Fault-Free Vertex-Pancyclicity in Twisted Cubes with Faulty Edges

Jung-Sheng Fu

***Abstract*—The *n*-dimensional twisted cube, denoted by TQ_n, a variation of the hypercube, possesses some properties superior to the hypercube. In this paper, we show that every vertex in TQ_n lies on a fault-free cycle of every length from 6 to 2^n, even if there are up to $n - 2$ link faults. We also show that our result is optimal.**

***Index Terms*—hypercubes, twisted hypercubes, fault-tolerant, vertex-pancyclic, interconnection network**

I. Introduction

A *graph G* is a triple consisting of a *vertex set* $V(G)$, an *edge set* $E(G)$, and a relation that associates with each edge two vertices called its endpoints [24]. We usually use a graph to represent the topology of an interconnection network (network for short). The hypercube is a popular interconnection network with many attractive properties such as regularity, symmetry, small diameter, strong connectivity, recursive construction, partition ability, and relatively low link complexity [22]. The twisted cube [13], as one of the important variations of the hypercube, and derived by changing some connections of the hypercube according to specific rules, possesses some desirable features: its diameter, wide diameter, and fault diameter are about half of those of the comparable hypercube [2]. An *n*-dimensional twisted cube is $(n - 3)$ fault-tolerant Hamiltonian connected [14] and $(n - 2)$ fault-tolerant pancyclic [18], whereas the hypercube is not. Moreover, its performance is superior to that of the hypercube [1]. Other previous works relating to the twisted cube can be found in [6], [9], [11], [12], [15], [16].

Linear arrays and rings, two of the most fundamental networks for parallel and distributed computation, are suitable for developing simple algorithms with low communication costs. Many efficient algorithms designed based on linear arrays and rings for solving a variety of algebraic problems and graph problems can be found in [17]. The *pancyclicity* of a network represents its power of embedding rings of all possible lengths. A graph G is called *m-pancyclic* whenever G contains a cycle of each length l for $m \le l \le |V(G)|$. A graph G is *m-vertex-pancyclic* (respectively, *m-edge-pancyclic*) if every vertex (respectively, edge) lies on a cycle of every length from m to $|V(G)|$. It is clear that if a graph G is m-edge-pancyclic, then it is m-vertex-pancyclic. The crossed cube [7], the twisted cube [8], and the möbius cube [23] are 4-edge-pancyclic. Besides, a 3-pancyclic graph is called pancyclic, a 3-edge-pancyclic graph is called edge-pancyclic, and a 3-vertex-pancyclic graph is called vertex-pancyclic. The alternating group graph [4] and the augmented cube [19] are edge-pancyclic.

Since faults may occur to networks, the fault tolerance of networks is an important issue in designing network topologies. Let $F_e \subset E(G)$ (respectively, $F_v \subset V(G)$) denote the faulty edges (respectively, the faulty vertices) in a graph G and let $F = F_e \cup F_v$. Suppose that $G - F$ is P, where P is pancyclic, vertex-pancyclic, or edge-pancyclic. Then, we call G $|F|$ fault-tolerant P. In addition, G is $|F|$-edge fault-tolerant P (respectively, $|F|$-vertex fault-tolerant P) if $F = F_e$ (respectively, if $F = F_v$). Note that if G is $|F|$ fault-tolerant P, then G is $|F|$-edge fault-tolerant P and $|F|$-vertex fault-tolerant P. Previously, the pancyclicity on various faulty networks was studied in [3], [10], [18], [19], [20], [21]. In [18], TQ_n has been shown to be $(n - 2)$ fault-tolerant pancyclic, where $n \ge 3$ is an odd integer.. In this paper, we show that $TQ_n - F_e$ is 6-vertex-pancyclic if $|F_e| \le n - 2$, $n \ge 3$ is an odd integer. That is, we show that TQ_n is $(n - 2)$-edge fault-tolerant 6-vertex-pancyclic. In addition, we also show that our result is optimal.

II. Preliminaries

Let G be a graph and let $u, v \in V(G)$. The degree of vertex v in G, written as $\deg_G(v)$, is the number of edges incident to v in G. In addition, $\delta(G) = \min\{\deg_G(v) | v \in V(G)\}$. A *path* $P[x_0, x_t] = \langle x_0, x_1, \cdots, x_t\rangle$ is a sequence of nodes such that two consecutive nodes are adjacent. t is the distance between nodes x_0 and x_t if $P[x_0, x_t]$ is a shortest path in G. We use $d_G(x_0, x_t)$ to denote the distance between x_0 and x_t in G, and use (u, v) to denote an edge whose endpoints are u and v. Moreover, a path $\langle x_0, x_1, \cdots, x_t\rangle$ may contain other subpaths, denoted as $\langle x_0, x_1, \cdots, x_i, P[x_i, x_j], x_j, \cdots, x_t\rangle$, where $P[x_i, x_j] = \langle x_i, x_{i+1}, \cdots, x_{j-1}, x_j\rangle$. A *cycle* is a path with $x_0 = x_t$ and $t \ge 3$. A cycle (respectively, path) in G is called a *Hamiltonian cycle* (respectively, *Hamiltonian path*) if it contains every vertex of G exactly once.

The vertex set of the twisted n-cube TQ_n is the set of all binary strings of length n, where n is odd. Let $b = b_{n-1}b_{n-2}\ldots b_0$ denote one vertex in TQ_n. For $i \in \{0, 1, \ldots, n - 1\}$, let the *i-th parity function* $P_i(b) = b_i \oplus b_{i-1} \oplus \ldots \oplus b_0$, where $\oplus$ denotes the exclusive-or operation. The TQ_n can be defined recursively as follows: TQ_1 is a complete graph with two vertices 0 and 1. Suppose that $n \ge 3$. We can decompose the vertices of TQ_n

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into four sets, $TQ_{n-2}^{0,0}$, $TQ_{n-2}^{0,1}$, $TQ_{n-2}^{1,0}$, and $TQ_{n-2}^{1,1}$, where $TQ_{n-2}^{i,j}$ consists of those vertices b with $b_{n-1} = i$ and $b_{n-2} = j$. For each $ij \in \{00, 01, 10, 11\}$, the induced subgraph of $TQ_{n-2}^{i,j}$ in TQ_n is isomorphic to TQ_{n-2}. Edges that connect these four subtwisted cubes can be described as follows: an $(n-1)$-edge joins vertices $b = b_{n-1}b_{n-2}\ldots b_0$ and $b^{(n-1)} = \overline{b_{n-1}}b_{n-2}\ldots b_0$. An $(n-2)$-edge joins vertices b and $b^{(n-2)}$, where $b^{(n-2)} = \overline{b_{n-1}}\,\overline{b_{n-2}}\ldots b_0$ when $P_{n-3}(b) = 0$, and $b^{(n-2)} = b_{n-1}\overline{b_{n-2}}\ldots b_0$ when $P_{n-3}(b) = 1$. Note that $(n-1)$-edges connect $TQ_{n-2}^{0,i}$ and $TQ_{n-2}^{1,i}$ and $(n-2)$-edges connect $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}$ and $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$, where i = 0 or 1. Fig. 1 depicts TQ_5, containing four sets, $TQ_3^{0,0}$, $TQ_3^{0,1}$, $TQ_3^{1,0}$, and $TQ_3^{1,1}$. Formally, TQ_n can be defined as follows.

