y^i \left[(y_j^1 - y_i) \int_0^{\frac{\pi}{2}} I_1(\theta) d\theta + \int_0^{\frac{\pi}{2}} e_j^2\{\theta\} I_2(\theta) d\theta \right] +$$
$$+ \beta^2 n_z^i \left[(z_j^1 - z_i) \int_0^{\frac{\pi}{2}} I_1(\theta) d\theta + \int_0^{\frac{\pi}{2}} e_j^3\{\theta\} I_2(\theta) d\theta \right],$$

$$B_{ij}^1 = n_x^i \left[(x_j^1 - x_i) \int_0^{\frac{\pi}{2}} \cos\theta I_2(\theta) d\theta + \int_0^{\frac{\pi}{2}} \cos\theta e_j^1\{\theta\} I_3(\theta) d\theta \right] +$$
$$+ \beta^2 \left\{ n_y^i \left[(y_j^1 - y_i) \int_0^{\frac{\pi}{2}} \cos\theta I_2(\theta) d\theta + \int_0^{\frac{\pi}{2}} \cos\theta e_j^2\{\theta\} I_3(\theta) d\theta \right] + \right.$$
$$\left. + n_z^i \left[(z_j^1 - z_i) \int_0^{\frac{\pi}{2}} \cos\theta I_2(\theta) d\theta + \int_0^{\frac{\pi}{2}} \cos\theta e_j^3\{\theta\} I_3(\theta) d\theta \right] \right\},$$

$$C_{ij}^1 = n_x^i \left[(x_j^1 - x_i) \int_0^{\frac{\pi}{2}} \sin\theta I_2(\theta) d\theta + \int_0^{\frac{\pi}{2}} \sin\theta e_j^1\{\theta\} I_3(\theta) d\theta \right] +$$
$$+ \beta^2 \left\{ n_y^i \left[(y_j^1 - y_i) \int_0^{\frac{\pi}{2}} \sin\theta I_2(\theta) d\theta + \int_0^{\frac{\pi}{2}} \sin\theta e_j^2\{\theta\} I_3(\theta) d\theta \right] + \right.$$
$$\left. + n_z^i \left[(z_j^1 - z_i) \int_0^{\frac{\pi}{2}} \sin\theta I_2(\theta) d\theta + \int_0^{\frac{\pi}{2}} \sin\theta e_j^3\{\theta\} I_3(\theta) d\theta \right] \right\},$$

$$\bar{e}_j(\theta) = (\bar{x}_j^2 - \bar{x}_j^1)\cos\theta + (\bar{x}_j^3 - \bar{x}_j^1)\sin\theta ,$$
$$\bar{e}_j(\theta) = \left(e_j^1(\theta), e_j^2(\theta), e_j^3(\theta)\right) ,$$

$$I_n(\theta) = \int_0^{\rho} \frac{r^n}{(ar^2 + 2br + c)^{\frac{3}{2}}} dr \,, n = 1,2,3 ,$$

$$a = \|\bar{e}_j(\theta)\|^2 , c = \|\bar{x}_j^1 - \bar{x}_i\|^2 , \; b = (\bar{x}_j^1 - \bar{x}_i) \cdot \bar{e}_j(\theta)$$

(b the dot product between $\bar{x}_j^1 - \bar{x}_i$ and $\bar{e}_j(\theta)$). (12)

Integrals $I_n(\theta)$ from the above relation are the same as in [9], where the case of an incompressible fluid was considered and so they have the same analytical expressions, given by:

$$I_1(\theta)=\frac{\sqrt{c}}{\Delta}-\frac{b\rho+c}{\Delta\sqrt{a\rho^2+2b\rho+c}},$$

$$I_2(\theta)=\frac{\left(b^2-\Delta\right)\rho+bc}{a\Delta\sqrt{a\rho^2+2b\rho+c}}-\frac{b\sqrt{c}}{a\Delta}+$$
$$+\frac{1}{a^{3/2}}\ln\frac{\sqrt{a}\sqrt{a\rho^2+2b\rho+c}+a\rho+b}{b+\sqrt{ac}},$$

$$I_3(\theta)=\frac{\sqrt{a\rho^2+2b\rho+c}}{a^2}-\frac{3b}{a^{5/2}}\ln\frac{\left|a\rho+b+\sqrt{a}\sqrt{a\rho^2+2b\rho+c}\right|}{b+\sqrt{ac}}+$$
$$+\frac{\sqrt{c}\left(b^2-2\Delta\right)}{a^2\Delta}+\frac{c\left(\Delta-b^2\right)+\rho b\left(3\Delta-b^2\right)}{a^2\Delta\sqrt{a\rho^2+2b\rho+c}}. \tag{13}$$

