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Two Dimensional Green's Functions for the Piezoelectric Half-Space

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***Abstract*—A novel formulation for two-dimensional self-similar anisotropic elastodyamic problems is generalized to piezoelectric materials. By making use of the formulation, the general solution of the displacements is expressed in terms of the eigenvalues and eigenvectors of a related eight-dimensional eigenvalue problem. Without the need of performing integral transforms as required in the well-known Cagniard-de Hoop method, the present formulation can be utilized to obtain expressions of analytical solutions directly. In the study, the method is applied to derive the explicit dynamic Green's functions in the piezoelectric half-space. Numerical examples for the quartz of the semi-infinite region are illustrated.**

***Index Terms*—dynamic Green's functions, piezoelectric half-space.**

I. Introduction

Because of the intrinsic anisotropic elastic features of piezoelectric materials, many analytic methods for piezoelectric solids are derived from those for the general anisotropic elasticity. The Green's functions, which relate the mechanical displacements and electric potential at a point to the concentrated forces or charges applied at another point, play an important role in understanding analytically mechanical or electrical behavior of loaded piezoelectric materials.

Lothe and Barnett [1] developed an integral formalism for surface waves in piezoelectric half-infinite solid. They [2] also considered the existence of surface waves in piezoelectric half-space subjected to various boundary conditions. Taylor and Crampin [3] considered the circumstances for the propagation of surface waves in a homogeneous anisotropic piezoelectric half-space. Their study revealed that the particular form of anisotropic symmetry with respect to the direction of propagation critically affects the properties of the surface waves. Peach [4] extended the results of Lothe and Barnett [1] for the anisotropic materials to those for the piezoelectric materials. He presented general existence theorems for surface waves on piezoelectric substrates. Gao and Noda [5] developed an exact solution for the static Green's functions of a half-infinite piezoelectric solid. Their work showed that the normal component of the electric displacement on the solid surface is not zero and is dependent on the applied loads and the electro-elastic constants of the piezoelectric material and air.

The Stroh formalism is widely recognized as an elegant and powerful analytic method in two-dimensional general anisotropic elastostatics [6]-[9]. A distinctive feature of the Stroh formalism is that the general solution is provided in terms of the eigenvalues and eigenvectors of a constant six-dimensional matrix. The general solution contains three arbitrary complex functions. These functions can often be determined by virtue of the orthogonality relations among the eigenvectors in conjunction with theories of analytic functions. The Stroh's formalism has been applied to yield the static Green's functions for various configurations (Ting, [8]). Generalization of the Stroh's formalism to piezoelectric materials has been given by Ting [8], leading to an eigenvalue problem of a constant eight-dimensional matrix. Wu [9] extended the Stroh's formalism to treat the self-similar elastodynamic problems for general anisotropic elastic material. The formulation is also based on a six-dimensional matrix, which, however, is a function of position and time. A major advantage of the novel formulation of Wu [9] is that solutions can be derived directly without the need of performing integral transforms. The formulation of Wu has been further extended to piezoelectric materials in the context of the quasi-static approximation to derive the dynamic Green's functions for an infinite piezoelectric medium (Wu and Chen, [10]). In this paper the dynamic surface Green's functions for a general piezoelectric half-space is considered. The surface is assumed traction-free mechanically and insulating electrically.

II. Formulation

The formulation of Wu and Chen [10] for self-similar elastodynamic problems for general piezoelectric materials is outlined in this section. For a linear piezoelectric solid, the mechanical stress σ_{ij}, the mechanical displacement u_i, the electric displacement D_i and the electric potential ϕ are related by

$$\sigma_{ij} = C_{ijks}u_{k,s} + e_{sij}\phi_{,s} , \quad (1)$$

$$D_i = e_{iks}u_{k,s} - \varepsilon_{is}\phi_{,s} , \quad (2)$$

where a subscript comma denotes partial differentiation with

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respect to spatial coordinates, repeated indices imply summation from 1 to 3, C_{ijks} are the elastic stiffness, and e_{iks}, and ε_{is} are, respectively, the piezoelectric stress constants and permittivity constants. In the absence of body forces and free charges the balance laws under quasi-static approximation require

$$\sigma_{ij,j} = \rho \ddot{u}_i , \tag{3}$$

$$D_{i,i} = 0 , \tag{4}$$

where ρ is the density and an overhead dot designates derivative with respect to time t.