Definition 1. The vertex set of TQ_n is $\{b_{n-1}b_{n-2}\ldots b_0 \mid b_i \in \{0, 1\}$ for all $0 \le i \le n-1\}$, where n is odd. Vertex $b = b_{n-1}b_{n-2}\ldots b_0$ is adjacent to vertex b^d, for all $0 \le d \le n-1$, where $b^d = b_{n-1}b_{n-2}\ldots\overline{b_d}\ldots b_0$ if (1) d is even or (2) d is odd and $P_{d-1}(b) = 1$, and $b^d = b_{n-1}b_{n-2}\ldots\overline{b_{d+1}}\,\overline{b_d}\ldots b_0$ if d is odd and $P_{d-1}(b) = 0$. The edge joining b and b^d is referred to as a d-edge.

Furthermore, we use b^{ij} to denote $(b^i)^j$. Note that it is possible that $b^{ij} \neq b^{ji}$. The following lemma shown in [14], [18],[5] will be used often.

Lemma 1. [14] TQ_n (respectively, $TQ_n^{0,i} \cup TQ_n^{1,i}$ for $i \in \{0, 1\}$) is $(n-2)$-Hamiltonian (respectively, $(n-1)$-Hamiltonian) and $(n-3)$-Hamiltonian connected (respectively, $(n-2)$-Hamiltonian connected), where $n \ge 3$ is an odd integer.

Lemma 2. [18] TQ_n (respectively, $TQ_n^{0,i} \cup TQ_n^{1,i}$ for $i \in \{0, 1\}$) is $(n-2)$ fault-tolerant 4-pancyclic (respectively, $(n-1)$ fault-tolerant 4-pancyclic), where $n \ge 3$ is an odd integer.

Lemma 3. [5] Let $u, v \in V(TQ_n)$ where $n \ge 3$ is an odd integer. There is a path $P[u, v]$ with length l, for any integer l between $d_{TQ_n}(u,v) + 2$ and $2^n - 1$.

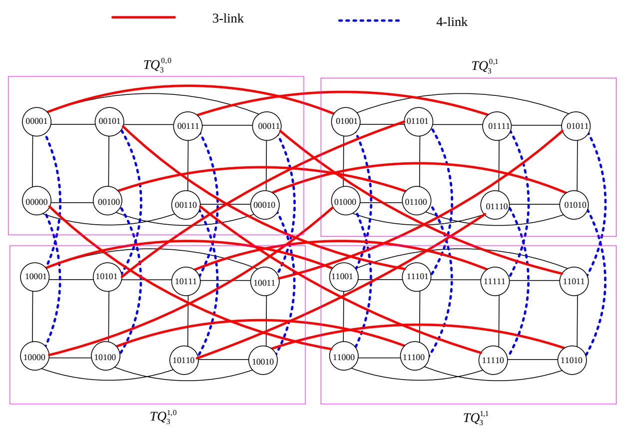

Fig 1. TQ_5 (contains $TQ_3^{0,0}$, $TQ_3^{0,1}$, $TQ_3^{1,0}$, and $TQ_3^{1,1}$).

III. Some Important Properties

In this section, we introduce some important properties of the twisted cube, which are needed to derive our main result. Because of the pages limitation, the proofs are omitted.

Lemma 4. Let $(x, y) \in E(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0})$, where $n \ge 5$ is an odd integer. Then $d_{TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}}(x^{(n-2)}, y^{(n-2)}) = 1$ if $P_{n-3}(x) = P_{n-3}(y)$ and $d_{TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}}(x^{(n-2)}, y^{(n-2)}) = 2$ if $P_{n-3}(x) \neq P_{n-3}(y)$.

Lemma 5. Let $(x, z), (z, y) \in E(TQ_{n-2}^{0,0})$,where $n \ge 5$ is an odd integer and x, y are distinct. If $P_{n-3}(z) \neq P_{n-3}(y)$ and $P_{n-3}(z) \neq P_{n-3}(x)$, then $d_{TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}}(x^{(n-2)}, y^{(n-2)}) = 2$ and $\{y^{(n-2)}, x^{(n-2)}\} \subset V(TQ_{n-2}^{0,1})$ or $\{y^{(n-2)}, x^{(n-2)}\} \subset V(TQ_{n-2}^{1,1})$.

Lemma 6. Let $z \in V(TQ_{n-2}^{0,0})$, $x = z^{(n-1)}$, and $(x, y) \in E(TQ_{n-2}^{1,0})$, where $n \ge 5$ is an odd integer. If $P_{n-3}(x) \neq P_{n-3}(y)$ then $(z^{(n-2)}, y^{(n-2)}) \in E(TQ_{n-2}^{0,1})$ or $(z^{(n-2)}, y^{(n-2)}) \in E(TQ_{n-2}^{1,1})$.

Lemma 7. Let $u, v \in V(TQ_{n-2}^{j,i})$ and $d_{TQ_{n-2}^{j,i}}(u,v) \le 2$, where $n \ge 5$ is an odd integer and $i, j \in \{0, 1\}$. For any integer l between $d_{TQ_{n-2}^{j,i}}(u,v) + 2$ and $2^{n-1} - 1$, there is a path $P[u, v]$ with length l in $TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i}$.

Lemma 8. Let $x \in V(TQ_{n-2}^{0,1})$, $y \in V(TQ_{n-2}^{1,1})$ and $d_{TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}}(x, y) = 2$, where $n \ge 5$ is an odd integer. There is a path $P[x, y]$ with length l in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$, for any integer l between 4 and $2^{n-1} - 1$.

Lemma 9. Let $z \in V(TQ_{n-2}^{0,0})$ and $F \subset \{(v, v^{(n-1)}) \mid v \in V(TQ_{n-2}^{0,0})\}$, where $n \ge 5$ is an odd integer and $|F| = n - 2$. There is a cycle C with length l in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F$ such that $z \in V(C)$ and $|\{(v, v^{(n-1)}) \mid v \in V(TQ_{n-2}^{0,0})\} \cap E(C)| = 2$, for any integer l between $2^{n-1} - 2$ and 2^{n-1}.

