Weakening the Singularities when Applying the BEM for 2D Compressible Fluid Flow

Ion Vladimirescu, Luminita Grecu

***Abstract* — One of the most important and difficult step in applying the BEM consists in the evaluation of integrals of singular kernels. We need to evaluate such integrals because the problem to solve is usually reduced to a singular boundary integral equation. The accuracy of the BEM and the well conditioning of the problem depend on the technique used to overpass this difficulty. This paper presents an application of a technique which can be successfully applied for the treatment of the singularities, namely the regularization technique, which implies the weakening of the singularities by using series expansions. For showing the efficiency of this technique we applied it for the treatment of the singularities that arise when the BEM is used to solve the problem of the 2D compressible fluid flow around an obstacle. The singular boundary integral equation is solved using quadratic isoparametric boundary elements. In order to test the method proposed, a computer code is developed, by using MathCAD programming. The results obtained show the great accuracy of this technique.**

***Index Terms*—boundary element method, singular boundary integral equation, regularization method, integrals of singular kernels**

I. INTRODUCTION

As in case of many numerical techniques used to solve problems of partial differential equations the Boundary Element Method (BEM) reduces the problem to a system of linear equations. First, by using a direct or an indirect technique, the problem is reduced to a singular boundary integral equation. In order to solve this singular boundary integral equation different discretization schemes can be used. Finally the problem is reduced to a system of linear equations. Its coefficients are regular or singular integrals.

The diagonal coefficients of the system arise from singular integrals and, in order to obtain a well conditioned problem, a great attention must be given to their evaluation. The analytical calculus is preferred but it is not always possible. So for evaluating the coefficients numerical methods must be used. Different numerical techniques can be applied for the evaluation of the non singular integrals, and usually software applications can deal with such calculus.

For the singular ones special treatments are necessary. The accuracy of the numerical solution depends on the techniques used to evaluate the integrals of singular kernels. In the scientific literature different techniques are presented [1], [2], [3], [9].

In this paper we apply the regularization method to treat the singularities that arise when considering the BEM to solve the problem of the 2D compressible fluid flow around obstacles.

In the herein paper we consider the same problem but for the numerical evaluation of the singular coefficients a method based on wakening the singularities is proposed. For solving the singular boundary integral higher order boundary elements, namely quadratic isoparametric boundary elements are used.

A computer code in MathCAD solves the problem for a particular case, when the exact solution is known, and this solution is in good agreement with the numerical one.

In paper [11] another method for the treatment of the singularities, based on the definition of the Cauchy Principal Value of an integral is proposed, and the numerical results show its efficiency. A comparison between the two techniques can also be done.

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I. Vladimirescu is with the University of Craiova, Faculty of Mathematics.

Luminita Grecu is with the University of Craiova, Faculty of Engineering and Management of Technological Systems Dr. Tr. Severin, (phone:+40252333431; fax: +40252-317219; e-mail: lumigrecu@ hotmail.com).

II. THE DISCRETE FORM OF THE BOUNDARY INTEGRAL EQUATION FOR THE 2D COMPRESSIBLE FLUID FLOW

The indirect BEM, which uses fundamental solutions of source type, applied to the problem of the 2D compressible fluid flow, offers the following equivalent boundary formulation for the mathematical model of the problem (see [4]):

$$\left(n_x^{0^2}+\beta^2 n_y^{0^2}\right)f(\bar{x}_0)+ \\ +\frac{1}{\pi}\int_C^{\prime} f(\bar{x})\frac{(x-x_0)n_x^0+\beta^2(y-y_0)n_y^0}{\left|\bar{x}-\bar{x}_0\right|^2}ds=2\beta n_x^0 , \quad (1)$$

where n_x^0, n_y^0 are the components of the normal unit vector outward the fluid (inward the body) in the point $\bar{x}_0$, $\beta=\sqrt{1-M^2}$ (for the subsonic flow, M= Mach number), and f is the unknown function, the intensity of the sources, presumed to satisfy a hölder condition. The sign " ' " denotes the Cauchy principal value of the integral (see [5]).