By virtue of letting $\phi = u_4$ and $D_i = \sigma_{4i}$, (1) and (2) can be expressed in terms of the generalized stress and generalized displacement as

$$\sigma_{Ij} = E_{IjKs} u_{K,s} , \tag{5}$$

where the upper case subscripts range from 1 to 4, lower case subscripts from 1 to 3 and generalized electric-mechanical constants E_{IjKs} are defined as

$$E_{IjKs} = \begin{cases} C_{ijks}, I,K = 1,2,3, \\ e_{sij}, I = 1,2,3, K = 4, \\ e_{iks}, I = 4, K = 1,2,3, \\ -\varepsilon_{is}, I = 4, K = 4. \end{cases}.$$

Equations (3) and (4) can also be combined as

$$\sigma_{Ij,j} = \rho \delta^*_{IK} \ddot{u}_K , \tag{6}$$

where $\delta^*_{IK} = \delta_{IK},\ I,K$=1,2,3 , δ_{IK} being the Kronecker's delta and $\delta^*_{IK} = 0,\ I,K = 4$. Substitution of (5) into (6) yields the governing equations in terms of the generalized displacement $\mathbf{u}$ as

$$E_{IjKs} u_{K,sj} = \rho \delta^*_{IK} \ddot{u}_K . \tag{7}$$

For two-dimensional problems in which the generalized displacement $\mathbf{u} = [u_1, u_2, u_3, \phi]^T$ are independent of x_3, (7) can be expressed as

$$\mathbf{Q}\mathbf{u}_{,11} + (\mathbf{R} + \mathbf{R}^T)\mathbf{u}_{,12} + \mathbf{T}\mathbf{u}_{,22} = \rho \hat{\mathbf{I}} \ddot{\mathbf{u}} , \tag{8}$$

where $\hat{\mathbf{I}}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{T}$ are 4×4 matrices given by

$$\hat{\mathbf{I}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix},\ \mathbf{Q} = \begin{bmatrix} \mathbf{Q}^E & \mathbf{e}_{11} \\ \mathbf{e}^T_{11} & -\varepsilon_{11} \end{bmatrix},\ \mathbf{R} = \begin{bmatrix} \mathbf{R}^E & \mathbf{e}_{21} \\ \mathbf{e}^T_{12} & -\varepsilon_{12} \end{bmatrix},$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}^E & \mathbf{e}_{22} \\ \mathbf{e}^T_{22} & -\varepsilon_{22} \end{bmatrix}, \tag{9}$$

in which $\mathbf{I}$ is the 3×3 identity matrix and the elements of 3×3 matrices $\mathbf{Q}^E$, $\mathbf{R}^E$, $\mathbf{T}^E$, and 3×1 matrices $\mathbf{e}_{ij}$ are

$$Q^E_{ik} = C_{i1k1},\ R^E_{ik} = C_{i1k2},\ T^E_{ik} = C_{i2k2},\ (\mathbf{e}_{ij})_s = e_{ijs} .$$

Consider the generalized displacement $\mathbf{u}$ in the following form

$$\mathbf{u}(x_1, x_2, t) = \tilde{\mathbf{u}}(w) + \psi(t)\mathbf{e}_4 , \tag{10}$$

where $w(x_1, x_2, t)$ is defined implicitly by

$$\Delta(w, x_1, x_2, t) = wt - x_1 - p(w)x_2 = 0 , \tag{11}$$

with $p(w)$ as an analytic function of w, $\psi(t)$ a function of t, and $\mathbf{e}_4 = [0, 0, 0, 1]^T$. With (10), (8) becomes

$$\frac{1}{\Delta'}\frac{\partial}{\partial w}\left\{[\mathbf{Q} - \rho w^2 \hat{\mathbf{I}} + p(w)(\mathbf{R} + \mathbf{R}^T) + p(w)^2\mathbf{T}]\frac{1}{\Delta'}\tilde{\mathbf{u}}'(w)\right\} = \mathbf{0} , \tag{12}$$

where $\tilde{\mathbf{u}}'(w)$ denotes the derivative of $\tilde{\mathbf{u}}(w)$ with respect to w, Δ' is given by

$$\Delta' = \frac{\partial \Delta(w, x_1, x_2, t)}{\partial w} = t - p'(w)x_2 , \tag{13}$$

and $p'(w)$ is the derivative of $p(w)$ with respect to w. Equation (12) shows that for the generalized displacement $\mathbf{u}$ given by (10) to be a solution to (8), $\tilde{\mathbf{u}}(w)$ must satisfy (12) and $\psi(t)$ is arbitrary.