IV. Edge-Fault-Tolerant 5-Vertex-Pancyclicity

In this section, by the aid of the lemmas in Section 3, we will show that TQ_n is $(n-2)$-edge fault-tolerant 6-vertex-pancyclic. We proceed by induction on n. First, we need to show that TQ_3 is 1-edge fault-tolerant 6-vertex-pancyclic. We have following lemma.

Lemma 10. $TQ_3 - \{e\}$ is 6-vertex-pancyclic for any $e \in E(TQ_3)$.

Proof. It is easy to see that TQ_3 is node-symmetry. Moreover, $TQ_3 - \{e\}$ is Hamiltonian (by Lemma 1) and a Hamiltonian cycle contains all nodes in TQ_3. Thus, we only need to show that vertex 000 lies on a cycle of length l in $TQ_3 - \{e\}$, for any integer between 6 and 7. The desired cycles of length l are as listed below:

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Length	The edge e	Desired Cycle
6	(000, 110), (001, 101), (010, 100), (010, 011), (010, 110), or (110, 111)	⟨000, 001, 011, 111, 101, 100, 000⟩
	(000, 001), (001, 011), (100, 101), or (101, 111)	⟨000, 100, 010, 011, 111, 110, 000⟩
	(000, 100), or (011, 111)	⟨000, 001, 101, 100, 010, 110, 000⟩
7	(000, 110), (001, 101), (010, 011), (100, 101), or (101, 111)	⟨000, 001, 011, 111, 110, 010, 100, 000⟩
	(000, 001), (010, 100), or (010, 110)	⟨000, 100, 101, 001, 011, 111, 110, 000⟩
	(000, 100), (001, 011), or (110, 111)	⟨000, 001, 101, 111, 011, 010, 110, 000⟩
	(011, 111)	⟨000, 001, 101, 111, 110, 010, 100, 000⟩

□

Lemma 10 provides the base case. There are two steps in inductive phase. First, suppose that TQ_n is $(n-2)$-edge fault-tolerant 6-vertex-pancyclic where $n \geq 3$ is an odd integer; we want to show that $TQ_n^{0,i} \cup TQ_n^{1,i}$ for $i \in \{0, 1\}$ is $(n-1)$-edge fault-tolerant 6-vertex-pancyclic. The second step is given that $TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i}$ for $i \in \{0, 1\}$ is $(n-3)$-edge fault-tolerant 6-vertex-pancyclic where $n \geq 5$ is an odd integer; we want to show that TQ_n is $(n-2)$-edge fault-tolerant 6-vertex-pancyclic. We format the first step as Lemma 11 and second step as Lemma 12. Since the proof of Lemma 11 is easier than and similar to that of Lemma 12, we only prove Lemma 12.

Lemma 11. If TQ_n is $(n-2)$-edge fault-tolerant 6-vertex-pancyclic, then $TQ_n^{0,i} \cup TQ_n^{1,i}$ for $i \in \{0, 1\}$ is $(n-1)$-edge fault-tolerant 6-vertex-pancyclic, where $n \geq 3$ is an odd integer.

Lemma 12. Suppose that $TQ_n^{0,i} \cup TQ_n^{1,i}$ for $i \in \{0, 1\}$ is $(n-3)$-edge fault-tolerant 6-vertex-pancyclic where $n \geq 5$ is an odd integer. Then TQ_n is $(n-2)$-edge fault-tolerant 6-vertex-pancyclic.

Proof. Assume that $F \subset E(TQ_n)$ and $|F| \leq n-2$. For any $z \in V(TQ_n)$ and any integer $l \in \{6, 7, \dots, 2^n\}$, we want to find a cycle C of length l in $TQ_n - F$, such that $z \in V(C)$. By Lemma 2, there exists a cycle C of length 2^n in $TQ_n - F$. Clearly, $z \in V(C)$. Thus, we need not find the cycle of length 2^n. Let $F_0 = F \cap E(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0})$, $F_1 = F \cap E(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$, and $F_c = F \cap \{(u, u^{(n-2)}) | u \in V(TQ_n)\}$. Without loss of generality, assume that $z \in V(TQ_{n-2}^{0,0})$. Two cases are considered:

Case 1. $|F_0| \leq n-3$. Three cases are further considered:

Case 1.1: $6 \leq l \leq 2^{n-1}$. By assumption, we have $TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i}$ for $i \in \{0, 1\}$ is $(n-3)$-edge fault-tolerant 6-vertex-pancyclic. Thus, for any integer $l \in \{6, \dots, 2^{n-1}\}$, there exists a cycle C of the length l in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$ such that $z \in V(C)$. Clearly, C is the desired cycle.

Case 1.2: $2^{n-1} + 1 \leq l \leq 2^{n-1} + 2$. We have two scenarios as follows:

Case 1.2.1: $|F_0| \leq n-5$. Let $(x, y) \in E(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}) - F$, where $(x, x^{(n-2)})$, $(y, y^{(n-2)}) \notin F_c$. Moreover, let $u \in V(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}) - \{x^{(n-2)}, y^{(n-2)}, z\}$. Since $|F_0| \leq n-5$ and $|F_0 \cup \{u\}| \leq n-4$, by Lemma 1, there exist a Hamiltonian path $P[x^{(n-2)}, y^{(n-2)}]$ with length l_0 in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$ (respectively, $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 \cup \{u\})$, where $l_0 = 2^{n-1} - 1$ (respectively, $2^{n-1} - 2$). Clearly, $z \in V(P[x^{(n-2)}, y^{(n-2)}])$. The desired cycle of length l can be constructed by $\langle x, x^{(n-2)}, P[x^{(n-2)}, y^{(n-2)}], y^{(n-2)}, y, x \rangle$ for any integer $l = l_0 + 3 \in \{2^{n-1} + 1, 2^{n-1} + 2\}$ (see Fig. 2(a)).