The above singular boundary integral equation, obtained in terms of primary variables of the problem is deduced in paper [4]. This singular boundary integral equation is solved by using different boundary elements: constant, linear and quadratic boundary elements, in papers [6], [7], [8]. In all cases the problem is reduced to linear systems of equations,

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the unknowns being the nodal values of the functions to be found. For solving them the evaluation of the coefficients involved, is necessary.

In this paper we consider the singular integrals that arise in case of using quadratic isoparametric boundary elements to solve the singular boundary equation (1). So it is necessary to show what they represent and how do they arise. That's why we make a short presentation of how to solve the singular boundary equation (1) by using quadratic boundary elements

Using quadratic isoparametric boundary elements to solve the singular boundary integral equation (1), we divide the boundary into N one-dimensional quadratic boundary elements, noted L_i , $i=\overline{1,N}$ each of them with three nodes: two extreme nodes and an interior one.

Considering that the discrete equation is satisfied in every node, we obtain:

$$\left(n_x^{j\,2}+\beta^2 n_y^{j\,2}\right)f\left(\bar{x}_j\right)+ +\frac{1}{\pi}\sum_{i=1}^{N}{}' \int_{L_i} f(\bar{x})\frac{\left(x-x_j\right)n_x^j+\beta^2\left(y-y_j\right)n_y^j}{\left\|\bar{x}-\bar{x}_j\right\|^2}ds=2\beta n_x^j \tag{2}$$

When using isoparametric boundary elements the geometry and the behavior of the unknown f , on a boundary element, are described using a quadratic model, with the same set of basic functions, noted N_1, N_2, N_3 . Using the intrinsic system of coordinates we have:

$$\left(n_x^{j2}+\beta^2 n_y^{j2}\right)f\left(\bar{x}_j\right)+\frac{1}{\pi}\sum_{i=1}^{N}\left(\sum_{l=1}^{3}a_{ij}^l f_l^i\right)=2\beta n_x^j \tag{3}$$

where

$$a_{ij}^l=\int_{-1}^{1}N_l\frac{\left([N]\{x^i\}-x_j\right)n_x^j+\beta^2\left([N]\{y^i\}-y_j\right)n_y^j}{\left\|[N]\{\bar{x}\}-\bar{x}_j\right\|^2}J(\xi)d\xi \tag{4}$$

$[N]=\left(N_1\ N_2\ N_3\right)$ a line matrix with,

$$N_1(\xi)=\frac{\xi(\xi-1)}{2},\quad N_2(\xi)=1-\xi^2,$$
$$N_3(\xi)=\frac{\xi(\xi+1)}{2},\quad \xi\in[-1,1],$$

$\{x^i\},\{y^i\}$ column matrices containing the global coordinates of the nodes (for L_i),

$f_l^i, l=\overline{1,3}, i=\overline{1,N}$ is the value for the node l of element L_i.

After evaluating the coefficients and returning to the global system of notation, we obtain the following linear algebraic system:

$$[A]\{f\}=\{B\},\ A\in M_{2N}(R),\ \{f\},\{B\}\in R^{2N}$$
$$B_j=2\pi\beta n_x^j. \tag{5}$$

$\{f\}$ being the column matrix to find, containing the nodal values of the unknown function

After solving this system, we find the nodal values of the intensity, and then the components of the other fields of interest.

III. The Regularization Method Applied for Obtaining the Expressions of the Singular Coefficients

For getting the matrix $[A]$ we need to evaluate the integrals that appear. Some are regular integrals, and for their evaluation usual numerical methods can be used, but others are singular integrals.

For the singular integrals we have applied a regularization method which uses new coordinates and new modified shape functions that allow the new integrands to have no or only weakly singularities, and so the use of the computer for their evaluations.

The regularization method we applied was inspired by the work of M. Bonnet (see [1]).