By letting $\tilde{\mathbf{u}}'(w)$ have the following form

$$\tilde{\mathbf{u}}'(w) = f(w)\mathbf{a}(w) , \tag{14}$$

where $f(w)$ is an arbitrary scalar function of w. It follows that $\mathbf{u}$ is a solution of (8) if

$$\mathbf{D}(p, w)\mathbf{a}(w) = \mathbf{0}, \tag{15}$$

where $\mathbf{D}(p, w)$ is given by

$$\mathbf{D}(p, w) = \mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T} - \rho w^2 \hat{\mathbf{I}} . \tag{16}$$

For non-trivial solutions of $\mathbf{a}(w)$ we must have

$$|\mathbf{D}(p, w)| = 0, \tag{17}$$

where $|\mathbf{D}|$ is the determinant of $\mathbf{D}$. Equation (17) provides eight eigenvalues of p as a function of w, denoted by $p_\alpha(w)$, $\alpha = 1, 2, ..., 8$. The corresponding function $w_\alpha = w_\alpha(y_1, y_2)$, where $y_1 = x_1 / t$, and $y_2 = x_2 / t$, can be determined from (11) with $p(w)$ replaced by $p_\alpha(w)$.

A graphical way for finding real $p's$ can be achieved by making use of the slowness surface, (s_1, s_2) space, where $s_1 = 1/w$ and $s_2 = p/w$. No real p_α exists for $t \to \infty$ or $w \to 0$. In this case p_α appear in four complex conjugated pairs. On the other hand as $t \to 0$ or $w \to \infty$, there are six real p_α. From (16) the other two complex roots and the corresponding $\mathbf{a}^*$ may be shown to be

$$p^* = \frac{-\varepsilon_{12} + i\varepsilon}{\varepsilon_{22}},\quad \overline{p}^*,\ \mathbf{a}^* = \mathbf{e}_4 , \tag{18}$$

where $\varepsilon = \sqrt{\varepsilon_{11}\varepsilon_{22} - \varepsilon^2_{12}}$ and $i = \sqrt{-1}$.

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The general solution of the generalized displacement satisfying (8) may be represented as

$$\mathbf{u}(x_1,x_2,t)_{,1}=\sum_{\alpha=1}^{8}\frac{f_\alpha(w_\alpha)}{\Delta'_\alpha}\mathbf{a}_\alpha(w_\alpha)\,, \tag{19}$$

$$\mathbf{u}(x_1,x_2,t)_{,2}=\sum_{\alpha=1}^{8}\frac{p_\alpha(w_\alpha)}{\Delta'_\alpha}f_\alpha(w_\alpha)\mathbf{a}_\alpha(w_\alpha)\,, \tag{20}$$

$$\dot{\mathbf{u}}(x_1,x_2,t)=-\sum_{\alpha=1}^{8}\frac{w_\alpha}{\Delta'_\alpha}f_\alpha(w_\alpha)\mathbf{a}_\alpha(w_\alpha)+\dot{\psi}(t)\mathbf{e}_4\,. \tag{21}$$

By substituting (19) and (20) into the constitutive laws, the general solutions of the generalized stress vectors $\mathbf{t}_1$ and $\mathbf{t}_2$, where $\mathbf{t}_1$ and $\mathbf{t}_2$ are given by $\mathbf{t}_1=(\sigma_{11},\sigma_{21},\sigma_{31},D_1)^T$ and $\mathbf{t}_2=(\sigma_{12},\sigma_{22},\sigma_{32},D_2)^T$, can be expressed, respectively, as

$$\mathbf{t}_1(x_1,x_2,t)=\sum_{\alpha=1}^{8}\frac{1}{\Delta'_\alpha}\left[\rho w_\alpha^2\hat{\mathbf{I}}\mathbf{a}_\alpha(w_\alpha)-p_\alpha(w_\alpha)\mathbf{b}_\alpha(w_\alpha)\right]f_\alpha(w_\alpha)\,, \tag{22}$$

$$\mathbf{t}_2(x_1,x_2,t)=\sum_{\alpha=1}^{8}\frac{f_\alpha(w_\alpha)}{\Delta'_\alpha}\mathbf{b}_\alpha(w_\alpha)\,, \tag{23}$$

where

$$\mathbf{b}_\alpha(w)=(\mathbf{R}^T+p_\alpha(w)\mathbf{T})\mathbf{a}_\alpha(w)=-\frac{1}{p}\left(\mathbf{Q}-\rho w^2\hat{\mathbf{I}}+p_\alpha(w)\mathbf{R}\right)\mathbf{a}_\alpha(w)\,, \tag{24}$$

The second identity in (24) follows from (15).