Case 1.2.2: $n-4 \leq |F_0| \leq n-3$. Thus, $|F_1| + |F_c| \leq 2$. By assumption, $TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i}$ for $i \in \{0, 1\}$ is $(n-3)$-edge fault-tolerant 6-vertex-pancyclic. Thus, for any integer l'_0 between $2^{n-1} - 2$ and 2^{n-1}, there exists a cycle C_0 of length l'_0 in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$ such that $z \in V(C_0)$ (since $2^{n-1} - 2 > 6$ when $n \geq 5$). Let $(x, y) \in E(C_0)$ such that $(x, x^{(n-2)})$, $(y, y^{(n-2)}) \notin F_c$, $(x^{(n-2)}, x^{(n-2)i}) \notin F_1$, and $(y^{(n-2)}, y^{(n-2)i}) \notin F_1$, for all $i \in \{0, 1, \dots, n-3, n-1\}$[1]. Let $P[x, y] = C - \{(x, y)\}$ and let l_0 be the length of $P[x, y]$. (Thus, l_0 can be any integer between $2^{n-1} - 3$ and $2^{n-1} - 1$.) By Lemma 4, the length of $P[x^{(n-2)}, y^{(n-2)}]$ is one or two. In addition, since $(x^{(n-2)}, x^{(n-2)i}) \notin F_1$, and $(y^{(n-2)}, y^{(n-2)i}) \notin F_1$, for all $i \in \{0, 1, \dots, n-3, n-1\}$, $E(P[y^{(n-2)}, x^{(n-2)}]) \cap F_1 = \varnothing$. The desired cycle of length l can be constructed by $\langle x, P[x, y], y, y^{(n-2)}, P[y^{(n-2)}, x^{(n-2)}], x^{(n-2)}, x \rangle$ for any integer $l = l_0 + 2 + 1 \in \{2^{n-1} + 1, 2^{n-1} + 2\}$ or $l = l_0 + 2 + 2 \in \{2^{n-1} + 1, 2^{n-1} + 2\}$ (see Fig. 2(b)). Note that the $l_0 \in \{2^{n-1} - 2, 2^{n-1} - 1\}$ (respectively, $l_0 \in \{2^{n-1} - 3, 2^{n-1} - 2\}$) when the length of $P[y^{(n-2)}, x^{(n-2)}]$ is one (respectively, two).

Case 1.3: $2^{n-1} + 3 \leq l \leq 2^n - 1$. We have three scenarios as follows:

Case 1.3.1: $|F_0| = n-3$. Thus, $|F_1| + |F_c| = 1$. Since $|F_0| = n-3$, by Lemma 1, there exists a Hamiltonian cycle C_0 in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$. For any integer $l_0 \in \{2, 3, \dots, 2^{n-1} - 2\}$, let $P[x, y]$ be a subpath with length l_0 in C_0 such that $z \in V(P[x, y])$, $\{(x, x^{(n-2)}), (y, y^{(n-2)})\} \cap F_c = \varnothing$[2]. Since $|F_1| \leq |F_1| + |F_c| = 1 \leq n-4$, by Lemma 1, there exists a Hamiltonian path $P[y^{(n-2)}, x^{(n-2)}]$ in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. For any integer $l = l_0 + 2 + 2^{n-1} - 1 \in \{2^{n-1} + 3, 2^{n-1} + 4, \dots, 2^n - 1\}$, the desired cycle of length l can be constructed by $\langle x, P[x, y], y, y^{(n-2)}, P[y^{(n-2)}, x^{(n-2)}], x^{(n-2)}, x \rangle$ (see Fig. 2(c)).

Case 1.3.2: $|F_0| \leq n-4$ and $|F_1| \leq n-3$. Since $|F_1| \leq n-3$, by Lemma 1, there exists a Hamiltonian cycle C_1 in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. For any integer $l_1 \in \{2, 3, \dots, 2^{n-1} - 2\}$, let $P[x, y]$ be the subpath with length l_1 in C_1 such that $(x,$

[1] Let $S = \{(v, t) | (v, t) \in E(C_0), (v, v^{(n-2)}) \in F_c \text{ or } (t, t^{(n-2)}) \in F_c\}$ and $T = \{(v, t) | (v, t) \in E(C_0), (v^{(n-2)}, v^{(n-2)i}) \in F_1 \text{ or } (t^{(n-2)}, t^{(n-2)j}) \in F_1$, for some $i, j \in \{0, 1, \dots, n-3, n-1\}\}$. We have $|S| \leq 2 \times |F_c|$ and $|T| \leq 4 \times |F_1|$. Since $|F_1| + |F_c| \leq 2$. We have $|S \cup T| \leq 8$. Thus, we have $|E(C_0) - (S \cup T)| \geq 2^{n-1} - 2 - 8 \geq 6$ (since $n \geq 5$). Thus, we can always find such an edge.

[2] Clearly, we have l_0 choices. We can always find such a path since $l_0 \geq 2 > |F_c|$.

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$x^{(n-2)}$), $(y, y^{(n-2)}) \notin F_c$ [3]. Since $|F_0| \le n-4$, by Lemma 1, there exists a Hamiltonian path $P[y^{(n-2)}, x^{(n-2)}]$ in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$ (certainly, $z \in V(P[y^{(n-2)}, x^{(n-2)}])$). For any integer $l = 2^{n-1} - 1 + 2 + l_1 \in \{2^{n-1}+3, 2^{n-1}+4, \ldots, 2^n - 1\}$, the desired cycle of length l can be constructed by $\langle x, P[x, y], y, y^{(n-2)}, P[y^{(n-2)}, x^{(n-2)}], x^{(n-2)}, x\rangle$ (see Fig. 2(d)).

Case 1.3.3: $|F_1| = n-2$ (thus, $|F_0| + |F_c| = 0$). Let $(x', y') \in F_1$. Then, $|F_1 - \{(x', y')\}| = n-3$. By Lemma 2, for any integer $l'_1 \in \{4, 5, \ldots, 2^{n-1}\}$, there exists a cycle C_1 of length l'_1 in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - (F_1 - \{(x', y')\})$. If $(x', y') \in E(C_1)$, then let $x = x'$ and $y = y'$. If $(x', y') \notin E(C_1)$, then randomly choose an edge $(x, y) \in E(C_1)$. Let $P[x, y] = C - \{(x, y)\}$ and let l_1 be the length of $P[x, y]$. (Thus, l_1 can be any integer between 3 and $2^{n-1} - 1$.) Let $u \in V(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}) - \{x^{(n-2)}, y^{(n-2)}, z\}$. Since $|F_0| = 0$ and $|F_0 \cup \{u\}| = 1 \le n-4$, by Lemma 1, there exist a Hamiltonian path $P[y^{(n-2)}, x^{(n-2)}]$ with length l_0 in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$ (respectively, $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 \cup \{u\})$, where $l_0 = 2^{n-1} - 1$ (respectively, $l_0 = 2^{n-1} - 2$). The desired cycle of length l can be constructed by $\langle x, P[x, y], y, y^{(n-2)}, P[y^{(n-2)}, x^{(n-2)}], x^{(n-2)}, x\rangle$ for any integer $l = l_0 + 2 + l_1 \in \{2^{n-1}+3, 2^{n-1}+4, \ldots, 2^n - 1\}$ (see Fig. 2(e)).