The non singular integral becomes:

$$\frac{1}{2\pi}\iint_{T_j} m(\bar{x})\frac{(x-x_i)n_x^i+\beta^2\left((y-y_i)n_y^i+(z-z_i)n_z^i\right)}{\left\|\bar{x}-\bar{x}_i\right\|^3}da=$$
$$=\frac{S_j}{\pi}\left[m_j^1 A_{ij}+m_j^2 B_{ij}^1+m_j^3 C_{ij}^1\right]$$

where $A_{ij}=A_{ij}^1-B_{ij}^1-C_{ij}^1$. (14)

IV. Evaluating the singular integrals

Considering now that the triangle, noted T_j, has a vertex in $\bar{x}_i$ we calculate the singular integrals occuring in (9) using the following relations:

$$\bar{x}=\bar{x}_i+\left(\bar{x}_j^2-\bar{x}_i\right)r\cos\theta+\left(\bar{x}_j^3-\bar{x}_i\right)r\sin\theta$$
$$m=m_i+\left(m_j^2-m_i\right)r\cos\theta+\left(m_j^3-m_i\right)r\sin\theta \tag{15}$$

where $\bar{x}_j^2,\bar{x}_j^3$ are the other two nodes of T_j, and m_j^2,m_j^3 are the values of the unknown function, m, in these nodes.
If S_j is the area of T_j, we have:

$$\frac{1}{2\pi}{}^{/}\iint_{T_j} m(\bar{x})\frac{(x-x_i)n_x^i+\beta^2\left((y-y_i)n_y^i+(z-z_i)n_z^i\right)}{\left\|\bar{x}-\bar{x}_i\right\|^3}da=$$
$$=\frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\rho}\left[m_i+\left(m_j^2-m_i\right)r\cos\theta\right]\frac{r\langle\bar{e}_j,\bar{n}\rangle}{r^3\left\|\bar{e}_j(\theta)\right\|^3}2S_j r\,dr\,d\theta+$$
$$+\frac{\bar{n}_i}{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\rho}\left[\left(m_j^3-m_i\right)r\sin\theta\right]\frac{r\langle\bar{e}_j,\bar{n}\rangle}{r^3\left\|\bar{e}_j(\theta)\right\|^3}2S_j r\,dr\,d\theta$$

where $\langle\bar{e}_j,\bar{n}\rangle \underset{not}{=} e_j^1(\theta)n_x^i+\beta^2\left(e_j^2(\theta)n_y^i+e_j^3(\theta)n_z^i\right)$, e_j^1,e_j^2,e_j^3 being components of

$$\bar{e}_j(\theta)=\left(\bar{x}_j^2-\bar{x}_i\right)\cos\theta+\left(\bar{x}_j^3-\bar{x}_i\right)\sin\theta$$

Using the fact that the finite part of $\int_0^{\rho}\frac{1}{r}dr$ is:

$$FP\int_0^{\rho}\frac{1}{r}dr=\ln(\rho)$$

we get the equivalent form:

$$\frac{1}{2\pi}{}^{/}\iint_{T_j} m(\bar{x})\frac{(x-x_i)n_x^i+\beta^2\left((y-y_i)n_y^i+(z-z_i)n_z^i\right)}{\left\|\bar{x}-\bar{x}_i\right\|^3}da=$$
$$=\frac{S_j}{\pi}\left[m_i A_{ij}^2+\left(m_j^2-m_i\right)B_{ij}^2+\left(m_j^3-m_i\right)C_{ij}^2\right],$$

where

$$A_{ij}^2=\int_0^{\frac{\pi}{2}}\frac{e_j^1(\theta)n_x^i+\beta^2\left(e_j^2(\theta)n_y^i+e_j^3(\theta)n_z^i\right)}{\left\|\bar{e}_j(\theta)\right\|^3}\ln\rho(\theta)d\theta$$

