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A Remark on Restart Phase of GMRES(k) Method with Iterative Refinement for Gaining Robustness of Convergence

Yuzo Kusakabe† Seiji Fujino‡ *

Abstract— **Restarted GMRES(k) method with iterative refinement process for approximated solutions was proposed by Imakura, *et al.* They referred to it as IR-GMRES(k) method. Convergence of this IRGMRES(k) method depends on accuracy of the initial approximated solutions adopted at restarting phase. Then it is crucial to examine how to give preferable initial approximated solutions. Therefore, in this paper, we consider reasonable givings for the initial approximated solutions when IR-GMRES(k) method restarts, and discuss validity and robustness of our proposed givings through several numerical experiments.**

Keywords:* *GMRES method, GMRES(m) method, restart cycle, iterative refinement, MR method, MR(2) method

1 Introduction

GMRES (k) (Generalized Minimal RESidual method with restart cycle k)[3] is known to be a strong tool for solving a linear system of equations $A\boldsymbol{x} = \boldsymbol{b}$, where $A \in R^{N\times N}$is a given nonsymmetric matrix. $\boldsymbol{x}$ and $\boldsymbol{b}$ are solution and right-hand vectors, respectively. Recently, IR-GMRES(k) (GMRES with Iterative Refinement) method has been proposed by Imakura, *et al*[1]. In IR-GMRES(k) method they devised a concept of linear systems, i.e., $A\boldsymbol{e}_k = \boldsymbol{r}_k$ on the error $\boldsymbol{e}_k$ and residual $\boldsymbol{r}_k$ at restarting phase. Moreover they predicted that for improvement of convergence of IR-GMRES(k) method, it is crucial to give an effective initial approximated solution to the above linear equations on the error $\boldsymbol{e}_k$.

Therefore, in this paper, we propose some variants of IR-GMRES(k) method, i.e., IR-GMRES(k)-MR, IR-GMRES(k)-MR(2) and IR-GMRES(k)-GCR(1) methods which utilize one-staged MR (Minimized Residual), two-staged MR(2) methods and GCR(1) method [2] for setting the initial approximated solutions, respectively. Moreover we discuss validity and robustness of these variants of IR-GMRES(k) methods through several numerical experiments.

*†Graduate School of Information Science and Electrical Engineering, Kyushu University, Email: kusakabe@zeal.cc.kyushuu.ac.jp ‡Research Institute for Information Technology, Kyushu University, Email: fujino@cc.kyushu-u.ac.jp

This paper is organized as follows. In section 2, we briefly note the conventional GMRES(k) method. In section 3, we describe some variants of IR-GMRES(k) method with iterative refinement. In section 4, we exhibit the givings for the initial approximated solutions of IR-GMRES(k) method. In section 5, we demonstrate validity through some numerical experiments, and in section 6 we draw some concluding remarks.

2 GMRES(k) method

We will focus on the iterative solution of a linear system of equations

$$A\boldsymbol{x} = \boldsymbol{b} \tag{1}$$

where A is a nonsingular reall $N \times N$ matrix and a right-hand side vector $\boldsymbol{b}$ is a given real N-vector. Starting from initial guess $\boldsymbol{x}_0$ of the solution, initial residual $\boldsymbol{r}_0$ is defined as $\boldsymbol{r}_0 = \boldsymbol{b} - A\boldsymbol{x}_0$.

It is well known that the GMRES method is one of iterative method using Krylov subspace condition as

$$K_i(A;\boldsymbol{r}_0) = \text{span}\{\boldsymbol{r}_0, A\boldsymbol{r}_0, \ldots, A^{i-1}\boldsymbol{r}_0\}, \tag{2}$$

$$\boldsymbol{x}_i = \boldsymbol{x}_0 + \boldsymbol{z}_i, \quad \boldsymbol{z}_i \in K_i(A;\boldsymbol{r}_0) \tag{3}$$

and minimum residual condition as

$$\boldsymbol{z}_i = \text{arg min}_{\boldsymbol{z}_i\in K_i(A;\boldsymbol{r}_0)}\text{min}||\boldsymbol{b} - A(\boldsymbol{x}_0 + \boldsymbol{z})||_2 \tag{4}$$

in order to compute the iterative solution $\boldsymbol{x}_i$.

GMRES method, however, requires much computation times and amount of memory as the iterations increase. Therefore, GMRES(k)method with restart is often used. In GMRES(k) method, approximated solution $\boldsymbol{x}_k$ given at k-th iteration is defined as new initial guess $\boldsymbol{x}_0$.