Case 2: $|F_0| = n-2$. Thus, $|F_1| + |F_c| = 0$. Two cases are further considered:

Case 2.1: $6 \le l \le 2^{n-1}+2$. Let $S_0 = \{(z, v) | v \in V(TQ_{n-2}^{0,0})$ and $P_{n-3}(v) = P_{n-3}(z)\}$, $S_1 = \{(z, v) | v \in V(TQ_{n-2}^{0,0})$ and $P_{n-3}(v) \ne P_{n-3}(z)\}$. Note that $|S_0 \cup S_1| = n-2$ and $S_0 \cap S_1 = \varnothing$. We have four scenarios as follows:

Case 2.1.1: $|S_0 - F_0| \ge 1$. That is, there exists an edge $(z, x) \in TQ_{n-2}^{0,0} - F_0$ such that $P_{n-3}(x) = P_{n-3}(z)$. By Lemma 4, we have $d_{TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}}(x^{(n-2)}, z^{(n-2)}) = 1$. By Lemma 7, for any integer $l_1 \in \{3, 4, \ldots, 2^{n-1} - 1\}$, there is a path $P[x^{(n-2)}, z^{(n-2)}]$ with length l_1 in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$. The desired cycle of length l can be constructed by $\langle z, x, x^{(n-2)}, P[x^{(n-2)}, z^{(n-2)}], z^{(n-2)}, z\rangle$ for any integer $l = 1 + 2 + l_1 \in \{6, 7, \ldots, 2^{n-1} + 2\}$ (see Fig. 3(a)).

Case 2.1.2: $F_0 \subset E(TQ_{n-2}^{0,0})$. Thus, $(z, z^{(n-1)}) \notin F_0$. Let $x = z^{(n-1)}$. Clearly, $P_{n-3}(z) = P_{n-3}(x)$. The rest of the discussion is the same as that of Case 2.1.1.

Case 2.1.3: $F_0 \not\subset E(TQ_{n-2}^{0,0})$ and $|S_0 - F_0| = 0$. Since $F_0 \not\subset E(TQ_{n-2}^{0,0})$, we have $|F_0 \cap (S_0 \cup S_1)| \le |F_0 \cap E(TQ_{n-2}^{0,0})| \le |F_0| - 1 = n-3$. Moreover, because $|S_0 \cup S_1| = n-2$ and $|S_0 - F_0| = 0$, we have $|S_1 - F_0| \ge 1$. That is, there exists an edge $(z, x) \in TQ_{n-2}^{0,0} - F_0$ such that $P_{n-3}(x) \ne P_{n-3}(z)$.

First, we discuss the situation that $7 \le l \le 2^{n-1}+2$. By Lemma 4, we have $d_{TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}}(x^{(n-2)}, z^{(n-2)}) = 2$. By Lemma 7, for any integer $l_1 \in \{4, 5, \ldots, 2^{n-1} - 1\}$, there is a path $P[x^{(n-2)}, z^{(n-2)}]$ with length l_1 in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$. The desired cycle of length l can be constructed by $\langle z, x, x^{(n-2)}, P[x^{(n-2)}, z^{(n-2)}], z^{(n-2)}, z\rangle$ for any integer $l = 1 + 2 + l_1 \in \{7, 8, \ldots, 2^{n-1} + 2\}$ (see Fig. 3(b)).

[3] Since $|E(C_1)| = 2^{n-1}$, we have at least 2^{n-1} choices. If such a path does not exist, then $|F| \ge 2^{n-1}/2 > n-2$ when $n \ge 5$, which is a contradiction.

Now, we discuss the situation that $l = 6$. First, consider that $|S_1 - F_0| \ge 2$, i.e., there exists two edges (z, x), $(z, y) \in TQ_{n-2}^{0,0} - F_0$ such that $P_{n-3}(z) \ne P_{n-3}(y)$ and $P_{n-3}(z) \ne P_{n-3}(x)$. By Lemma 5, we have $d_{TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}}(x^{(n-2)}, y^{(n-2)}) = 2$. Let $P[y^{(n-2)}, x^{(n-2)}]$ have length 2 . The desired cycle of length 6 can be constructed by $\langle x, z, y, y^{(n-2)}, P[y^{(n-2)}, x^{(n-2)}], x^{(n-2)}, x\rangle$ (see Fig. 3(c)). Then, consider that $|S_1 - F_0| = 1$. We have $|F_0 \cap (S_0 \cup S_1)| = |F_0 \cap E(TQ_{n-2}^{0,0})| = n-3$. Thus, if $e \in E(TQ_{n-2}^{0,0}) - (S_0 \cup S_1)$, then $e \notin F_0$. Let $y \in \{x^0, x^2\} - \{z\}$. Clearly, since $(x, y) \in E(TQ_{n-2}^{0,0}) - (S_0 \cup S_1)$ and $P_{n-3}(y) \ne P_{n-3}(x)$. Since $P_{n-3}(x) \ne P_{n-3}(y)$ and $P_{n-3}(x) \ne P_{n-3}(z)$, by Lemma 5, we have $d_{TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}}(z^{(n-2)}, y^{(n-2)}) = 2$. Let $P[y^{(n-2)}, z^{(n-2)}]$ have length 2. The desired cycle of length 6 can be constructed by $\langle z, x, y, y^{(n-2)}, P[y^{(n-2)}, z^{(n-2)}], z^{(n-2)}, z\rangle$ (exchange x with z in Fig. 3(c)).

Case 2.2: $2^{n-1} + 3 \le l \le 2^n - 1$. We have two scenarios as follows:

Case 2.2.1: $F_0 \subset \{(v, v^{(n-1)}) | v \in TQ_{n-2}^{0,0}\}$. By Lemma 9, there is a cycle C_0 with length l'_0 in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$ such that $z \in V(C_0)$ and $|\{(v, v^{(n-1)}) | v \in TQ_{n-2}^{0,0}\} \cap E(C_0)| = 2$, for any integer $l'_0 \in \{2^{n-1} - 2, 2^{n-1} - 1, 2^{n-1}\}$. Let $(x, y) \in E(C_0) - \{(v, v^{(n-1)}) | v \in TQ_{n-2}^{0,0}\}$. Clearly, we have $x, y \in TQ_{n-2}^{0,0}$ or $x, y \in V(TQ_{n-2}^{1,0})$. Let $P[x, y] = C_0 - \{(x, y)\}$ and $l_0 = |E(P[x, y])|$. Thus, l_0 can be any integer between $2^{n-1} - 3$ and $2^{n-1} - 1$.

