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A Model of Forming a Simple Path in a Level of a Complete K-ary Tree Maximizing Total Shortening Path Length

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Abstract—**This study proposes a model of adding edges of forming a simple path to a level of depth N in a complete K-ary ($K \geq 3$) tree of height H under giving priority to edges between two nodes of which the deepest common ancestor is deeper. An optimal depth N^* is obtained by maximizing the total shortening path length which is the sum of shortening lengths of shortest paths between every pair of all nodes in the complete K-ary tree.**

Keywords: complete K-ary tree, simple path, shortest path, organization structure

1 Introduction

The pyramid organization structure based on the principle of unity of command can be expressed as a rooted tree, if we let nodes and edges in the rooted tree correspond to members and relations between members in the organization respectively [2, 6]. Then the path between each node in the rooted tree is equivalent to the route of communication of information between each member in the organization. Moreover, adding edges to the rooted tree is equivalent to forming additional relations other than that between each superior and his subordinates.

The purpose of our study is to obtain an optimal set of additional relations to the pyramid organization such that the communication of information between every member in the organization becomes the most efficient. This means that we obtain a set of additional edges to the rooted tree minimizing the sum of lengths of shortest paths between every pair of all nodes.

We have obtained an optimal depth for each of the following three models of adding relations in a level to the organization structure which is a complete K-ary tree of height H: (i) a model of adding an edge between two nodes with the same depth, (ii) a model of adding edges between every pair of nodes with the same depth, and (iii) a model of adding edges between every pair of siblings with the same depth [3]. A complete K-ary tree is a rooted tree in which all leaves have the same depth and all internal nodes have $K (K = 2, 3, \ldots)$ children [1].

This study proposes a model of adding edges of forming a simple path to a level of depth $N (N = 1, 2, \ldots, H)$ in a complete K-ary ($K \geq 3$) tree of height $H (H = 1, 2, \ldots)$ under giving priority to edges between two nodes of which the deepest common ancestor is deeper as follows.

Step 1: Add new edges between nodes of which the depth of the deepest common ancestor is $N - 1$ so as to form simple paths of every child of each node of depth $N - 1$.

Step 2: Add new edges between nodes of which the depth of the deepest common ancestor is $N - 2$ so as to form simple paths of every descendant of each node of depth $N - 2$.

Step 3: Repeat Step 2 while the depth of the deepest common ancestor is $N - 3, N - 4, \ldots, 1, 0$.

If $l_{i,j} (= l_{j,i})$ denotes the path length, which is the number of edges in the shortest path from a node v_i to a node v_j $(i, j = 1, 2, \ldots, (K^{H+1} - 1)/(K - 1))$ in the complete K-ary tree of height H, then $\sum_{i<j} l_{i,j}$ is the total path length. Furthermore, if $l'_{i,j}$ denotes the path length from v_i to v_j after adding edges in this model, $l_{i,j} - l'_{i,j}$ is called the shortening path length between v_i and v_j, and $\sum_{i<j}(l_{i,j} - l'_{i,j})$ is called the *total shortening path length.* Minimizing the total path length is equivalent to maximizing the total shortening path length.

In Section 2 the total shortening path length is formulated when new edges are added to a level of depth N in a complete K-ary tree of height H as *Step 1–3.* In Section 3 an optimal depth N^* which maximizes the total shortening path length is obtained.

2 Formulation of Total Shortening Path Length

When we add a new edge between two nodes with depth N of which the depth of the deepest common ancestor is $N - i \, (i = 1, 2, \ldots, N)$, the sum of shortening path lengths can be formulated by summing up the following three equations:

$$A_{H,N}(i) = m(H - N)^2(2i - 1), \tag{1}$$

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$$\begin{aligned} & B_{H,N}(i) \\ = \ & 2m(H-N)\sum_{j=1}^{i-1}\{m(H-N+j-1)(K-1)+1\} \\ & \times(2i-2j-1), \end{aligned} \qquad (2)$$

