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Matchings Extend to Perfect Matchings on Hypercube Networks

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Abstract—**In this work, we investigate in the problem of perfect matchings with prescribed matchings in the n-dimensional hypercube network Q_n. We obtain the following contributions: For any arbitrary matching with at most $n-1$ edges, it can be extended to a perfect matching of Q_n for $n \geq 1$. Furthermore, for any arbitrary non-forbidden matching with n edges, it also can be extended to a perfect matching of Q_n for $n \geq 1$. It is shown by J. Fink in 2007 that any arbitrary perfect matching of the n-dimensional hypercube Q_n, $n \geq 2$, can be extended to a Hamiltonian cycle. Therefore, it leads to a further result that for any arbitrary non-forbidden matching with at most n edges, it can be extended to a Hamiltonian cycle of Q_n for $n \geq 2$.**

Keywords: matching, perfect matching, Hamiltonian cycle, hypercube

1 Introduction

The underlying topology of an interconnection network is usually modeled as a graph $G = (V, E)$, in which the vertex set $V(G)$ represents processors and the edge set $E(G)$ represents connections between processors. We use graphs and networks interchangeably. For the fundamental graph definitions and notations we follow [2]. $G = (V, E)$ is a simple graph if V is a finite set and E is a subset of $\{(a, b)|(a, b)$ is an unordered pair of $V\}$. We say that V is the vertex set and E is the edge set. The neighborhood of v, $N(v)$, is $\{x|(v, x) \in E\}$. Two vertices are *adjacent* if $(a, b) \in E$. A *path* is a sequence of adjacent edges $(v_0, v_1), (v_1, v_2), \cdots, (v_{m-1}, v_m)$, written as $\langle v_0, v_1, \cdots, v_m\rangle$, in which all the vertices $v_0, v_1, \cdots, v_m$ are distinct. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. Hence, the length of a path or cycle is the number of edges in the path or cycle. Let u and v be two vertices in a graph, the *distance* from u to v, denoted by $dist(u, v)$, is the minimum length of any path from u to v. A cycle is a *Hamiltonian cycle* if it traverses every vertex of G exactly once [4, 5].

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A *matching* M in graph G is a set of pairwise non-adjacent edges, that is, every vertex is incident with at most one edge of M. A vertex is *matched* if it is incident to an edge in the matching M. Otherwise the vertex is *unmatched*. A *maximum matching* is a matching that contains the largest possible number of edges. The *matching number* of a graph is the size of a maximum matching. A *perfect matching* is a matching which matches all vertices of the graph. That is, every vertex of the graph is incident to exactly one edge of the matching. A matching is a *forbidden matching* if every vertex of $N(v)$ is matched for some unmatched vertex $v \in V(G)$. Otherwise, it is a *non-forbidden matching*. A perfect matching is also a *minimum-size edge cover*. For any graph G without isolated vertices, the sum of the *edge covering number* and the matching number equals to the number of vertices. If the graph G has a perfect matching, then both of the edge cover number and the matching number are equal to $|V(G)|/2$ [11].

Perfect matchings can be applied to *network disclosure attacks*. The *Perfect Matching Disclosure Attack* performs a high rate of success when tracing messages sent through a threshold mix in arbitrary scenarios [12]. Another related work concerning perfect matchings is *matching preclusion*. A matching M in graph G is called an *almost perfect matching* if M contains exactly $(|V(G)|-1)/2$ edges. An edge set F in graph G is called a *matching preclusion set* if $G \setminus F$ has neither a perfect matching nor an almost perfect matching. If a network has a large size of matching preclusion set, each vertex (process) in this network will has a specific partner in the event of edge (link) faults and the network will be robust. On the side, matching preclusion is also related to the problem of *connectivity* of networks [3, 10].