3 IR-GMRES(k) method

First of all, we describe an outline of IR-GMRES(k) method (see [1] in detail by Imakura, *et al.*) This IR-GMRES(k) method may be regarded as an extension of iterative refinement on approximated solution with complete LU decomposition which is often used in usual direct method. That is, minimum residual condition of $\boldsymbol{z}_i$ used when we solve a linear system $A\boldsymbol{x} = \boldsymbol{b}$ by initial guess $\boldsymbol{x}_k$ of GMRES(k) method at restart phase may be rewritten as follows :

$$\begin{aligned} \boldsymbol{z}_i &= \arg\min_{\boldsymbol{z}_i \in K_i(A;\boldsymbol{r}_0)} ||\boldsymbol{b} - A(\boldsymbol{x}_k + \boldsymbol{z})||_2 & (5) \\ &= \arg\min_{\boldsymbol{z}_i \in K_i(A;\boldsymbol{r}_0)} ||\boldsymbol{r}_k - A\boldsymbol{z}||_2 & (6) \\ &= \arg\min_{\boldsymbol{z}_i \in K_i(A;\boldsymbol{r}_0)} ||\boldsymbol{r}_k - A(\boldsymbol{0} + \boldsymbol{z})||_2. & (7) \end{aligned}$$

Here, we can regard eqn.(7) as minimum residual condition of $\boldsymbol{z}_i$ on unknown error $\boldsymbol{e}$ with initial guess $\boldsymbol{0}$.

$$A\boldsymbol{e} = \boldsymbol{r}_k \tag{8}$$

Moreover, we give initial guess $\boldsymbol{e}_0$ such that the following inequality is satisfied,

$$||\boldsymbol{r}_k - A\boldsymbol{e}_0||_2 \leq ||\boldsymbol{r}_k||_2 \tag{9}$$

We show algorithm of IR-GMRES(k) method as follows:

algorithm of IR-GMRES(k) method

1. Solve $A\boldsymbol{x} = \boldsymbol{b}$ by GMRES(k) method with an initial guess $\boldsymbol{x}_0$, and get $\boldsymbol{x}_k$.
2. Set $\boldsymbol{r}_k = \boldsymbol{b} - A\boldsymbol{x}_k$. (10)
3. Solve $A\boldsymbol{e} = \boldsymbol{r}_k$ by GMRES(k) method with an initial guess $\boldsymbol{e}_0$, and get $\boldsymbol{e}_k$. (11)
4. Update $\boldsymbol{x}_k = \boldsymbol{x}_k + \boldsymbol{e}_k$.
5. Repeat (go to (10))

At eqn. (11) of the above algorithm of IR-GMRES(k) method, $\boldsymbol{e}_k$ is updated by the below equation. Here upper-subscript of (l) of vectors as $\boldsymbol{e}_k^{(l)}$ means the number of restart.

$$\boldsymbol{e}_k^{(l)} = \boldsymbol{e}_0^{(l)} + \boldsymbol{z}_k^{(l)}. \tag{12}$$

Moreover we give initial guess $\boldsymbol{e}_0$ in the algorithm of IRGMRES(k) method so that the following eqns. (13) and (14) are satisfied.

$$\begin{aligned} \boldsymbol{e}_0^{(l)} &= \alpha \boldsymbol{z}_k^{(l-1)} + \beta \boldsymbol{z}_k^{(l-2)}, & (13) \\ \alpha, \beta &= \arg\min_{\alpha,\beta \in R} ||\boldsymbol{r}_k^{(l)} - A\boldsymbol{e}_0^{(l)}||_2. & (14) \end{aligned}$$

4 How to give the initial approximated solutions

For the purpose of improving convergence rate of IR-GMRES(k) method, it is significant to construct the following recursion of (15) so that initial guess $\boldsymbol{e}_0$ minimizes the residual 2-norm $||\boldsymbol{r}_k - A\boldsymbol{e}_0||_2$:

$$\begin{aligned} \boldsymbol{r}_{k+1}^{(l)} &= \boldsymbol{r}_k^{(l)} - A\boldsymbol{e}_0^{(l)}, & (15) \\ ||\boldsymbol{r}_{k+1}^{(l)}||_2 &= ||\boldsymbol{r}_k^{(l)} - A\boldsymbol{e}_0^{(l)}||_2 \leq ||\boldsymbol{r}_k^{(l)}||_2 & (16) \end{aligned}$$

We omit upper-subscript of (l) of vectors as below.