If $\{x^{(n-2)}, y^{(n-2)}\} \subset V(TQ_{n-2}^{1,0})$ or $\{x^{(n-2)}, y^{(n-2)}\} \subset V(TQ_{n-2}^{1,1})$, then $d_{TQ_{n-2}^{0,1}}(x^{(n-2)}, y^{(n-2)}) = 1$. By Lemma 7, for any integer $l_1 \in \{3, 4, \ldots, 2^{n-1} - 1\}$, there is a path $P[y^{(n-2)}, x^{(n-2)}]$ with length l_1 in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$. The desired cycle of length l can be constructed by $\langle x, P[x, y], y, y^{(n-2)}, P[y^{(n-2)}, x^{(n-2)}], x^{(n-2)}, x\rangle$ for any integer $l = l_0 + 2 + l_1 \in \{2^{n-1}+2, 2^{n-1}+3, \ldots, 2^n - 1\}$ (see Fig. 2(e) with $2^{n-1} - 3 \le l_0 \le 2^{n-1} - 1$).

If $x \in V(TQ_{n-2}^{0,1})$, $y \in V(TQ_{n-2}^{1,1})$ or $y \in V(TQ_{n-2}^{0,1})$, $x \in V(TQ_{n-2}^{1,1})$, then $d_{TQ_{n-2}^{0,1}}(x^{(n-2)}, y^{(n-2)}) = 2$. By Lemma 8, there is a path $P[y^{(n-2)}, x^{(n-2)}]$ with length l_1 in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$ for any integer l_1 between 4 and $2^{n-1} - 1$. The desired cycle of length l can be constructed by $\langle x, P[x, y], y, y^{(n-2)}, P[y^{(n-2)}, x^{(n-2)}], x^{(n-2)}, x\rangle$ for any integer $= l_0 + 2 + l_1 \in \{2^{n-1}+3, 2^{n-1}+4, \ldots, 2^n - 1\}$ (see Fig. 2(e) with $2^{n-1} - 3 \le l_0 \le 2^{n-1} - 1$).

Case 2.2.2: $F_0 \not\subset \{(v, v^{(n-1)}) | v \in V(TQ_{n-2}^{0,0})\}$. Let $(x', y') \in F_0 - \{(v, v^{(n-1)}) | v \in V(TQ_{n-2}^{0,0})\}$. Since $|F_0 - \{(x', y')\}| = n-3$ and $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}$ is $(n-3)$-edge fault-tolerant 6-vertex-pancyclic, there exists a cycle C_0 of length l'_0 in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x', y')\})$ such that $z \in V(C_0)$ for any integer l'_0 between 6 and 2^{n-1}. If $(x', y') \in E(C_0)$, then let $x = x'$ and $y = y'$; otherwise, let $(x, y) \in E(C_0) - \{(v, v^{(n-1)}) | v \in V(TQ_{n-2}^{0,0})\}$. Clearly, we have $x, y \in V(TQ_{n-2}^{0,0})$ or $x, y \in V(TQ_{n-2}^{1,0})$. The rest of the discussion is the same as Case 2.2.1.

□

By Lemma 10, Lemma 11, and Lemma 12, we have following theorem.

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Theorem 1. TQ_n is $(n-2)$ edge fault-tolerant 6-vertex-pancyclic, where $n \geq 3$ is an odd integer.

The result of Theorem 1 is optimal since there are distributions of $n-2$ edge faults over a TQ_n such that no fault-free cycles of length four or five contain some specific vertex z in the faulty TQ_n. Consider that $z = 0^n$ (n consecutive 0's). First, we show that the situation that no fault-free cycles of length four contain z. Suppose that (z, z^d) is faulty if and only if $d \notin \{0, n-2\}$. Thus, every fault-free cycle contains z must contains (z, z^0) and $(z, z^{(n-2)})$. Assume that there exists a fault-free cycle of length four contains (z, z^0) and $(z, z^{(n-2)})$, then this cycle should be $\langle z, z^0, u, z^{(n-2)}, z\rangle$ for some vertex $u \in V(TQ_n)$. If $u \in V(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0})$, then $(u, z^{(n-2)})$ is an edge that connects $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}$ and $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$, i.e., $(u, z^{(n-2)})$ is an $(n-2)$-edge, which is a contradiction since $(z, z^{(n-2)})$ is already an $(n-2)$-edge (see Fig. 4(a)). If $u \notin V(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0})$, i.e., $u \in V(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$, then (z^0, u) is an $(n-2)$-edge. That is, $u = 010^{n-3}1$. However, it is easy to see that there is no edge between vertex u ($= 010^{n-3}1$) and $z^{(n-2)}$ ($= 110^{n-2}$) (see Fig. 4(b)). As a result, we can conclude no fault-free cycles of length four contain z.

Then, we show the situation that no fault-free cycles of length five contain z. Suppose that (z, z^d) is faulty if and only if $d \notin \{1, n-2\}$. Thus, every cycle contains z must contains (z, z^1) and $(z, z^{(n-2)})$. Assume that there exists a fault-free cycle of length five contains (z, z^1) and $(z, z^{(n-2)})$, then this cycle should be $\langle z, z^1, u, v, z^{(n-2)}, z\rangle$ for some vertex $u, v \in V(TQ_n)$. If $u, v \in V(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0})$, then $(v, z^{(n-2)})$ is an $(n-2)$-edge, which is a contradiction (since $(z^{(n-2)}, z)$ is already an $(n-2)$-edge) (see Fig. 4(c)). If $u, v \in V(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$, then $u = z^{1(n-2)} = 110^{n-5}110$. Note that $z^{(n-2)} = 110^{n-2}$. It is not difficult to see that there is an edge between u and $z^{(n-2)}$. Therefore, $\langle u, v, z^{(n-2)}, u\rangle$ is a cycle of length three, which is a contradiction since there is no cycle of three in TQ_n (see Fig. 4(d)). If $u \in V(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0})$ and $v \in V(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$, then two cases are further considered:

Case 1: $u \in V(TQ_{n-2}^{0,0})$ and $v \in V(TQ_{n-2}^{1,1})$. Let $u = 00u_{n-3}u_{n-4}\ldots u_0$, then $v = 11u_{n-3}u_{n-4}\ldots u_0$. Note that $(v, z^{(n-2)}) = (11u_{n-3}u_{n-4}\ldots u_0, 110^{n-2}) \in E(TQ_{n-2}^{1,1})$. Thus, it is not difficult to see that $(00u_{n-3}u_{n-4}\ldots u_0, 0^n) = (u, z) \in E(TQ_{n-2}^{0,0})$. Consequently, $\langle z, z^1, u, z\rangle$ is a cycle of length three, which is a contradiction (see Fig. 4(e)).

Case 2: $u \notin V(TQ_{n-2}^{0,0})$ or $v \notin V(TQ_{n-2}^{1,1})$. That is, $u = z^{1(n-1)} = 100^{n-5}110$ or $v = z^{(n-2)(n-1)} = 010^{n-2}$. If $u = 100^{n-5}110$, then $v = u^{(n-2)} = 010^{n-5}110$. Thus, there is not an edge between $z^{(n-2)}$ ($= 110^{n-2}$) and v, which is a contradiction (see Fig. 4(f)). With similar discussion, we can prove that there is not an edge between z^1 and u when $v = 010^{n-2}$.