Using the Taylor series we can replace the shape functions by some expressions with Taylor polynomials and, after we simplify some factors, we get some modified shape functions, in fact, new integrands, that have no or only weakly singularities. For their evaluations we can use the computer and a math application, for example MathCAD.

Let $\bar{x}_j\in L_i$, $\bar{x}_j=\sum_{l=1}^{3}N_l(\eta)\bar{x}_l$,

where $\eta\in[-1,1]$ is the value of the local coordinate ξ, that makes $\bar{x}=\bar{x}_j$.

We have:

$$\left\|\bar{x}-\bar{x}_j\right\|^2=\left\|\sum_{l=1}^{3}\left(N_l(\xi)-N_l(\eta)\right)\bar{x}_l\right\|^2 \tag{6}$$

Using the Taylor series we get:

$$N_l(\xi)=N_l(\eta)+N_l'(\eta)(\xi-\eta)+N_l''(\eta)(\xi-\eta)^2\Rightarrow$$
$$N_l(\xi)-N_l(\eta)=N_l'(\eta)(\xi-\eta)+\frac{1}{2}N_l''(\eta)(\xi-\eta)^2=,$$
$$=(\xi-\eta)\hat{N}_l(\xi,\eta) \tag{7}$$

where $\hat{N}_l$, depends on ξ but also of η, and it is named the modified shape function associated with N_l.

It is given by:

$$\hat{N}_l(\xi,\eta)=N_l'(\eta)+\frac{1}{2}N_l''(\eta)(\xi-\eta). \tag{8}$$

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Based on this technique we obtain the expressions for the modified shape functions. Denoting by $\rho = \xi - \eta$ we have:

$$\hat{N}_1(\rho,\eta) = \eta - \frac{1}{2} + \frac{1}{2}\rho,$$

$$\hat{N}_2(\xi,\eta) = -2\eta - \rho,$$

$$\hat{N}_3(\xi,\eta) = \eta + \frac{1}{2} + \frac{1}{2}\rho \tag{9}$$

So we get:

$$\left\|\bar{x} - \bar{x}_j\right\|^2 = \left\|\sum_{l=1}^{3} \rho \hat{N}_l(\rho,\eta)\bar{x}_l\right\|^2 = \rho^2 \left\|\sum_{l=1}^{3} \hat{N}_l(\rho,\eta)\bar{x}_l\right\|^2 \underset{not}{=} \rho^2 \hat{N}_{ij}$$

$$d\xi = d\rho, \quad \xi \in [-1,1] \Rightarrow \rho \in [-1-\eta, 1-\eta].$$

After some calculus we get the following expressions for the coefficients involved:

$$a_{ij}^l = \int_{-1-\eta}^{1-\eta} \left[N_l(\eta) + \rho\hat{N}_l(\rho,\eta)\right] \frac{\left(\sum_{l=1}^{3}\hat{N}_l x_l^i\right) n_x^j + \beta^2 \left(\sum_{l=1}^{3}\hat{N}_l y_l^i\right) n_y^j}{\rho \hat{N}_{ij}} J(\rho) d\rho$$

$$= \int_{-1-\eta}^{1-\eta} N_l(\eta) \frac{\left(\sum_{l=1}^{3}\hat{N}_l x_l^i\right) n_x^j + \beta^2 \left(\sum_{l=1}^{3}\hat{N}_l y_l^i\right) n_y^j}{\rho \hat{N}_{ij}} J(\rho) d\rho +$$

$$+ \int_{-1-\eta}^{1-\eta} \hat{N}_l(\eta) \frac{\left(\sum_{l=1}^{3}\hat{N}_l x_l^i\right) n_x^j + \beta^2 \left(\sum_{l=1}^{3}\hat{N}_l y_l^i\right) n_y^j}{\hat{N}_{ij}} J(\rho) d\rho \tag{10}$$

where

$$J(\rho) = \sqrt{4a_i\rho^2 + 2(4a_i\eta + b_i)\rho + 4a_i\eta^2 + 2b_i\eta + aa_i}$$

As one can sees only the first of the last two integrals in (10) has a weakly singularity the second being a regular integral. By introducing the expressions of the modified shape functions in the above relations we can write the singular integral as a sum of two integrals, only one having a weakly singularity.