An alternative method for determining $p_\alpha(y_1,y_2)$ is given by substituting (11) into (16) and rewrite $\mathbf{D}$ as

$$\mathbf{D}(p,y_1,y_2)=\hat{\mathbf{Q}}+p(\hat{\mathbf{R}}+\hat{\mathbf{R}}^T)+p^2\hat{\mathbf{T}}\,, \tag{25}$$

where

$$\hat{\mathbf{Q}}=\begin{bmatrix}\mathbf{Q}^E-\rho y_1^2\mathbf{I} & \mathbf{e}_{11}\\ \mathbf{e}_{11}^T & -\varepsilon_{11}\end{bmatrix},\ \hat{\mathbf{R}}=\begin{bmatrix}\mathbf{R}^E-\rho y_1y_2\mathbf{I} & \mathbf{e}_{21}\\ \mathbf{e}_{12}^T & -\varepsilon_{12}\end{bmatrix},$$
$$\hat{\mathbf{T}}=\begin{bmatrix}\mathbf{T}^E-\rho y_2^2\mathbf{I} & \mathbf{e}_{22}\\ \mathbf{e}_{22}^T & -\varepsilon_{22}\end{bmatrix}.$$

The function $p_\alpha(y_1,y_2)$ can be directly obtained by $|\mathbf{D}(p,y_1,y_2)|=0$. The corresponding $w_\alpha(y_1,y_2)$ is simply given by (11) and the associated eigenvector $\mathbf{a}_\alpha(y_1,y_2)$ is determined by (15). Introduce the vector $\hat{\mathbf{b}}_\alpha(y_1,y_2)$ given by

$$\hat{\mathbf{b}}_\alpha(y_1,y_2)=(\hat{\mathbf{R}}^T+p_\alpha\hat{\mathbf{T}})\mathbf{a}_\alpha=-\frac{1}{p_\alpha}(\hat{\mathbf{Q}}+p_\alpha\hat{\mathbf{R}})\mathbf{a}_\alpha\,, \tag{26}$$

The second line of (26) follows from (15). The vector $\hat{\mathbf{b}}_\alpha(y_1,y_2)$ is related to $\mathbf{b}_\alpha(w)$ by

$$\hat{\mathbf{b}}_\alpha(y_1,y_2)=\mathbf{b}_\alpha(w)-\rho wy_2\hat{\mathbf{I}}\mathbf{a}_\alpha(w)\,. \tag{27}$$

Equation (26) can be cast into the following eight-dimensional eigenvalue problem

$$\hat{\mathbf{N}}\hat{\xi}=p\hat{\xi}\,, \tag{28}$$

where

$$\hat{\mathbf{N}}=\begin{pmatrix}\hat{\mathbf{N}}_1 & \hat{\mathbf{N}}_2\\ \hat{\mathbf{N}}_3 & \hat{\mathbf{N}}_1^T\end{pmatrix},\ \hat{\xi}=\begin{pmatrix}\mathbf{a}\\ \hat{\mathbf{b}}\end{pmatrix},\ \hat{\mathbf{N}}_1=-\hat{\mathbf{T}}^{-1}\hat{\mathbf{R}}^T,\ \hat{\mathbf{N}}_2=\hat{\mathbf{T}}^{-1},$$
$$\hat{\mathbf{N}}_3=\hat{\mathbf{R}}\hat{\mathbf{T}}^{-1}\hat{\mathbf{R}}^T-\hat{\mathbf{Q}}\,.$$

The p and $\hat{\xi}$ are the eigenvalue and right eigenvector, respectively, of $\hat{\mathbf{N}}$. Since $\hat{\mathbf{N}}_2$ and $\hat{\mathbf{N}}_3$ are symmetric, the left eigenvector, $\hat{\eta}$, of $\hat{\mathbf{N}}$ defined by

$$\hat{\mathbf{N}}^T\hat{\eta}=p\hat{\eta}\,, \tag{29}$$

is given by

$$\hat{\eta}=\begin{pmatrix}\hat{\mathbf{b}}\\ \mathbf{a}\end{pmatrix}.$$

If the eigenvalues p_α and p_β are distinct, the corresponding left and right eigenvectors satisfy orthogonality relations

$$\hat{\eta}_\alpha^T\hat{\xi}_\beta=\mathbf{a}_\alpha^T\hat{\mathbf{b}}_\beta+\hat{\mathbf{b}}_\alpha^T\mathbf{a}_\beta=0,\ \alpha\neq\beta\,. \tag{30}$$

III. Dynamic Surface Green's Function

Consider a piezoelectric half-space $x_2\geq 0$. The surface at $x_2=0$ is assumed traction-free mechanically and insulating electrically. A line impulse force $\mathbf{h}\delta(t)$ and a line impulse charge $q\delta(t)$, $\delta(t)$ being the Dirac delta function, are applied at the origin. The surface is mechanically traction-free ($\mathbf{t}_2^E=\mathbf{0}$) and electrically insulating ($D_2=0$), where $\mathbf{t}_2^E=(\sigma_{12}\quad\sigma_{22}\quad\sigma_{32})^T$. Then the corresponding conditions on the boundary $x_2=0$ are given by

$$\mathbf{t}_2(x_1,t)=-\delta(x_1)\delta(t)\mathbf{F}\,, \tag{31}$$

where $\mathbf{F}=(h_1,h_2,h_3,-q)^T$.