4.1 IR-GMRES(k)-MR method

In the previous paper[2], we devised an variant of IR-GMRES(k) method based minimized residual of the MR method as the initial guess $\boldsymbol{e}_0$. We referred to IR-GMRES(k)-MR method. Since it holds that $\boldsymbol{r}_{k+1} = (I - \alpha A)\boldsymbol{r}_k$, we can gain the following equations.

$$\begin{aligned} \boldsymbol{e}_0 &= \alpha \boldsymbol{r}_k, & (17) \\ \alpha &= \arg\min_{\alpha \in R} ||\boldsymbol{r}_k - \alpha A\boldsymbol{r}_k||_2. & (18) \end{aligned}$$

4.2 IR-GMRES(k)-MR(2) method

In the same paper, we devised the 2nd variant of IR-GMRES(k) method which gives initial guess $\boldsymbol{e}_0$ as (19). We referred to IR-GMRES(k)-MR(2) method. This variant utilizes parameter α of IR-GMRES(k)-MR method. However this variant consists of square of parameter α. Since it holds that $\boldsymbol{r}_{k+1} = (I - \alpha A)^2\boldsymbol{r}_k$, we can gain the following equations.

$$\begin{aligned} \boldsymbol{e}_0 &= 2\alpha \boldsymbol{r}_k - \alpha^2 A\boldsymbol{r}_k, & (19) \\ \alpha &= \arg\min_{\alpha \in R} ||\boldsymbol{r}_k - \alpha A\boldsymbol{r}_k||_2. & (20) \end{aligned}$$

4.3 IR-GMRES(k)-GCR(1) method

Similarly, in the same paper, we devised the 3rd variant of IR-GMRES(k) method. We referred to IR-GMRES(k)-GCR(1) method. This variant fixes as $\alpha = 1$ and uses one parameter β only of GCR(1) method as initial guess $\boldsymbol{e}_0$. Since it holds that $\boldsymbol{r}_{k+1} = \boldsymbol{r}_k - A(\boldsymbol{r}_k - \beta A\boldsymbol{r}_k)$, we can gain also the following equations.

$$\begin{aligned} \boldsymbol{e}_0 &= \boldsymbol{r}_k - \beta A\boldsymbol{r}_k, & (21) \\ \beta &= \arg\min_{\beta \in R} ||\boldsymbol{r}_k - \beta A\boldsymbol{r}_k||_2. & (22) \end{aligned}$$

5 Numerical experiments

In this section, we present some numerical experiments that focus on how the proposed variant IR-GMRES(k) method can overcome the original restarted GMRES(k) and IR-GMRES(k) methods. All computations were done in double precision floating point arithmetics of Fortran95, and performed on HITACHI SR11000 model J1 with CPU of POWER5, clock of 1.9GHz, main memory of 128GB and OS of AIX 5.3. Optimum option -64 -Oss -nolimit -noscope -noparallel was used. In all experiments the right-hand side vector is taken to be a vector as $\boldsymbol{b} = A\hat{\boldsymbol{x}}$, where a solution vector $\boldsymbol{x}$ is defined by $\hat{\boldsymbol{x}} = (1, \cdots, 1)^T$. Stopping criterion for successful convergence of GMRES(k) and IR-GMRES(k) methods is less than 10^{-12} of the relative residual 2-norm $||\boldsymbol{r}_{n+1}||_2/||\boldsymbol{r}_0||_2$. In all cases the iteration was started with the initial guess solution $\boldsymbol{x}_0 = \boldsymbol{0}$. All matrices were normalized with diagonal scaling. The maximum iteration was fixed as 10000. The restart cycle k was set as 20, 50, 100, 200, 500, 1000. Table 1 tabulates the characteristics of test matrices.

Table 1 shows the characteristics of test matrix.

Table 1: The characteristics of test matrix.

matrix	n	nnz	ave. nnz	analytical field
bridge*	64,461	4,373,817	67.85	structural
poisson3Da	13,514	352,762	26.10	
poisson3Db	85,623	2,374,949	27.74	
xenon1	48,600	1,181,120	24.30	
xenon2	157,464	3,866,688	24.56	
sme3Da*	12,504	874,887	69.97	
sme3Db*	29,067	2,081,063	71.60	
ecl32	51,993	380,415	7.32	thermal
epb1	14,734	95,053	6.45	
epb2	25,228	175,027	6.94	
epb3	84,617	463,625	5.48	
ex10hs*	2,548	57,308	22.49	hydro-dynamic
ex11*	16,614	1,096,948	66.03	
raefsky3	21,200	1,488,768	70.22	
raefsky4*	19,779	1,328,611	67.17	
venkat50*	62,424	1,717,777	27.52	
add20	2,395	17,319	7.23	electrical
add32	4,960	23,884	4.82	
memplus	17,758	126,150	7.10	
wang3	26,064	177,168	6.80	
wang4	26,068	177,196	6.80	