$$B_{ij}^2=\int_0^{\frac{\pi}{2}}\frac{\left[e_j^1(\theta)n_x^i+\beta^2\left(e_j^2(\theta)n_y^i+e_j^3(\theta)n_z^i\right)\right]\cos\theta}{\left\|\bar{e}_j(\theta)\right\|^3}\rho(\theta)d\theta$$

$$C_{ij}^2=\int_0^{\frac{\pi}{2}}\frac{\left[e_j^1(\theta)n_x^i+\beta^2\left(e_j^2(\theta)n_y^i+e_j^3(\theta)n_z^i\right)\right]\sin\theta}{\left\|\bar{e}_j(\theta)\right\|^3}\rho(\theta)d\theta\cdot \tag{16}$$

Denoting by $A_{ij}'=A_{ij}^2-B_{ij}^2-C_{ij}^2$, we further get:

$$\frac{1}{2\pi}{}^{/}\iint_{T_j} m(\bar{x})\frac{(x-x_i)n_x^i+\beta^2\left((y-y_i)n_y^i+(z-z_i)n_z^i\right)}{\left\|\bar{x}-\bar{x}_i\right\|^3}da=$$
$$=\frac{S_j}{\pi}\left[m_i A_{ij}'+m_j^2 B_{ij}^2+m_j^3 C_{ij}^2\right] \tag{17}$$

and finally equation (9) has the form:

$$m_i A_i+\sum_{j\in A_1}\frac{S_j}{\pi}\left(m_j^1 A_{ij}+m_j^2 B_{ij}^1+m_j^3 C_{ij}^1\right)+$$
$$+\sum_{j\in A_2}\frac{S_j}{\pi}\left(m_j^2\overline{B}_{ij}^2+m_j^3\overline{C}_{ij}^2\right)=2\beta n_i^x$$

where

$$A_i=\left(n_x^i\right)^2+\beta^2\left(\left(n_y^i\right)^2+\left(n_x^i\right)^2\right)+\frac{1}{\pi}\left(\sum_{j\in A_2}S_j A_{ij}'\right)\cdot \tag{18}$$

Returning to the global system of notation the problem is reduced to the following system of equations:

$$\sum_{j=1}^{N}\widetilde{A}_{ij}f_j=2U_\infty n_i^x \tag{19}$$

After solving system (19) we may compute the velocity for the N nodes choosen for the boundary discretization and for any other point in the fluid domain.

V. Testing the Method

In order to test the method we shall consider the uniform motion of an incompressible fluid in the presence of a sphere of radius R, centered in the origin of the system of coordinates. In this case the integral equation (7) can be solved analytically. A solution of this equation can be found in [10].

Using the spherical coordinates for the nodal points, so expressing the position of a point through relation: $\bar{x} = R\left(\sin q_1 \cos q_2 \bar{i} + \sin q_1 \sin q_2 \bar{j} + \cos q_1 \bar{k}\right)$, and the method of successive approximations to integrate equation (7), the exact solution is obtained It has the following expression:

$$m(q_1, q_2) = \frac{3}{2} U_\infty \cos q_1$$

Comparisons between the analytical values of the intensity m, on the sphere, and the values calculated by means of the boundary element method (with a computer code in MATHCAD) are performed in Fig.2. The boundary mesh is represented by 24 planar triangles and has 14 control points.

We can observe that the calculated and analytical values of the intensity are very close evan if the number of nodes on the boundary is not very big, fact that validates the computer code and proves the efficiency of the method proposed in this paper. Better results can be achieved with more nodes on the boundary or using higher order boundary elements.

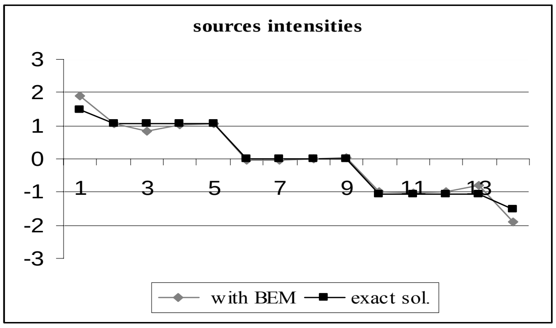

Fig.2. The sources intensities for the 14 control points

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