As in the case of unbounded media (Wu and Chen, [10]), the generalized stresses are homogeneous of degree -2 and the generalized displacement $\mathbf{u}$ homogeneous of degree -1. Thus the fictitious generalized displacement $\mathbf{u}^*$ given by

$$\mathbf{u}^*(x_1,x_2,t)=\int_{-\infty}^{t}\mathbf{u}(x_1,x_2,\tau)d\tau\,, \tag{32}$$

is homogeneous of degree 0. The conditions for the corresponding fictitious generalized stress $\mathbf{t}_2^*$ is

$$\mathbf{t}_2^*(x_1,t)=\int_{-\infty}^{t}\mathbf{t}_2(x_1,\tau)d\tau=-\delta(x_1)H(t)\mathbf{F}\,. \tag{33}$$

The general expression for the generalized stress vector $\mathbf{t}_2^*$ as given by (23) can be rewritten in the following matrix form:

$$\mathbf{t}_2^*(x_1,x_2,t)=2\,\mathrm{Re}\left(\mathbf{B}(\mathbf{w})\left\langle\frac{1}{\Delta'}\right\rangle\mathbf{f}(\mathbf{w})\right), \tag{34}$$

where $\mathbf{B}(\mathbf{w})=\left[\mathbf{b}_1(w_1)\quad\mathbf{b}_2(w_2)\quad\mathbf{b}_3(w_3)\quad\mathbf{b}_4(w_4)\right]$,

$$\left\langle \frac{1}{\Delta'} \right\rangle = diag\left[\frac{1}{\Delta'_1} \quad \frac{1}{\Delta'_2} \quad \frac{1}{\Delta'_3} \quad \frac{1}{\Delta'_4} \right] ,$$

$\mathbf{f}(\mathbf{w}) = \left[f_1(w_1) \quad f_2(w_2) \quad f_3(w_3) \quad f_4(w_4) \right]^T$. For $t > 0$, (33) and (34) yield

$$2\,\mathrm{Re}\big(\mathbf{q}(y_1)\big) = -\delta(y_1)\mathbf{F} , \tag{35}$$

where

$$\mathbf{q}(y_1) = \mathbf{B}(y_1)\mathbf{f}(y_1) . \tag{36}$$

The analytic function $\mathbf{q}(\eta)$ with $\eta = y_1 + iy_2$ satisfying (35) is given by

$$\mathbf{q}(\eta) = \frac{1}{2\pi i \eta}\mathbf{F} . \tag{37}$$

Therefore,

$$\mathbf{f}(w_\alpha) = \frac{1}{2\pi i w_\alpha}\mathbf{B}^{-1}(w_\alpha)\mathbf{F} . \tag{38}$$

Let $\mathbf{e}_\alpha$ be the unit vector in α -direction and the matrix $\mathbf{I}_\alpha = \mathbf{e}_\alpha \mathbf{e}_\alpha^T$. The analytic function $\mathbf{f}(\mathbf{w})$ is obtained as

$$\mathbf{f}(\mathbf{w}) = \frac{1}{2\pi i}\left\langle \frac{1}{w} \right\rangle \sum_{\alpha=1}^{4} \mathbf{I}_\alpha \mathbf{B}^{-1}(w_\alpha)\mathbf{F} , \tag{39}$$

where the 4×4 matrix $\mathbf{B}(w_\alpha) = \left[\mathbf{b}_1(w_\alpha) \quad \mathbf{b}_2(w_\alpha) \quad \mathbf{b}_3(w_\alpha) \quad \mathbf{b}_4(w_\alpha) \right]$, the vector $\mathbf{b}_\beta(w_\alpha) = \left(\mathbf{R}^T + p_\beta(w_\alpha)\mathbf{T} \right)\mathbf{a}_\beta(w_\alpha)$, $p_\beta(w_\alpha)$ and $\mathbf{a}_\beta(w_\alpha)$ are the eigenvalues and eigenvectors, respectively, of $\hat{\mathbf{N}}\hat{\xi} = p\hat{\xi}$ with $w = w_\alpha$. Equation (39) can also be expressed as the following form:

$$f_k(w_k) = \frac{1}{2\pi i w_k}\mathbf{e}_k^T \left(\sum_{\alpha=1}^{4} \mathbf{I}_\alpha \mathbf{B}^{-1}(w_\alpha) \right)\mathbf{F} = \frac{1}{2\pi i w_k}\mathbf{e}_k^T \mathbf{B}^{-1}(w_k)\mathbf{F} \cdot \tag{40}$$