Tables 2-6 present convergence properties of GMRES(k), IR-GMRES(k), IR-GMRES(k)-MR(2), IR-GMRES(k)-GCR(1) method for test matrices. However, in case of seven matrices with mark " $*$ " as shown in Table 1, tested all iterative methods did not converge until maximum iteration. It must be noted that convergence of IR-GMRES(k)-MR method deteriorated always as compared with that of IR-GMRES(k)-MR(2) method. Then results of IR-GMRES(k)-MR method were omitted in Tables 2-6. In Tables 2-6, "TRR" means values of True Relative Residual of $||\boldsymbol{b} - A\boldsymbol{x}_{n+1}||_2/||\boldsymbol{b} - A\boldsymbol{x}_0||_2$ for the converged solutions $\boldsymbol{x}_{n+1}$. "max" also means that the iterative method did not converge until maximum iteration. The bold figure means the fastest case for each matrix. The italic figure of IR-GMRES(k) method with $k = 100$ and $k = 200$ for matrix poisson3Da as shown in Table 2 means a case of too many iterations. Similarly, the italic figure of IR-GMRES(k) method with $k = 100$ for matrix wang3 as shown in Table 6 means also a case of too many iterations. From Tables 2-6 we can observe the following facts.

- When the restart cycle k is small as $k = 20$ or $k = 50$, IR-GMRES(k)-MR(2) method works well compared with the other GMRES(k) and IR-GMRES(k) methods in view of convergence rate.
- When the restart cycle k is large as $k = 500$ or $k = 1000$, convergence rate of all tested GMRES(k) and IR-GMRES(k) methods are competitive each other.
- IR-GMRES(k) method sometimes requires much computation cost for initial approximated solution at restart phase such as matrices poisson3Da with $k = 100, 200$ and wang3 with $k = 20$. The degradation of convergence rate leads to increase of computation time. Parameter $\beta = 0$ of IR-GMRES(k) method in eqn. (13) causes this degradation.

Table 2: Iterations, computation time, True Relative Residual and used memory of some GMRES(k) methods.(cont'd)

matrix	method	k	itr.	time	TRR	memory [MB]
poisson 3Da	GMRES	20	531	1.01	-12.01	6.77
		50	421	0.94	-12.01	9.88
		100	286	0.81	-12.04	15.10
		200	208	0.85	-12.03	25.64
		500	206	0.88	-12.04	58.18
		1000	206	0.87	-12.04	115.47
	IR-GMRES	20	584	1.20	-12.01	7.08
		50	420	0.97	-12.03	10.19
		100	*356*	1.00	-12.02	15.41
		200	*390*	1.60	-12.03	25.95
		500	206	0.88	-12.04	58.49
		1000	206	0.87	-12.04	115.78
	IR-GMRES-MR(2)	20	533	1.10	-12.01	7.08
		50	397	0.91	-12.03	10.19
		100	283	**0.80**	-12.03	15.41
		200	206	0.85	-12.01	25.95
		500	206	0.88	-12.04	58.49
		1000	206	0.87	-12.04	115.78
	IR-GMRES-GCR(1)	20	495	1.02	-12.04	7.08
		50	419	0.96	-12.01	10.19
		100	285	0.81	-12.00	15.41
		200	208	0.85	-12.04	25.95
		500	206	0.88	-12.04	58.49
		1000	206	0.87	-12.04	115.78
poisson 3Db	GMRES	20	1399	20.40	-12.03	44.49
		50	989	17.28	-12.00	64.11
		100	769	17.09	-12.01	96.83
		200	574	18.04	-12.03	162.39
		500	427	22.69	-12.03	359.97
		1000	427	23.03	-12.03	692.34
	IR-GMRES	20	1502	23.90	-12.00	46.45
		50	996	18.05	-12.01	66.07
		100	769	17.36	-12.01	98.79
		200	574	18.14	-12.03	164.35
		500	427	22.90	-12.03	361.93
		1000	427	23.26	-12.03	694.30
	IR-GMRES-MR(2)	20	1109	17.65	-12.02	46.45
		50	956	17.39	-12.01	66.07
		100	751	**16.83**	-12.02	98.79
		200	570	17.96	-12.02	164.35
		500	427	23.22	-12.03	361.93
		1000	427	22.90	-12.03	694.30
	IR-GMRES-GCR(1)	20	1214	19.06	-12.00	46.45
		50	984	17.82	-12.01	66.07
		100	768	17.10	-12.01	98.79
		200	573	18.06	-12.01	164.35
		500	427	22.84	-12.03	361.93
		1000	427	23.21	-12.03	694.30