and

$$\begin{aligned} C_{H,N}(i) &= \sum_{j=1}^{i-2}\{m(H-N+j-1)(K-1)+1\} \\ & \times \sum_{k=1}^{i-j-1}\{m(H-N+k-1)(K-1)+1\} \\ & \times(2i-2j-2k-1), \end{aligned} \qquad (3)$$

where $m(h)$ denotes the number of nodes of a complete K-ary tree of height $h\,(h=0,1,2,\ldots)$, and we define

$$\sum_{j=1}^{0}\cdot = 0\,, \qquad (4)$$

and

$$\sum_{j=1}^{-1}\cdot = 0\,. \qquad (5)$$

When one edge between two nodes of which the depth of the deepest common ancestor is $N-i$ in *Step 2–3* is added, the sum of shortening path lengths is obtained by adding $D_{H,N}(i)+E_{H,N}(i)$ to $A_{H,N}(i)+B_{H,N}(i)+C_{H,N}(i)$. $D_{H,N}(i)$ and $E_{H,N}(i)$ which are the sum of additional shortening path lengths of using former added edges are given by

$$D_{H,N}(i) = 2m(H-N)\sum_{j=1}^{i-2}m(H-N+j-1)+P_{H,N}(i), \qquad (6)$$

and

$$E_{H,N}(i) = \begin{cases} 0 & (i=1) \\ 3m(H-N)^2 & (i=2) \\ 2m(H-N)^2+Q_{H,N}(i) & (i\geq 3) \end{cases}, \qquad (7)$$

where

$$P_{H,N}(i) = \begin{cases} 0 & (i\leq 3) \\ m(H-N)^2(i-3) & (i\geq 4) \end{cases}, \qquad (8)$$

$$Q_{H,N}(i) = 2m(H-N)\,m(H-N+i-2), \qquad (9)$$

and Equations (4) and (5) apply.

From the above equations, the total shortening path length of this model $S_{H,N}$ is formulated by

$$\begin{aligned} S_{H,N} &= \sum_{i=1}^{N}(K-1)K^{N-i}\big\{A_{H,N}(i)+B_{H,N}(i) \\ & + C_{H,N}(i)+D_{H,N}(i)+E_{H,N}(i)\big\}. \end{aligned} \qquad (10)$$

3 An Optimal Adding Depth

Since the number of nodes of a complete K-ary tree of height h is

$$m(h) = \frac{K^{h+1}-1}{K-1}, \qquad (11)$$

$S_{H,N}$ of Equation (10) becomes the following.

When $N=1$, then we have

$$S_{H,1} = \frac{1}{K-1}\left(K^{2H}-2K^{H}+1\right). \qquad (12)$$

When $N=2$, then we have

$$\begin{aligned} S_{H,2} &= \frac{1}{K-1}\big\{(3K+4)K^{2H-2}-(4K+10)K^{H-1} \\ & + K+6\big\}. \end{aligned} \qquad (13)$$

When $N\geq 3$, then we have

$$\begin{aligned} S_{H,N} &= \frac{1}{(K-1)^3}\Big[(-NK+N+1)K^{2H-2N+2} \\ & + \frac{1}{2}\big\{(N^2+N)K^3+(-N^2+N-2)K^2 \\ & + (-N^2-N)K+N^2-N\big\}K^{2H-N} \\ & + (4NK-4N-2)K^{H-N+1} \\ & + \big\{-2NK^2+(-2N+2)K+4N\big\}K^{H} \\ & + (K+4)K^{N}-5NK-K+5N-4\Big]. \end{aligned} \qquad (14)$$

Let

$$\Delta S_{H,N} \equiv S_{H,N+1}-S_{H,N}, \qquad (15)$$

for $N=1,2,\ldots,H-1$, so that we heve the following results.