The fictitious generalized velocity $\dot{\mathbf{u}}^*$ is given by

$$\begin{aligned} \dot{\mathbf{u}}^*(x_1, x_2, t) &= -\frac{1}{\pi}\mathrm{Im}\left\{ \sum_{k=1}^{4} \frac{1}{\Delta'_k}\mathbf{a}_k(w_k)\mathbf{e}_k^T \left(\sum_{\alpha=1}^{4} \mathbf{I}_\alpha \mathbf{B}^{-1}(w_\alpha) \right) \right\}\mathbf{F} + \dot{\psi}^*(t)\mathbf{e}_4 , \\ &= -\frac{1}{\pi}\mathrm{Im}\left\{ \sum_{k=1}^{4} \frac{1}{\Delta'_k}\mathbf{a}_k(w_k)\mathbf{e}_k^T \mathbf{B}^{-1}(w_k) \right\}\mathbf{F} + \dot{\psi}^*(t)\mathbf{e}_4 \end{aligned} \tag{41}$$

The function $\dot{\psi}^*(t)$ is determined by requiring $\dot{\mathbf{u}}^* \to 0$ as $t \to 0^+$ in (41). The result is

$$\dot{\mathbf{u}}^*(x_1, x_2, t) = \left[-\frac{F_4}{\pi t \varepsilon} + \dot{\psi}^*(t) \right]\mathbf{e}_4 , \tag{42}$$

or

$$\dot{\phi}^*(t) = -\frac{F_4}{\pi t \varepsilon} + \dot{\psi}^*(t) , \tag{43}$$

where $w^* = y_1 + p^* y_2$.

If $\dot{\phi}^*(t)$ is required to be bounded at $t = 0$, the function $\dot{\psi}^*(t)$ must be in the following form

$$\dot{\psi}^*(t) = \frac{F_4}{\pi t \varepsilon} + c(t) , \tag{44}$$

where $c(t)$ is a regular function of t. Since only the spatial variation of the electric potential $\dot{\phi}^*(t)$ is of interest, we can let $c(t) = 0$ (Wu and Chen, [10]). The actual generalized displacement $\mathbf{u}$, which is the same as the fictitious generalized velocity $\dot{\mathbf{u}}^*$, is obtained as

$$\mathbf{u}(x_1, x_2, t) = \dot{\mathbf{u}}^*(x_1, x_2, t) = \mathbf{G}_{sf}^+(x_1, x_2, t)\mathbf{F} , \tag{45}$$

where $\mathbf{G}_{sf}^+$ is the free surface Green's tensor for $t > 0$ and can be expressed as

$$\begin{aligned} \mathbf{G}_{sf}^+(x_1, x_2, t) &= -\frac{1}{\pi}\mathrm{Im}\left\{ \sum_{k=1}^{n_s} \frac{1}{\Delta'_k}\mathbf{a}_k(w_k)\left[\mathbf{e}_k^T \left(\sum_{\alpha=1}^{4} \mathbf{I}_\alpha \mathbf{B}^{-1}(w_\alpha) \right) \right] \right\} + \frac{1}{\pi t \varepsilon}\mathbf{e}_4\mathbf{e}_4^T . \\ &= -\frac{1}{\pi}\mathrm{Im}\left\{ \sum_{k=1}^{n_s} \frac{1}{\Delta'_k}\mathbf{a}_k(w_k)\mathbf{e}_k^T \mathbf{B}^{-1}(w_k) \right\} + \frac{1}{\pi t \varepsilon}\mathbf{I}_4 \end{aligned} \tag{46}$$

Since as $t \to 0^+$, the fictitious generalized displacement is

$$\mathbf{u}^*(x_1, x_2, t) = \frac{F_4}{\pi \varepsilon}\mathrm{Re}\left\{ \log(x_1 + p^* x_2) \right\}\mathbf{e}_4 , \tag{47}$$

while $\mathbf{u}^*(x_1, x_2, t) = \mathbf{0}$ as $t \to 0^-$. The Green's function $\mathbf{G}_{sf}(x_1, x_2, t)$ for $t > 0^-$ is given by

$$\mathbf{G}_{sf}(x_1, x_2, t) = \mathbf{G}_{sf}^+(x_1, x_2, t) + \frac{\delta(t)}{\pi \varepsilon}\mathrm{Re}\left\{ \log(x_1 + p^* x_2) \right\}\mathbf{I}_4 . \tag{48}$$