Table 3: Iterations, computation time, True Relative Residual and used memory of some GMRES(k) methods.(cont'd)

matrix	method	k	itr.	time	TRR	memory [MB]
xenon1	GMRES	20	max	51.16	-9.06	23.35
		50	6466	41.45	-12.00	34.49
		100	3638	33.40	-12.00	53.09
		200	2389	35.39	-12.00	90.40
		500	1916	58.89	-12.00	203.24
		1000	1727	91.66	-12.00	394.38
	IR-GMRES	20	max	55.46	-10.21	24.46
		50	6092	40.17	-12.00	35.60
		100	3559	33.14	-12.00	54.20
		200	2380	35.73	-12.00	91.51
		500	1916	58.26	-12.00	204.36
		1000	1727	90.46	-12.00	395.49
	IR-GMRES-MR(2)	20	max	55.40	-10.86	24.46
		50	5866	38.96	-12.00	35.60
		100	3504	**32.76**	-12.00	54.20
		200	2364	35.06	-12.00	91.51
		500	1911	58.59	-12.00	204.36
		1000	1725	91.32	-12.00	395.49
	IR-GMRES-GCR(1)	20	max	55.41	-9.21	24.46
		50	6389	42.46	-12.00	35.60
		100	3617	33.63	-12.00	54.20
		200	2385	35.67	-12.00	91.51
		500	1915	58.19	-12.00	204.36
		1000	1727	91.75	-12.00	395.49
xenon2	GMRES	20	max	181.50	-7.75	76.09
		50	8503	197.53	-12.00	112.15
		100	4704	152.70	-12.00	172.27
		200	2894	**144.94**	-12.00	292.64
		500	2293	232.01	-12.00	654.66
		1000	2172	411.12	-12.00	1261.07
	IR-GMRES	20	max	194.85	-8.63	79.69
		50	7986	193.34	-12.00	115.75
		100	4603	151.64	-12.00	175.88
		200	2886	146.50	-12.00	296.25
		500	2293	234.29	-12.00	658.26
		1000	2172	409.47	-12.00	1264.68
	IR-GMRES-MR(2)	20	max	196.74	-9.92	79.69
		50	7537	180.63	-12.00	115.75
		100	4543	151.59	-12.00	175.88
		200	2852	145.96	-12.00	296.25
		500	2285	234.94	-12.00	658.26
		1000	2168	404.84	-12.00	1264.68
	IR-GMRES-GCR(1)	20	max	194.67	-7.87	79.69
		50	8398	201.41	-12.00	115.75
		100	4672	155.25	-12.00	175.88
		200	2889	148.98	-12.00	296.25
		500	2292	233.55	-12.00	658.26
		1000	2172	410.92	-12.00	1264.68
ecl32	GMRES	20	max	32.47	-8.42	14.87
		50	9305	43.58	-12.00	26.79
		100	6181	46.17	-12.00	46.68
		200	2075	28.08	-12.01	86.58
		500	530	**15.59**	-12.02	207.19
		1000	521	16.91	-12.02	411.27
	IR-GMRES	20	max	34.81	-5.48	16.06
		50	max	47.65	-11.99	27.98
		100	4662	34.95	-12.00	47.87
		200	2086	27.86	-12.00	87.77
		500	535	15.94	-12.04	208.38
		1000	521	16.75	-12.02	412.46
	IR-GMRES-MR(2)	20	max	34.75	-8.34	16.06
		50	max	47.66	-9.96	27.98
		100	6858	51.63	-12.00	47.87
		200	2839	38.27	-12.00	87.77
		500	528	15.95	-12.02	208.38
		1000	521	17.12	-12.02	412.46
	IR-GMRES-GCR(1)	20	max	34.64	-8.81	16.06
		50	max	47.64	-7.91	27.98
		100	4426	33.24	-12.00	47.87
		200	2060	27.45	-12.00	87.77
		500	530	15.90	-12.03	208.38
		1000	521	16.82	-12.02	412.46

Table 4: Iterations, computation time, True Relative Residual and used memory of some GMRES(k) methods.(cont'd)