When $N=1$, then we have

$$\begin{aligned} \Delta S_{H,1} &= \frac{1}{K-1}\big\{(-K^2+3K+4)K^{2H-2} \\ & - (2K+10)K^{H-1}+K+5\big\}. \end{aligned} \qquad (16)$$

If $K=3$ and $H=2$, then

$$\Delta S_{H,1} \ < \ 0\,, \qquad (17)$$

if $K=3$ and $H\geq 3$, then

$$\Delta S_{H,1} \ > \ 0\,, \qquad (18)$$

and if $K\geq 4$, then

$$\Delta S_{H,1} \ < \ 0\,. \qquad (19)$$

When $N=2$, then we have

$$\begin{aligned} \Delta S_{H,2} &= \frac{1}{K-1}\big\{(-3K^3+2K^2+8K+4)K^{2H-4} \\ & - (2K^2+6K+14)K^{H-2}+K^2+5K+5\big\} \\ &< \ 0\,. \end{aligned} \qquad (20)$$

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When $N \geq 3$, then we have

$$\begin{aligned}\Delta S_{H,N} &= \frac{1}{(K-1)^3}\Big[\big\{NK^3-(N+1)K^2 \\ &\quad -(N+1)K+N+2)\big\}K^{2H-2N} \\ &\quad +\frac{1}{2}\{-(N^2+N)K^4+(2N^2+2N+4)K^3 \\ &\quad -2K^2-(2N^2+2N+2)K+N^2+N\} \\ &\quad \times K^{2H-N-1} \\ &\quad +\big\{-4NK^2+(8N+6)K-4N-6\big\}K^{H-N} \\ &\quad +\big(-2K^2-2K+4\big)K^H \\ &\quad +\big(K^2+3K-4\big)K^N-5K+5\Big] \\ &< 0. \end{aligned} \tag{21}$$

From the above results, the optimal adding depth N^* which maximizes $S_{H,N}$ can be obtained and is given in Theorem 3.1.

Theorem 3.1
(i) *If $K = 3$, then we have the following*:
(a) *If $H = 1, 2$, then the optimal adding depth is $N^* = 1$.*
(b) *If $H \geq 3$, then $N^* = 2$.*
(ii) *If $K \geq 4$, then $N^* = 1$.*

Proof.
(i) Assume $K = 3$.
(a) If $H = 1$, then $N^* = 1$ trivially. If $H = 2$, then $N^* = 1$ since $\Delta S_{2,1} < 0$.
(b) If $H \geq 3$, then $N^* = 2$ since $\Delta S_{H,1} > 0$ and $\Delta S_{H,N} < 0$ for $N \geq 2$.
(ii) Assume $K \geq 4$.
If $H = 1$, then $N^* = 1$ trivially. If $H \geq 2$, then $N^* = 1$ since $\Delta S_{H,N} < 0$. □

Table 1 shows the optimal adding depth N^* and the total shortening path lengths S_{H,N^*} in the case of $K = 3, 4, 5$ and $H = 1, 2, \ldots, 10$.

4 Conclusions

This study considered the addition of relations to a pyramid organization structure such that the communication of information between every member in the organization becomes the most efficient. For a model of adding edges of forming a simple path to a level of depth N in a complete K-ary ($K \geq 3$) tree of height H under giving priority to edges between two nodes of which the deepest common ancestor is deeper, we obtained an optimal depth N^* which maximizes the total shortening path length in Theorem 3.1. This result indicates the most efficient way of adding relations of forming a simple path in a level is to use the first level or the second level depending on the number of subordinates and the number of levels in the organization structure.

Table 1: Optimal adding depth N^*.

K	H	N^*	S_{H,N^*}
3	1	1	2
	2	1	32
	3	2	432
	4	2	4446
	5	2	41760
	6	2	381150
	7	2	3446352
	8	2	31065246
	9	2	279731520
	10	2	2518016670
4	1	1	3
	2	1	75
	3	1	1323
	4	1	21675
	5	1	348843
	6	1	5589675
	7	1	89467563
	8	1	1431612075
	9	1	22906317483
	10	1	366503176875
5	1	1	4
	2	1	144
	3	1	3844
	4	1	97344
	5	1	2439844
	6	1	61027344
	7	1	1525839844
	8	1	38146777344
	9	1	953673339844
	10	1	23841853027344

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