Therefore, we can conclude that TQ_n is neither $(n-2)$-edge fault-tolerant 4-vertex-pancyclic nor $(n-2)$-edge fault-tolerant 5-vertex-pancyclic, where $n \geq 5$ is an odd integer

V. Discussion and Conclusion

Linear arrays and rings, two of the most fundamental networks for parallel and distributed computation, are suitable for developing simple algorithms with low communication costs. The *pancyclicity* of a network represents its power of embedding rings of all possible lengths. In this paper, using inductive proofs, we showed that TQ_n is $(n-2)$-edge fault-tolerant 6-vertex-pancyclic. In other words, every vertex of an TQ_n with at most $n-2$ faulty edges lies on a fault-free cycle of every length from 6 to 2^n. In addition, we also showed that our result is optimal.

TQ_n is $(n-2)$ fault-tolerant 4-pancyclic (thus, $(n-2)$-edge fault-tolerant 4-pancyclic), where $n \geq 3$ is an odd integer [18]. We have shown that TQ_n is $(n-2)$-edge fault-tolerant 6-vertex-pancyclic, but not $(n-2)$-edge fault-tolerant 5-vertex-pancyclic. A topic for further research is to explore the vertex-pancyclicity and/or edge-pancyclicity of twisted cubes in the presence of hybrid faults. Moreover, we have an interesting open problem as follows. Suppose a graph G is m fault-tolerant i-pancyclic and m fault-tolerant j-vertex-pancyclic. Clearly, we have $j \leq i$. What kind of G can cause $j = i$? Or is it impossible?

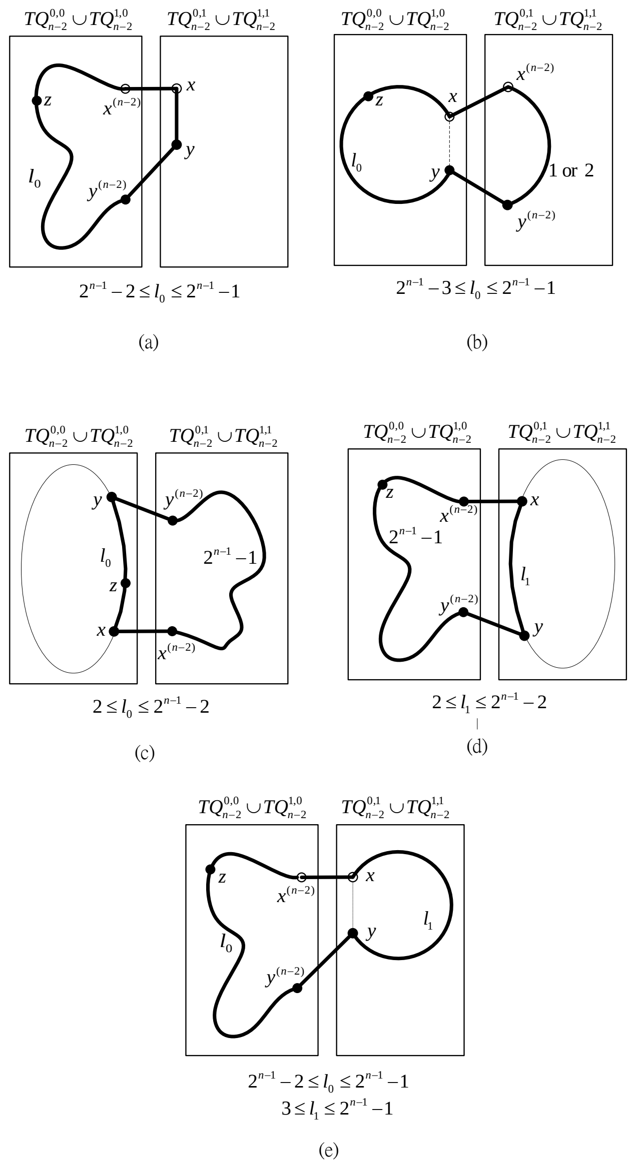

Fig 2. Construction of cycles in Cases 1 and 2.2 in Lemma 12.

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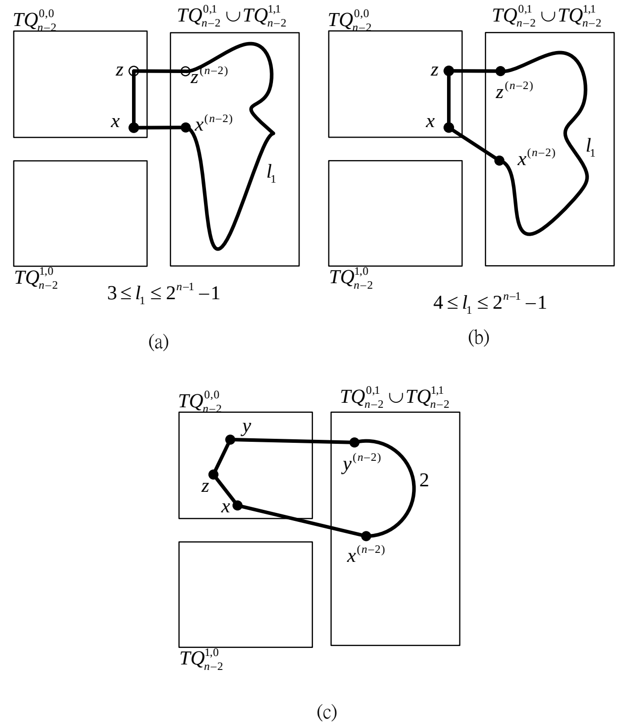

Fig 3. Construction of cycles in Cases 2.1 in Lemma 12.