matrix	method	k	itr.	time	TRR	memory [MB]
epb1	GMRES	20	2388	1.94	-12.00	4.07
		50	1513	1.83	-12.00	7.46
		100	1302	2.49	-12.00	13.14
		200	1135	3.64	-12.02	24.61
		500	901	6.74	-12.01	59.95
		1000	636	6.80	-12.02	121.89
	IR-GMRES	20	2458	2.12	-12.00	4.41
		50	1459	1.80	-12.00	7.80
		100	1286	2.46	-12.00	13.48
		200	1135	3.64	-12.01	24.95
		500	901	6.74	-12.01	60.28
		1000	636	6.70	-12.02	122.23
	IR-GMRES-MR(2)	20	1584	**1.36**	-12.00	4.41
		50	1450	1.79	-12.00	7.80
		100	1285	2.46	-12.02	13.48
		200	1134	3.62	-12.00	24.95
		500	899	6.69	-12.01	60.28
		1000	636	6.70	-12.02	122.23
	IR-GMRES-GCR(1)	20	2033	1.75	-12.00	4.41
		50	1422	1.74	-12.01	7.80
		100	1311	2.52	-12.00	13.48
		200	1130	3.62	-12.02	24.95
		500	901	6.72	-12.01	60.28
		1000	636	6.73	-12.02	122.23
epb2	GMRES	20	801	1.15	-12.02	7.11
		50	562	1.19	-12.01	12.90
		100	495	1.63	-12.00	22.58
		200	425	2.38	-12.01	42.06
		500	394	4.62	-12.03	101.41
		1000	394	4.57	-12.03	203.39
	IR-GMRES	20	759	1.16	-12.01	7.69
		50	571	1.22	-12.00	13.48
		100	496	1.63	-12.02	23.16
		200	425	2.36	-12.01	42.64
		500	394	4.61	-12.03	101.99
		1000	394	4.53	-12.03	203.97
	IR-GMRES-MR(2)	20	729	1.11	-12.00	7.69
		50	546	1.17	-12.01	13.48
		100	483	1.57	-12.01	23.16
		200	420	2.37	-12.01	42.64
		500	394	4.60	-12.03	101.99
		1000	394	4.54	-12.03	203.97
	IR-GMRES-GCR(1)	20	697	**1.06**	-12.01	7.69
		50	554	1.19	-12.02	13.48
		100	493	1.61	-12.02	23.16
		200	424	2.37	-12.02	42.64
		500	394	4.53	-12.03	101.99
		1000	394	4.60	-12.03	203.97
epb3	GMRES	20	max	44.24	-4.53	22.42
		50	max	68.88	-6.36	41.80
		100	max	119.06	-9.72	74.14
		200	max	218.63	-10.43	138.93
		500	6777	332.12	-12.00	334.21
		1000	5279	499.52	-12.00	662.74
	IR-GMRES	20	max	46.84	-4.54	24.35
		50	max	70.15	-5.86	43.74
		100	max	120.70	-8.83	76.08
		200	max	217.98	-11.99	140.87
		500	6928	346.49	-12.00	336.15
		1000	5279	502.89	-12.00	664.68
	IR-GMRES-MR(2)	20	max	46.64	-4.72	24.35
		50	max	69.74	-5.35	43.74
		100	max	120.44	-11.26	76.08
		200	9577	**207.57**	-12.00	140.87
		500	6820	336.69	-12.00	336.15
		1000	5271	502.04	-12.00	664.68
	IR-GMRES-GCR(1)	20	max	46.70	-4.66	24.35
		50	max	69.67	-5.76	43.74
		100	max	119.87	-10.41	76.08
		200	9789	212.88	-12.00	140.87
		500	7163	358.14	-12.00	336.15
		1000	5272	497.52	-12.00	664.68

Table 5: Iterations, computation time, True Relative Residual and used memory of some GMRES(k) methods.(cont'd)

matrix	method	k	itr.	time	TRR	memory [MB]
raefsky3	GMRES	20	max	45.37	-7.72	21.33
		50	max	50.09	-8.52	26.20
		100	max	59.97	-9.02	34.34
		200	max	85.39	-9.04	50.75
		500	max	159.76	-11.40	100.88
		1000	9556	**259.58**	-12.00	187.49
	IR-GMRES	20	max	49.29	-7.45	21.81
		50	max	51.65	-8.29	26.68
		100	max	60.74	-8.89	34.83
		200	max	85.70	-8.99	51.23
		500	max	159.17	-11.34	101.37
		1000	9532	261.27	-12.00	187.98
	IR-GMRES-MR(2)	20	max	49.26	-8.21	21.81
		50	max	51.74	-8.77	26.68
		100	max	60.73	-9.16	34.83
		200	max	85.65	-9.14	51.23
		500	max	160.02	-11.52	101.37
		1000	9531	261.94	-12.00	187.98
	IR-GMRES-GCR(1)	20	max	49.41	-8.23	21.81
		50	max	51.85	-8.78	26.68
		100	max	60.72	-9.16	34.83
		200	max	85.51	-9.16	51.23
		500	max	159.92	-11.48	101.37
		1000	9547	262.12	-12.00	187.98
add20	GMRES	20	333	0.06	-12.02	0.69
		50	238	0.06	-12.00	1.25
		100	211	0.07	-12.01	2.22
		200	188	0.11	-12.06	4.28
		500	188	0.11	-12.06	11.38
		1000	188	0.11	-12.06	26.25
	IR-GMRES	20	331	0.06	-12.00	0.74
		50	238	0.06	-12.00	1.31
		100	211	0.07	-12.02	2.28
		200	188	0.11	-12.06	4.34
		500	188	0.10	-12.06	11.43
		1000	188	0.11	-12.06	26.30
	IR-GMRES-MR(2)	20	288	**0.05**	-12.00	0.74
		50	236	0.06	-12.08	1.31
		100	209	0.07	-12.01	2.28
		200	188	0.11	-12.06	4.34
		500	188	0.11	-12.06	11.43
		1000	188	0.11	-12.06	26.30
	IR-GMRES-GCR(1)	20	330	0.06	-12.01	0.74
		50	239	0.06	-12.04	1.31
		100	211	0.07	-12.01	2.28
		200	188	0.11	-12.06	4.34
		500	188	0.10	-12.06	11.43
		1000	188	0.11	-12.06	26.30
add32	GMRES	20	107	0.04	-12.00	1.28
		50	96	0.04	-12.03	2.43
		100	93	0.06	-12.02	4.38
		200	93	0.06	-12.02	8.40
		500	93	0.06	-12.02	21.36
		1000	93	0.06	-12.02	46.02
	IR-GMRES	20	105	0.04	-12.02	1.39
		50	96	0.04	-12.03	2.55
		100	93	0.06	-12.02	4.50
		200	93	0.06	-12.02	8.51
		500	93	0.06	-12.02	21.48
		1000	93	0.06	-12.02	46.14
	IR-GMRES-MR(2)	20	97	**0.03**	-12.01	1.39
		50	95	0.04	-12.07	2.55
		100	93	0.06	-12.02	4.50
		200	93	0.06	-12.02	8.51
		500	93	0.06	-12.02	21.48
		1000	93	0.06	-12.02	46.14
	IR-GMRES-GCR(1)	20	108	0.04	-12.01	1.39
		50	96	0.04	-12.07	2.55
		100	93	0.06	-12.02	4.50
		200	93	0.06	-12.02	8.51
		500	93	0.06	-12.02	21.48
		1000	93	0.06	-12.02	46.14