The *hypercube* is a popular network because of its attractive properties, including regularity, symmetry, small diameter, strong connectivity, recursive construction, partitionability, and relatively low link complexity [1, 9]. It has been widely used in parallel systems, such as the *Connection machines*, *Symult S-series*, *Intel iPSC*, *iPSC/2*, and *SGI Origin 2000* [6, 13, 14]. The formal definition of an n-dimensional hypercube is given as follows. Each vertex v in Q_n can be distinctly labeled by a binary n-bit string, $v = v_n v_{n-1} \cdots v_1$. For $1 \leq i \leq n$, we use v^i to denote the binary string $v_n \cdots \bar{v}_i \cdots v_1$.

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The Q_n consists of all n-bit binary strings representing its vertices. Two vertices u and v are adjacent if and only if $v = u^i$ with some i. An n-dimensional hypercube Q_n can be constructed from two identical $(n-1)$-dimensional hypercubes, $Q_{n-1}^{i,0}$ and $Q_{n-1}^{i,1}$, for some $1 \leq i \leq n$, where $V(Q_{n-1}^{i,0}) = \{v_n v_{n-1} \cdots v_1 | v_i = 0\}$ and $V(Q_{n-1}^{i,1}) = \{v_n v_{n-1} \cdots v_1 | v_i = 1\}$. The vertex set of Q_n is $V(Q_n) = Q_{n-1}^{i,0} \cup Q_{n-1}^{i,1}$, and the edge set is $E(Q_n) = E(Q_{n-1}^{i,0}) \cup E(Q_{n-1}^{i,1}) \cup P$ where P is a set of edges connecting the vertices of $Q_{n-1}^{i,0}$ and $Q_{n-1}^{i,1}$ in a one to one fashion. An n-dimensional hypercube can be represented as $Q_n = Q_{n-1}^{i,0} \bigoplus Q_{n-1}^{i,1}$ with $i \in 1, 2, \cdots, n$. We need some more terms, let v be a vertex of $Q_{n-1}^{i,0}$ and $Q_{n-1}^{i,1}$, we use v' to denote the corresponding vertex of v in $Q_{n-1}^{i,1}$ and $Q_{n-1}^{i,0}$, respectively. That is, (v, v') is an edge of Q_n, one of u, v is in $Q_{n-1}^{i,0}$, and the other is in $Q_{n-1}^{i,1}$. We notice that the hypercube Q_n is vertex-symmetric and edge-symmetric for $n \geq 1$.

Some results on perfect matchings of hypercubes are discussed in [7, 8, 10]. In this paper, we shall show that for any arbitrary matching with at most $n-1$ edges of the hypercube Q_n for $n \geq 1$, it can be extended to a perfect matching of Q_n. Furthermore, for any arbitrary matching with exactly n edges of the hypercube Q_n for $n \geq 1$, it can be extended to a perfect matching of Q_n, except it is a forbidden matching of Q_n. It has shown that any perfect matching of the hypercube Q_n, $n \geq 2$, can be extended to a Hamiltonian cycle [7]. As a result, we have a further result that any non-forbidden matching with at most n edges of the hypercube Q_n, $n \geq 2$, can be extended to a Hamiltonian cycle.

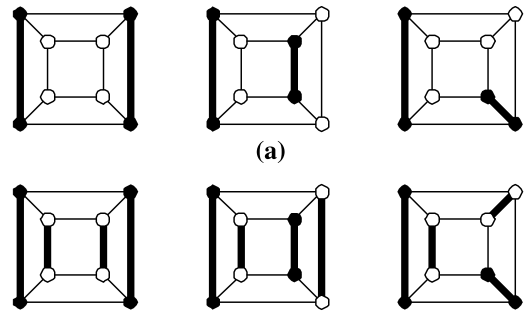
The rest of this paper is organized as follows. Section 2 shows the perfect matchings with prescribed matchings in the n-dimensional hypercube Q_n for $n \geq 1$. In Section 3, we give the conclusion remarks.