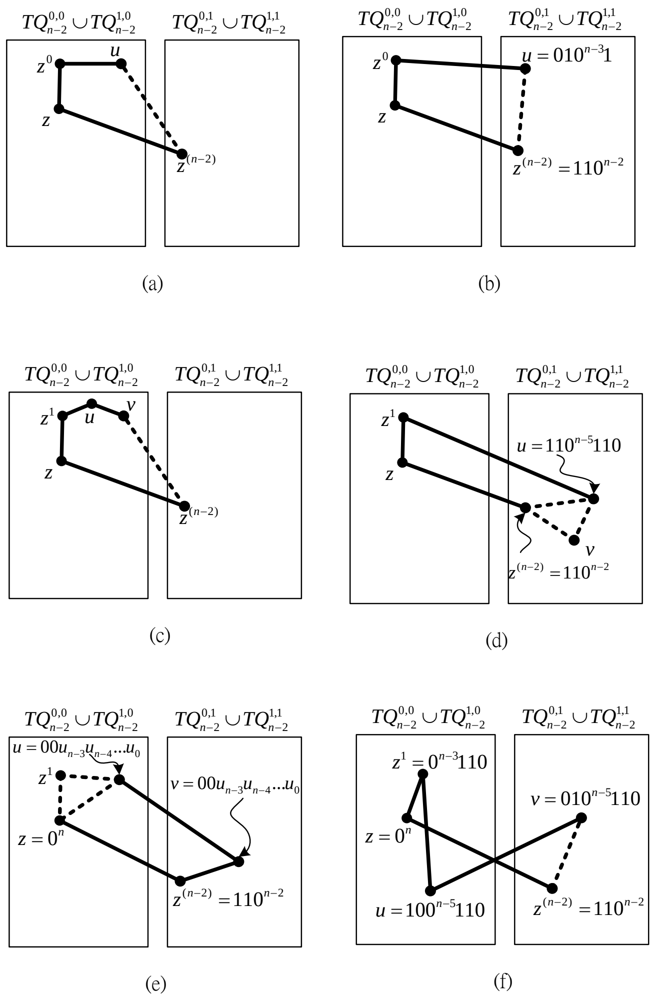

Fig. 4. The explanation that TQ_n is neither $(n-2)$-edge fault-tolerant 4-vertex-pancyclic nor $(n-2)$-edge fault-tolerant 5-vertex-pancyclic.

REFERENCES

[1] S. Abraham and K. Padmanabhan, "The twisted cube topology for multiprocessors: a study in network asymmetry," *Journal of Parallel and Distributed Computing*, vol. 13, pp. 104-110, 1991.

[2] C. P. Chang, J. N. Wang, L. H. Hsu, "Topological properties of twisted cubes," *Information Sciences*, vol.113, 147-167, 1999.

[3] J. M. Chang and J. S. Yang, "Fault-tolerant cycle-embedding in alternating group graphs ," *Applied Mathematics and Computation*, vol. 197, pp. 760-767, 2008.

[4] J. M. Chang, J. S. Yang, Y. L. Wang, and Yuwen Cheng, "Panconnectivity, fault- tolerant Hamiltonicity and Hamiltonian-connectivity in alternating group graphs," *Networks*, vol. 44, no. 4, pp. 302-310, 2004.

[5] J. Fan, X. Jia, and X. Lin, "Optimal embeddings of paths with various length in twisted cubes," *IEEE Transactions on Parallel and Distributed Systems*, vol. 18, no. 4, pp. 511-521, 2007.

[6] J. Fan, X. Jia, and X. Lin, "Embedding of cycles in twisted cubes with edge-pancyclic," *Algorithmica*, vol. 51, no. 3, pp. 264-282. 2008.

[7] J. Fan, X. Lin, and X. Jia, "Node-pancyclicity and edge-pancyclicity of crossed cubes," *Infromation Processing Letters*, vol. 93, pp. 133-138, 2004.

[8] J. Fan, X. Lin, X. Jia, and R. W. H. Lau, "Edge-pancyclicity of twisted cubes," *Lecture Notes in Computer Science* (v 3827 LNCS, *Algorithms and Computation - 16th International Symposium*), *ISAAC 2005*, *Proceedings*, 2005, p 1090-1099.

[9] J. S. Fu, "Fault-free Hamiltonian cycles in twisted cubes with conditional link faults," *Theoretical Computer Science*, vol. 407, no. 1-3, pp. 318-329, 2008.

[10] S. Y. Hsieh and N. W. Chang, "Hamiltonian path embedding and pancyclicity on the Möbius cube with both faulty nodes and faulty edges," *IEEE Transactions on Computers*, vol. 55, no. 7, pp. 854- 863, 2006.

[11] S. Y. Hsieh and C. W. Lee, "Conditional edge-fault hamiltonicity of matching composition networks" *IEEE Transactions on Parallel and Distributed Systems*, vol. 20, no. 4, pp. 581-592, 2009.

[12] S. Y. Hsieh and C. W. Lee, "Pancyclicity of restricted hypercube-like networks under the conditional fault model," *SIAM Journal on Discrete Mathematics*, accepted.

[13] P. A. J. Hibers, M. R. J. Koopman, and J. L. A. van de Snepscheut, "The twisted cube," in *Proceedings of the Conference on Parallel Architectures and Languages Europe, Lecture Notes in Computer Science*, pp. 152 – 159, Springer, 1987.

[14] W. T. Huang, J. M. Tan, C. N. Hung, and L. H. Hsu, "Fault-tolerant Hamiltonianicity of twisted cubes," *Journal of Parallel and Distributed Computing*, vol. 62, pp. 591-604, 2002.

[15] P. L. Lai and H. C. Hsu, "The two-equal-disjoint path cover problem of matching composition network," *Information Processing Letters*, vol. 107, no. 1, pp. 18-23, 2008.

[16] C. J. Lai and C. H. Tsai, "Embedding a family of meshes into twisted cubes," *Information Processing Letters*, vol. 108, no. 5, pp. 326-330, 2008.

[17] F. T. Leighton, *Introduction to Parallel Algorithms and Architecture: Arrays· Trees· Hypercubes*, Morgan Kaufman, CA, 1992.

[18] T. K. Li, M.C. Yang, J.M. Tan, and L.H. Hsu, "On embedding cycle in faulty twisted cubes," *Information Sciences*, Vol. 176, pp. 676-690, 2006.

[19] M. Ma, G. Liu, and J. M. Xu, "Panconnectivity and edge-fault-tolerant pancyclicity of augmented cubes," *Parallel Computing*, vol. 33, pp. 36-42, 2007.

[20] J. H. Park, "Panconnectivity and edge-pancyclicity of faulty recursive circulant $G(2^m,4)$," *Theoretical Computer Science*, vol. 390, pp. 70-80, 2008.

[21] J. -H. Park, H. -S. Lim, and H. -C. Kim, "Panconnectivity and pancyclicity of hypercube-like interconnection networks with faulty elements," *Theoretical Computer Science*, vol. 377, pp. 170-180, 2007.

[22] Y. Saad and M. H. Schultz, "Topological properties of hypercubes," *IEEE Transactions on Computers*, vol. 37, no. 7, pp. 867-872, Jul. 1988.

[23] M. Xu and J. M. Xu, "Edge-pancyclicity of Mobius cubes," *Information processing Letters*, vol. 96, no. 4, pp. 136-140, 2005.

[24] D. B. West, *Introduction to Graph Theory* (2nd *Edition*), Prenctice Hall, Upper Saddle River, 2001.

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