Table 6: Iterations, computation time, True Relative Residual and used memory of some GMRES(k) methods.

matrix	method	k	itr.	time	TRR	memory [MB]
memplus	GMRES	20	1059	1.19	-12.02	5.04
		50	480	0.76	-12.01	9.12
		100	390	0.93	-12.04	15.95
		200	345	1.29	-12.02	29.73
		500	319	2.04	-12.01	71.99
		1000	319	2.06	-12.01	145.47
	IR-GMRES	20	950	1.14	-12.00	5.44
		50	480	0.77	-12.00	9.53
		100	390	0.94	-12.04	16.36
		200	345	1.28	-12.02	30.14
		500	319	2.05	-12.01	72.40
		1000	319	2.05	-12.01	145.87
	IR-GMRES-MR(2)	20	796	0.95	-12.01	5.44
		50	463	**0.74**	-12.01	9.53
		100	386	0.93	-12.05	16.36
		200	344	1.28	-12.01	30.14
		500	319	2.05	-12.01	72.40
		1000	319	2.05	-12.01	145.87
	IR-GMRES-GCR(1)	20	1076	1.29	-12.01	5.44
		50	483	0.77	-12.04	9.53
		100	390	0.94	-12.03	16.36
		200	345	1.28	-12.01	30.14
		500	319	2.04	-12.01	72.40
		1000	319	2.06	-12.01	145.87
wang3	GMRES	20	933	1.38	-12.02	7.30
		50	581	1.29	-12.01	13.28
		100	426	1.42	-12.01	23.29
		200	317	1.68	-12.01	43.40
		500	260	2.07	-12.02	104.67
		1000	260	2.07	-12.02	209.83
	IR-GMRES	20	1025	1.61	-12.00	7.90
		50	639	1.43	-12.01	13.88
		100	*455*	1.53	-12.02	23.88
		200	452	2.54	-12.01	44.00
		500	260	2.07	-12.02	105.27
		1000	260	2.05	-12.02	210.43
	IR-GMRES-MR(2)	20	742	**1.16**	-12.01	7.90
		50	552	1.25	-12.04	13.88
		100	427	1.42	-12.02	23.88
		200	315	1.67	-12.01	44.00
		500	260	2.07	-12.02	105.27
		1000	260	2.08	-12.02	210.43
	IR-GMRES-GCR(1)	20	756	1.18	-12.01	7.90
		50	579	1.29	-12.00	13.88
		100	425	1.44	-12.02	23.88
		200	317	1.66	-12.01	44.00
		500	260	2.05	-12.02	105.27
		1000	260	2.05	-12.02	210.43
wang4	GMRES	20	max	14.82	-1.91	7.30
		50	729	1.59	-12.04	13.29
		100	397	1.35	-12.04	23.29
		200	289	1.50	-12.03	43.41
		500	242	1.78	-12.04	104.69
		1000	242	1.81	-12.04	209.86
	IR-GMRES	20	max	15.76	-1.90	7.90
		50	754	1.68	-12.00	13.88
		100	397	1.36	-12.04	23.89
		200	290	1.51	-12.02	44.01
		500	242	1.81	-12.04	105.28
		1000	242	1.81	-12.04	210.46
	IR-GMRES-MR(2)	20	max	15.75	-1.91	7.90
		50	682	1.51	-12.01	13.88
		100	394	**1.34**	-12.01	23.89
		200	287	1.49	-12.03	44.01
		500	242	1.80	-12.04	105.28
		1000	242	1.79	-12.04	210.46
	IR-GMRES-GCR(1)	20	1002	1.58	-12.03	7.90
		50	679	1.50	-12.01	13.88
		100	394	**1.34**	-12.02	23.89
		200	288	1.50	-12.01	44.01
		500	242	1.78	-12.04	105.28
		1000	242	1.80	-12.04	210.46