2 Perfect Matchings of Hypercubes

Lemma 1 *For the 3-dimensional hypercube Q_3, let M be an arbitrary matching with at most two edges in the hypercube Q_3. Then, the matching M can be extended to a perfect matching of Q_3.*

Proof. First, suppose that M contains exactly zero or one edge, it is clear that M can be extended to a perfect matching of Q_3. Now, suppose that M has exactly two edges. Figure 1.(a) shows all the non-isomorphic cases that M contains exactly two edges in the hypercube Q_3. Figure 1.(b) shows the perfect matchings of Q_3. As a result, the proof of this lemma is complete. ◇

Theorem 1 *Assume that $n \geq 1$ is an integer. Let M be an arbitrary matching with at most $n-1$ edges in the hypercube Q_n. Then, the matching M can be extended to a perfect matching of Q_n.*



(a)

(b)

Figure 1: Perfect matchings of the hypercube Q_3.

Proof. We prove it by induction on n. Suppose that $n = 1, 2$, it is clear that for any matching with at most $n-1$ edges, it can be extended to a perfect matching of Q_n. Suppose that $n = 3$, by Lemma 1, for any matching with at most two edges, it can be extended to a perfect matching of Q_3.

Assume that the theorem is true for n, which means that for any arbitrary matching M with at most $n-1$ edges, M can be extended to a perfect matching of Q_n. Now, we shall show that in the hypercube Q_{n+1}, let M' be an arbitrary matching with $m \leq n$ edges in Q_{n+1}, the matching M' can be extended to a perfect matching of Q_{n+1}.

By the definition of hypercubes, for each edge in Q_{n+1}, the labels of two endpoint vertices are different with exactly one bit. Since M' contains at most n edges, there exists one dimension $i \in 1, 2, \cdots, n+1$ in $Q_{n+1} = Q_n^{i,0} \bigoplus Q_n^{i,1}$ such that each edge of M' is distributed in $Q_n^{i,0}$ or $Q_n^{i,1}$. Let $M_0 = M' \cap Q_n^{i,0}$ and $M_1 = M' \cap Q_n^{i,1}$. That is to say that M' has $m_0 = |M_0|$ edges in $Q_n^{i,0}$, $m_1 = |M_1|$ edges in $Q_n^{i,1}$, and $m_0 + m_1 = m$. We may without loss of generality assume that $m_0 \geq m_1$. We divide the proof into the following two cases.

Case 1: $m_0 \geq m_1 > 0$.
Notice that $m_0 < n$ and $m_1 < n$ since $m_0 \geq m_1 > 0$. See Figure 2.(a). By inductive hypothesis, M_0 can be extended to a perfect matching of $Q_n^{i,0}$, say PM_0, and M_1 can be extended to a perfect matching of $Q_n^{i,1}$, say PM_1. See Figure 2.(b). Therefore, $PM_0 \cup PM_1$ is a perfect matching of Q_{n+1}.

Case 2: $m_1 = 0$.
That is, all the n edges of M' are distributed in $Q_n^{i,0}$. Suppose that $m_0 < n$, by inductive hypothesis, M' can be extended to a perfect matching of $Q_n^{i,0}$, and so M' can be extended to a perfect matching of Q_{n+1}. Hence, we may only consider the case that $m_0 = n$.

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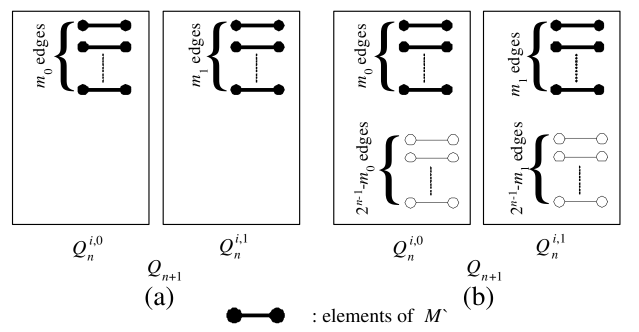

Figure 2: Case 1: $m_0 \geq m_1 > 0$.