We further consider the singular integrals for each of the three nodes of the current boundary element.

1. If $\bar{x}_j$ is the first node of L_i, so if $j = 2i - 1$, we have $\eta = -1$, and further we obtain:

$$\hat{N}_1(\rho,-1) = \frac{\rho - 3}{2},$$

$$\hat{N}_2(\rho,-1) = 2 - \rho,$$

$$\hat{N}_3(\rho,\eta) = \frac{\rho - 1}{2}, \tag{11}$$

$$J(\rho) = \sqrt{4a_i\rho^2 + 2(-4a_i + b_i)\rho + 4a_i - 2b_i + aa_i}$$

Denoting by:

$$p_i = \frac{4x_2^i - 3x_1^i - x_3^i}{2}, \quad P_i = \frac{4y_2^i - 3y_1^i - y_3^i}{2}$$

we get:

$$\hat{N}_{ij} = \left(\frac{\rho}{2}m_i + p_i\right)^2 + \left(\frac{\rho}{2}M_i + P_i\right)^2. \tag{12}$$

Taking into account that $N_1(-1) = 1$, $N_2(-1) = 0$, $N_3(-1) = 0$, and denoting by an_{ij}^1, as_{ij}^1 the nonsingular part, respectively the singular one , with a weak singularity, we obtain:

$$a_{ij}^1 = an_{ij}^1 + as_{ij}^1,$$

where

$$an_{ij}^1 = \int_0^2 \frac{m_i n_x^j + \beta^2 M_i n_y^j}{2\hat{N}_{ij}} J(\rho) d\rho +$$

$$+ \int_0^2 \frac{(\rho - 3)\left[\left(\frac{\rho}{2}m_i + p_i\right)n_x^j + \beta^2\left(\frac{\rho}{2}M_i + P_i\right)n_y^j\right]}{2\hat{N}_{ij}} J(\rho) d\rho,$$

$$as_{ij}^1 = \int_0^2 \frac{p_i n_x^j + \beta^2 P_i n_y^j}{\rho \hat{N}_{ij}} J(\rho) d\rho$$

$$a_{ij}^2 = \int_0^2 \frac{(2 - \rho)\left[\left(\frac{\rho}{2}m_i + p_i\right)n_x^j + \beta^2\left(\frac{\rho}{2}M_i + P_i\right)n_y^j\right]}{\hat{N}_{ij}} J(\rho) d\rho$$

$$a_{ij}^3 = \int_0^2 \frac{(\rho - 1)\left[\left(\frac{\rho}{2}m_i + p_i\right)n_x^j + \beta^2\left(\frac{\rho}{2}M_i + P_i\right)n_y^j\right]}{2\hat{N}_{ij}} J(\rho) d\rho \tag{13}$$

2. If $\bar{x}_j$ is the second node of L_i, so if $j = 2i$, we have $\eta = 0$. We deduce in this case the following expressions:

$$\hat{N}_1(\rho,-1) = \frac{\rho - 1}{2},$$

$$\hat{N}_2(\xi,-1) = -\rho,$$

$$\hat{N}_3(\xi,\eta) = \frac{\rho + 1}{2},$$

$$J(\rho) = \sqrt{4a_i\rho^2 + 2b_i\rho + aa_i}. \tag{14}$$

Denoting

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$$p_i = \frac{x_3^i - x_1^i}{2},\ P_i = \frac{y_3^i - y_1^i}{2}$$

we get:

$$\hat{N}_{ij} = \left(\frac{\rho}{2}m_i + p_i\right)^2 + \left(\frac{\rho}{2}M_i + P_i\right)^2,$$

and after some calculus the following expressions:

$$a_{ij}^1 = \int_{-1}^{1} \frac{(\rho-1)\left[\left(\frac{\rho}{2}m_i + p_i\right)n_x^j + \beta^2\left(\frac{\rho}{2}M_i + P_i\right)n_y^j\right]}{2\hat{N}_{ij}} J(\rho)d\rho$$