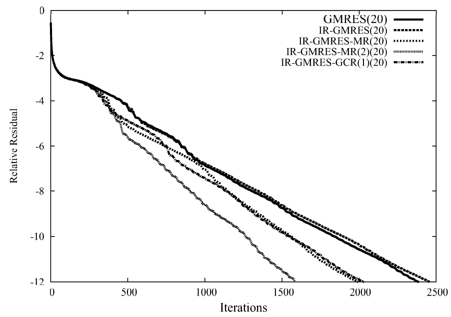

Figure 1: History of relative residual 2-norm of five types of GMRES($k = 20$) method for matrix epb1.

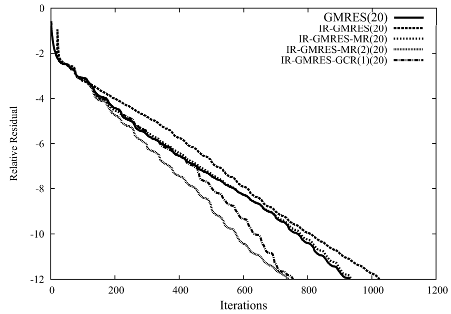

Figure 2: History of relative residual 2-norm of five types of GMRES($k = 20$) method for matrix wang3.

Figs.1-3 depict history of relative residual 2-norm of $\log_{10}(||\boldsymbol{r}_{n+1}||_2/||\boldsymbol{r}_0||_2)$ of GMRES(k) and some IRGMRES(k) methods for matrices epb1, wang3 with $k = 20$ and poisson3Da with $k = 100$. From Fig.2, we can see that residual of GMRES(k) and IR-GMRES(k)-MR(1), IR-GMRES(k)-MR(2) and IR-GMRES(k)-GCR(1) methods converges monotonously. On the other hand, the original IR-GMRES(k) method in green line converges irregularly as shown in Figs.2-3.

Fig.4 exhibits variation of iterations of GMRES(k) and

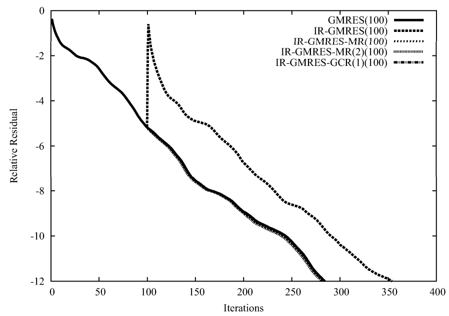

Figure 3: History of relative residual 2-norm of five types of GMRES($k = 100$) method for matrix poisson3Da.

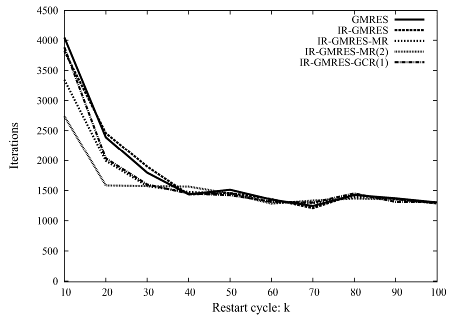

Figure 4: Variation of iteration counts of five types of GMRES ($k = 20$) method when restart cycle k varies as $k = 10, 20, \ldots, 100$ for matrix epb1.

IR-GMRES(k) methods when $k = 10, 20, \ldots, 100$ for matrix epb1. When the restart cycle k is more than 50, variation of convergence rate of GMRES(k) and IR-GMRES(k) methods is steady.

6 Concluding remarks

We showed that when the restart cycle k is small, convergence rate of IR-GMRES(k) methods are superior to that of the conventional GMRES(k) method. Moreover, we presented also that when the restart cycle k is large, convergence of IR-GMRES(k) methods is competitive as that of the conventional GMRES(k) method.

In conclusion, it must be noted that the proposed IR-GMRES(k)-MR(2) method outperforms well among some IR-GMRES(k) methods. Further investigations may be needed to give more effective initial approximated solution so as to minimize the residual 2-norm $||\boldsymbol{r}_k - A\boldsymbol{e}_0||_2$.

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