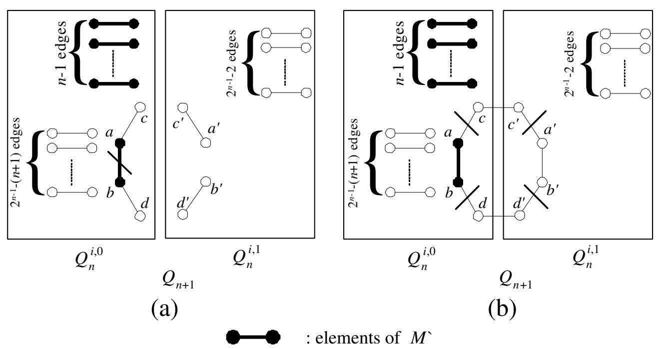

Figure 3: Case 2: $m_1 = 0$.

Let (a, b) be an edge in M'. By inductive hypothesis, $M' \setminus \{(a, b)\}$ can be extended to a perfect matching of $Q_n^{i,0}$, say PM_0, since $m - 1 \leq n - 1$. If PM_0 contains the edge (a, b), it is clear that M' can be extended to a perfect matching of Q_{n+1}. Here, we may consider that PM_0 do not contain the edge (a, b). Let (a, c) and (b, d) be two edges of PM_0. By inductive hypothesis with $n \geq 3$, $\{(a', c'), (b', d')\}$ can be extended to a perfect matching of $Q_n^{i,1}$, say PM_1. See Figure 3.(a). Therefore, $(PM_0 \cup PM_1 \cup \{(a, b), (a', b'), (c, c'), (d, d')\}) \setminus \{(a, c), (b, d), (a', c'), (b', d')\}$ is a perfect matching of Q_{n+1} as Figure 3.(b), and this theorem follows. $\diamondsuit$

Lemma 2 *For the 3-dimensional hypercube Q_3, let M be an arbitrary matching with exactly three edges in the hypercube Q_3. Then, the matching M can be extended to a perfect matching of Q_3, except M is a forbidden matching of Q_3.*

Proof. Figure 4.(a) shows all the non-isomorphic cases that M contains exactly three edges in the hypercube Q_3. Figure 4.(b) shows the perfect matchings of Q_3 with the matching M. As a result, the proof of this lemma is complete. $\diamondsuit$

Lemma 3 *For the 3-dimensional hypercube Q_3, let M be an arbitrary matching with at most one edge in the hypercube Q_3, and x, y be any two unmatched vertices with $dist(x, y)$ is odd. Then, the matching M can be extended to a perfect matching of $Q_3 \setminus \{x, y\}$.*

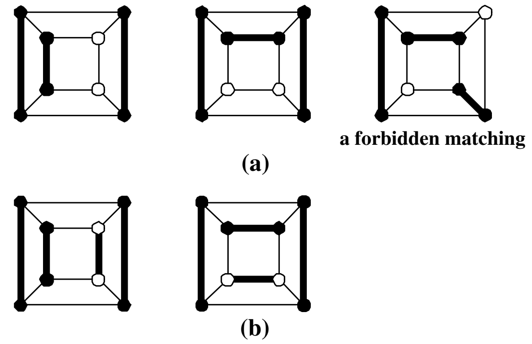

Figure 4: Perfect matchings of the hypercube Q_3.

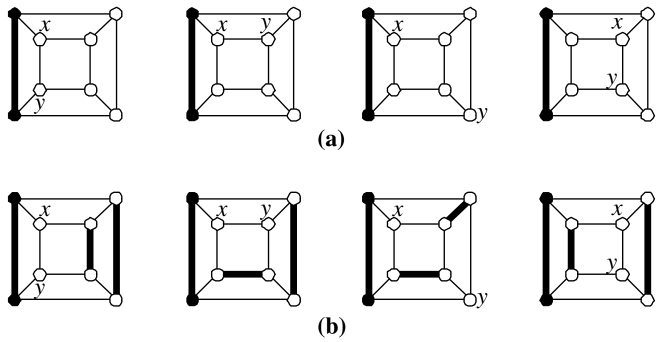

Figure 5: Perfect matchings of the hypercube $Q_3 \setminus \{x, y\}$.