IV. Numerical Examples

The Green's functions given by (48) were computed next for quartz, which is a crystal of trigonal 32 symmetry class. The Green's functions may be expressed in the following dimensionless form:

$$\bar{G}_{ij}(\psi, \tau) = \begin{cases} (\pi C_0 r / c_0) G_{ij}(x_1, x_2, t), & i, j = 1, 2, 3, \\ (\pi e_0 r / c_0) G_{ij}(x_1, x_2, t), & i = 4, j = 1, 2, 3 \text{ or } i = 1, 2, 3 , j = 4 \\ (\pi \varepsilon_0 r / c_0) G_{ij}(x_1, x_2, t), & i = 4, j = 4 \end{cases} , \tag{49}$$

where $c_0 = \sqrt{C_0 / \rho}$, $\tau = t c_0 / r$, $r = \sqrt{x_1^2 + x_2^2}$, $\psi = \tan^{-1}(x_2 / x_1)$. Here C_0, e_0 and $\varepsilon_0 = e_0^2 / C_0$, respectively, are certain reference elastic constant, piezoelectric stress constant and permittivity. The elastic stiffness constants $\mathbf{C}$, the piezoelectric stress constants $\mathbf{e}$, and dielectric constants $\boldsymbol{\varepsilon}$ of quartz used for calculations were [11]:

$$\mathbf{C} = \begin{bmatrix} 86.74 & 6.97 & 11.9 & -17.91 & 0 & 0 \\ 6.97 & 86.74 & 11.9 & 17.91 & 0 & 0 \\ 11.9 & 11.9 & 107.2 & 0 & 0 & 0 \\ -17.91 & 17.91 & 0 & 57.93 & 0 & 0 \\ 0 & 0 & 0 & 0 & 57.93 & -17.91 \\ 0 & 0 & 0 & 0 & -17.91 & 39.885 \end{bmatrix} \mathrm{GPa} , \tag{50}$$

$$\mathbf{e}=\begin{bmatrix} -0.171 & 0.171 & 0 & 0.0406 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0406 & 0.171 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ C/m}^2, \quad (51)$$

and

$$\varepsilon=\begin{bmatrix} 39.21 & 0 & 0 \\ 0 & 39.21 & 0 \\ 0 & 0 & 41.03 \end{bmatrix}\times 10^{-12} \text{ Farads/m}. \quad (52)$$

Figure 1 displays the wave surface of quartz in the infinite region. The three bulk wavefronts are denoted by L, FT, and ST. Some head wavefronts are designated as H_i ($i=1\sim 5$). Figure 2 shows the components of Green's functions, $(\bar{G})_{11}$, $(\bar{G})_{12}$, and $(\bar{G})_{13}$ for the observational angle $\psi=0^\circ$. In figure 2, the surface wave is denoted by SAW and the pseudo-surface wave is denoted by PSAW. Figure 3 shows the components of Green's functions ($(\bar{G})_{1j}$, $j=1,2,3$) for the observational angle $\psi=36^\circ$.

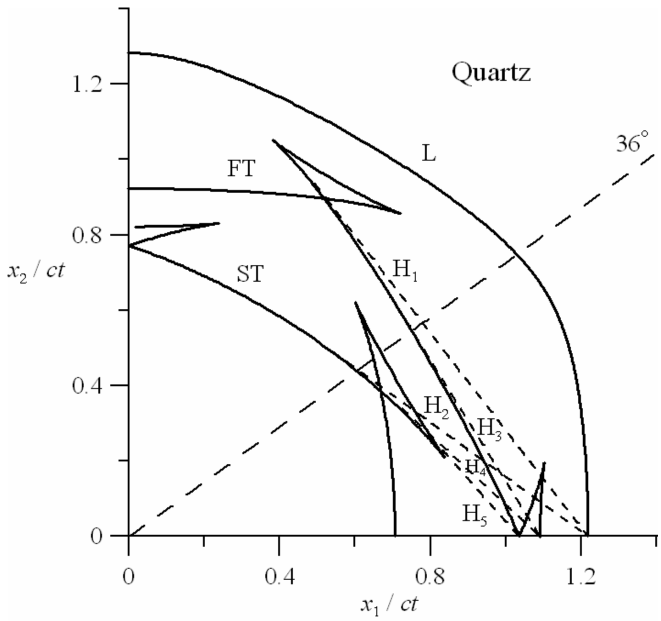

Figure 1. Wavefronts and the angle of observation for quartz.

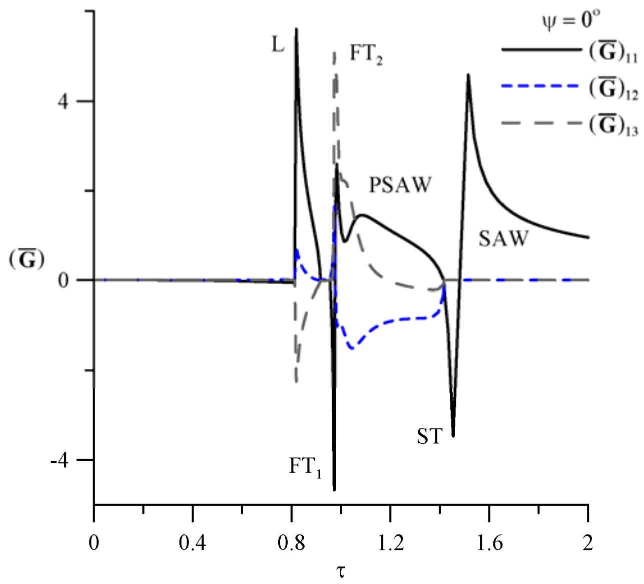

Figure 2. The components of Green's functions $(\bar{\mathbf{G}})_{11}$, $(\bar{\mathbf{G}})_{12}$, and $(\bar{\mathbf{G}})_{13}$ for $\psi=0^\circ$.