$$a_{ij}^2 = an_{ij}^2 + as_{ij}^2$$

$$an_{ij}^2 = \int_{-1}^{1} \frac{\frac{m_i}{2}n_x^j + \beta^2\frac{M_i}{2}n_y^j}{\hat{N}_{ij}} J(\rho)d\rho -$$

$$- \int_{-1}^{1} \frac{\rho\left[\left(\frac{\rho}{2}m_i + p_i\right)n_x^j + \beta^2\left(\frac{\rho}{2}M_i + P_i\right)n_y^j\right]}{\hat{N}_{ij}} J(\rho)d\rho$$

$$as_{ij}^2 = \int_{0}^{2} \frac{p_i n_x^j + \beta^2 P_i n_y^j}{\rho\hat{N}_{ij}} J(\rho)d\rho$$

$$a_{ij}^3 = \int_{-1}^{1} \frac{(\rho+1)\left[\left(\frac{\rho}{2}m_i + p_i\right)n_x^j + \beta^2\left(\frac{\rho}{2}M_i + P_i\right)n_y^j\right]}{2\hat{N}_{ij}} J(\rho)d\rho \tag{15}$$

3. If $\overline{x}_j$ is the third node of L_i, so if $j = 2i+1$ ($\eta = 1$), we obtain:

$$\hat{N}_1(\rho,-1) = \frac{\rho+1}{2},$$
$$\hat{N}_2(\xi,-1) = -2-\rho,$$
$$\hat{N}_3(\xi,\eta) = \frac{\rho+3}{2},$$
$$J(\rho) = \sqrt{4a_i\rho^2 + 2(4a_i + b_i)\rho + 4a_i + 2b_i + aa_i} \tag{16}$$

Denoting by

$$p_i = \frac{-4x_2^i + x_1^i + 3x_3^i}{2},$$
$$P_i = \frac{-4y_2^i + y_1^i + 3y_3^i}{2},$$

we analogous get:

$$a_{ij}^1 = \int_{0}^{2} \frac{(\rho+1)\left[\left(\frac{\rho}{2}m_i + p_i\right)n_x^j + \beta^2\left(\frac{\rho}{2}M_i + P_i\right)n_y^j\right]}{2\hat{N}_{ij}} J(\rho)d\rho$$

$$a_{ij}^2 = \int_{0}^{2} \frac{-(2+\rho)\left[\left(\frac{\rho}{2}m_i + p_i\right)n_x^j + \beta^2\left(\frac{\rho}{2}M_i + P_i\right)n_y^j\right]}{\hat{N}_{ij}} J(\rho)d\rho$$

$$a_{ij}^3 = an_{ij}^3 + as_{ij}^3$$

$$an_{ij}^3 = \int_{0}^{2} \frac{m_i n_x^j + \beta^2 M_i n_y^j}{2\hat{N}_{ij}} J(\rho)d\rho +$$

$$+ \int_{0}^{2} \frac{(\rho+3)\left[\left(\frac{\rho}{2}m_i + p_i\right)n_x^j + \beta^2\left(\frac{\rho}{2}M_i + P_i\right)n_y^j\right]}{2\hat{N}_{ij}} J(\rho)d\rho$$

$$as_{ij}^3 = \int_{0}^{2} \frac{p_i n_x^j + \beta^2 P_i n_y^j}{\rho\hat{N}_{ij}} J(\rho)d\rho \tag{17}$$

So all coefficients in relation (3) are represented by simple integrals, and can be evaluated using a computer and a math software application or ordinary numerical methods. Of course the accuracy will depend of the numerical integration scheme we choose.