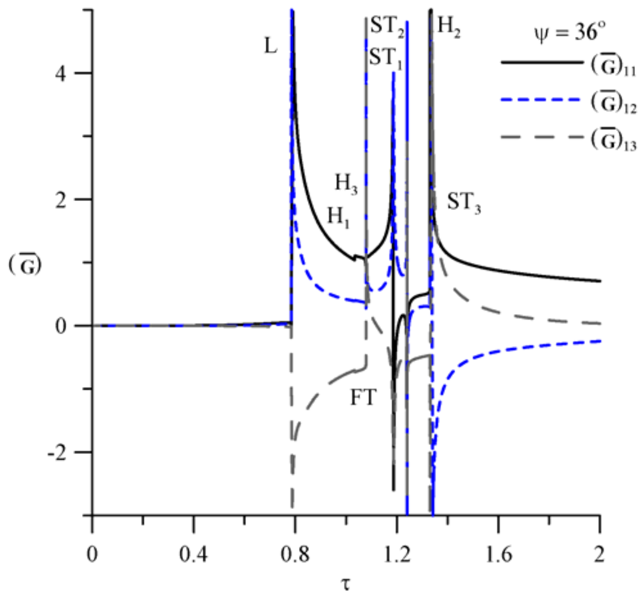

Figure 3. The components of Green's functions $(\bar{\mathbf{G}})_{11}$, $(\bar{\mathbf{G}})_{12}$, and $(\bar{\mathbf{G}})_{13}$ for $\psi=36^\circ$.

V. Concluding Remarks

A novel formulation developed by Wu [9] for two-dimensional anisotropic elastodynamics is extended to treat general piezoelectric materials. The present formulation does not require integral transforms and can be used to acquire the general solutions of displacement or stress fields in the time domain directly. The formulation is applied to derive analytic expressions for dynamic Green's functions of general half-space piezoelectric solids. The Green's functions can be simply calculated using the eigenvalues and eigenvectors of a related eight by eight matrix. Numerical

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examples provided for the piezoelectric material-quartz show that the dynamic responses can be accurately computed by the proposed formulation.

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REFERENCES

[1] J. Lothe, and D. M. Barnett, "On the existence of surface-wave solutions for anisotropic half-spaces with free surface." *J. Appl. Phys.* 47, pp. 428-433, 1976.

[2] J. Lothe, and D. M. Barnett, "Integral formalism for surface waves in piezoelectric crystals. Existence considerations." *J. Appl. Phys.* 47, pp. 1799-1807, 1976.

[3] D. B. Taylor, and S. Crampin, "Surface waves in anisotropic media: propagation in a homogeneous piezoelectric halfspace." *Proc. R. Soc. Lond. A* 364, pp. 161-179, 1978.

[4] R. Peach, "On the existence of surface acoustic waves on piezoelectric substrates." *IEEE Trans. Ultrason. Ferroelect. Freq. Contr.*. Vol 48, No. 5, pp. 1308-1320, 2001.

[5] C. F. Gao, and N. Noda, "Green's functions of a half-infinite piezoelectric body: exact solutions." *Acta Mechanica* 172, pp. 169-179, 2004.

[6] A. N. Stroh, "Dislocations and cracks in anisotropic elasticity," *Phil. Mag.* 3, pp. 625-646, 1958.

[7] A. N. Stroh, "Steady state problems in anisotropic elasticity," *J. Math. Phys.* 41, pp. 77-103, 1962.

[8] T. C. T. Ting, "*Anisotropic elasticity: theory and application*." Oxford University Press, 1996.

[9] K.-C. Wu, "Extension of Stroh's formalism to self-similar problems in two-dimensional elastodynamics." *Proc. R. Soc. Lond. A* 456, pp. 869-890, 2000.

[10] K.-C. Wu, and S.-H. Chen, "Two dimensional dynamic Green's functions for piezoelectric materials," CMES-*Computer modeling in engineering & sciences*. Vol. 20, No 3, pp. 147-156, 2007.

[11] B. A. Auld, "*Acoustic fields and waves in solids*." John Wiley and Sons, 1973.