Returning to the global system of notation, by using relations: $f_{2i+1} = f_3^i = f_1^{i+1}$, $f_{2i} = f_2^i$, and notating:

$$a_{ij} = \begin{cases} a_{pj}^2, & pt.\ \ i = 2p \\ a_{pj}^1 + a_{(p-1)j}^3, & pt.\ \ i = 2p-1 \end{cases},\ p = \overline{2,N}$$

$$a_{1j} = a_{1j}^1 + a_{Nj}^3, \tag{18}$$

we get the following form of equation (3):

$$\left(n_x^{j\,2} + \beta^2 n_y^{j\,2}\right)f_j + \frac{1}{\pi}\sum_{i=1}^{2N} a_{ij} f_i = 2\beta n_x^j \tag{19}$$

The final form of the linear system of equations to solve is:

$$\sum_{i=1}^{2N} A_{ij} f_i = 2\pi\beta n_x^j, \tag{20}$$

$$\text{where } A_{ij} = \begin{cases} a_{ij}, & pt.\, i \neq j \\ a_{jj} + \pi\left(n_x^{j\,2} + \beta^2 n_y^{j\,2}\right), & pt.\, i = j \end{cases}. \tag{21}$$

Denoting by $A = \left(A_{ij}\right)_{1 \le i,j \le n}$, and $B_j = 2\pi\beta n_x^j$ we can write the system as in (5):

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$$[A]\{f\} = \{B\}$$

As a conclusion we can say that, using the new variables and the modified shape functions, the coefficients a_{ij}^{l}, given by the singular integrals, may be written, each of them, like a sum of two integrals: a usual one and an integral with a weakly singularity. These coefficients depend only on the coordinates of the nodes and they can be evaluated with a computer.

After solving the system we get the nodal values of the unknown function f and we can further evaluate other fields of interest as for example the velocity field and the local pressure coefficient, c_p.

IV. Numerical results and conclusions

In order to test the technique we used for the treatment of the singularities, and the computer code we made using MathCAD programming, based on it, we make an analytical checking. We shall consider the problem of an incompressible ideal fluid flow around a circular obstacle. In this case the problem has an exact solution given for example in paper [10].

Comparisons between the numerical solution obtained with the computer code and the exact solution of the problem can be made. Considering for example the case of the local pressure coefficient, the solutions are performed in Fig. 1.

We use for the boundary discretization 20 nodes.

As we can see Fig.1 shows a very good agreement between the two kinds of solutions, because the calculated and analytical values are very close, taking into account the small number of nodes.

A more detailed representation is made in Fig. 2, where only the first 7 nodes are considered.

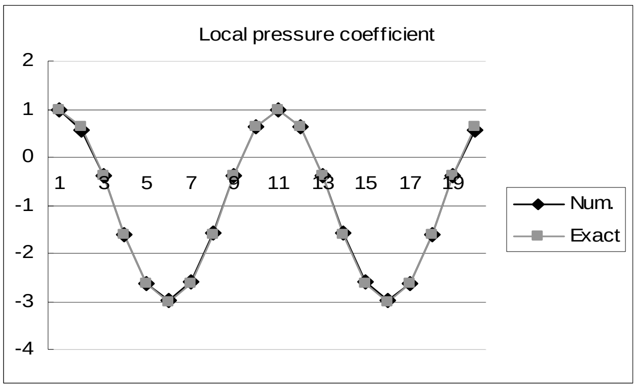

Fig.1. The numerical and the exact solution for the circular obstacle (20 nodes on the boundary)

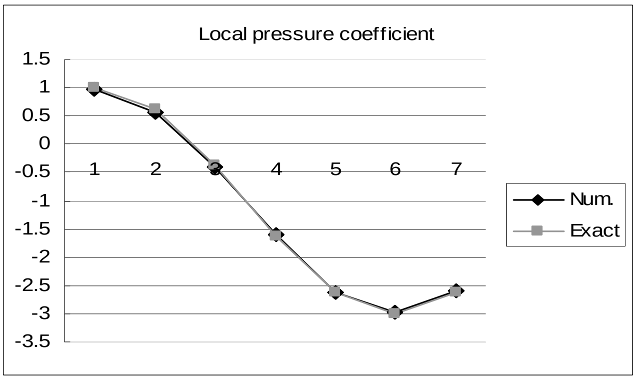

Fig.2. The numerical and the exact solution for the first 7 nodes

The error that appears is performed in Fig.3.

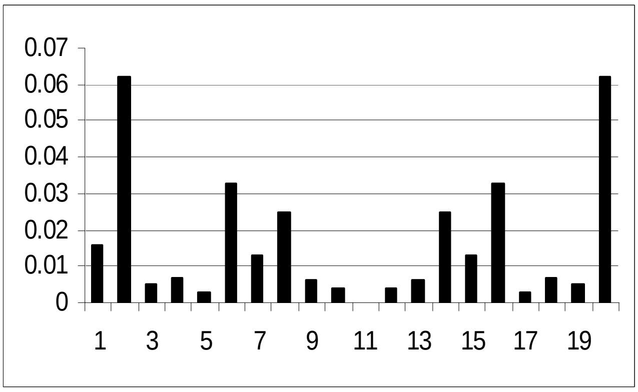

Fig.3. Errors in case of 20 nodes on the boundary

The same particular case is considered in paper [11] in order to check another method (CPV) that can be used for the treatment of the singularities.

In Fig. 4 we make a comparison between the numerical solutions got with the two techniques and the exact solution. We use the term "the numerical CPV solution" for the numerical solution obtained when using the technique presented in paper [11].

We consider in both cases the same number of nodes for the boundary discretization: 20.

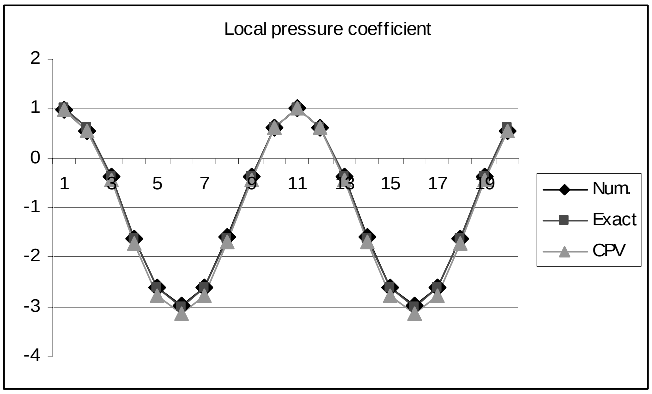

Fig.4. The numerical, the exact, and the numerical CPV solution for the circular obstacle (20 nodes on the boundary)

As we can see both techniques are efficient and of great accuracy, but better results are obtained using the regularization method.

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In order to better observe the accuracy of the two mentioned numerical solutions, in Fig. 5 there are represented the errors that appear in both situations

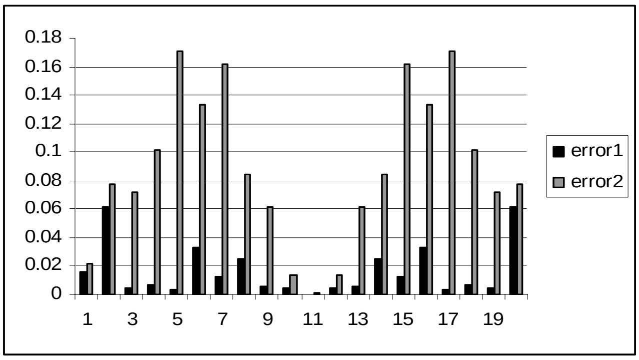

Fig.5. Errors in case of 20 nodes on the boundary (error1 – for the regularization technique, error2-for CPV)

Similar comparisons can be made for other fields of interest, as for example for the components of the velocity field on the boundary, as well as in the fluid domain.

Different number of nodes can be chosen for the boundary discretization and of course by using a bigger number of nodes the numerical solution can be improved.

As we can see from the herein paper if using suitable methods for the evaluation of integrals of singular kernels we can obtain very good numerical results when using BEM, even for a quite small number of nodes used for the boundary discretization.

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