Proof. Firstly, suppose that $|M| = 0$, it is clear that for any two unmatched vertices x, y with $dist(x, y)$ is odd, there is a perfect matching of $Q_3 \setminus \{x, y\}$. Now, suppose that $|M| = 1$. Figure 5.(a) shows all the non-isomorphic cases that M contains exactly one edge and x, y be any two unmatched vertices with $dist(x, y)$ is odd in the hypercube Q_3. Figure 5.(b) shows the perfect matchings of $Q_3 \setminus \{x, y\}$ with the matching M. Hence, the proof of this lemma is complete. $\diamondsuit$

Lemma 4 *Assume that $n \geq 3$ is an integer. Let M be an arbitrary matching with at most $n-2$ edges in the hypercube Q_n, and x, y be any two unmatched vertices with $dist(x, y)$ is odd. Then, the matching M can be extended to a perfect matching of $Q_n \setminus \{x, y\}$.*

Proof. We prove it by induction on n. Suppose that $n = 3$, by Lemma 3, for any matching with at most one edge and x, y be any two unmatched vertices with $dist(x, y)$ is odd, it can be extended to a perfect matching of $Q_3 \setminus \{x, y\}$.

Assume that the theorem is true for n, which means that for any arbitrary matching M with at most $n - 2$ edges and x, y be any two unmatched vertices with $dist(x, y)$ is

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odd, M can be extended to a perfect matching of $Q_n \setminus \{x, y\}$. Now, we shall show that in the hypercube Q_{n+1}, let M' be an arbitrary matching with $m \leq n-1$ edges in Q_{n+1} and x, y be any two unmatched vertices with $dist(x, y)$ is odd, the matching M' can be extended to a perfect matching of $Q_{n+1} \setminus \{x, y\}$.

By the definition of hypercubes, for each edge in Q_{n+1}, the labels of two endpoint vertices are different with exactly one bit. Since M' contains at most $n-1$ edges, there exists one dimension $i \in 1, 2, \cdots, n+1$ of $Q_{n+1} = Q_n^{i,0} \bigoplus Q_n^{i,1}$ such that each edge of M' is distributed in $Q_n^{i,0}$ or $Q_n^{i,1}$. Let $M_0 = M' \cap Q_n^{i,0}$ and $M_1 = M' \cap Q_n^{i,1}$. That is to say that M' has $m_0 = |M_0|$ edges in $Q_n^{i,0}$, $m_1 = |M_1|$ edges in $Q_n^{i,1}$, and $m_0 + m_1 = m$. We may without loss of generality assume that $m_0 \geq m_1$ and divide the proof into the following three cases.

Case 1: Both x and y are in $Q_n^{i,1}$.
By Theorem 1, M_0 can be extended to a perfect matching of $Q_n^{i,0}$, say PM_0, since $m_0 \leq m \leq n-1$. Since $m_0 \geq m_1$ and $m \leq n-1$, $m_1 \leq n-2$. By inductive hypothesis, M_1 can be extended to a perfect matching of $Q_n^{i,1} \setminus \{x, y\}$, say PM_1'. Therefore, $PM_0 \cup PM_1'$ is a perfect matching of $Q_{n+1} \setminus \{x, y\}$.

Case 2: Both x and y are in $Q_n^{i,0}$.
Case 2.1: $m_0 \leq n-2$.
By inductive hypothesis, M_0 can be extended to a perfect matching of $Q_n^{i,0} \setminus \{x, y\}$, say PM_0'. By Theorem 1, M_1 can be extended to a perfect matching of $Q_n^{i,1}$, say PM_1. Therefore, $PM_0' \cup PM_1$ is a perfect matching of $Q_{n+1} \setminus \{x, y\}$.

Case 2.2: $m_0 = n-1$.
Let (a, b) be an edge of M_0, by inductive hypothesis, $M_0 \setminus \{(a, b)\}$ can be extended to a perfect matching of $Q_n^{i,0} \setminus \{x, y\}$, say PM_0', since $m_0 - 1 = n-2$. Firstly, suppose that $(a, b) \in PM_0'$. By Theorem 1, $Q_n^{i,1}$ has a perfect matching, say PM_1. Then, $PM_0' \cup PM_1$ is a perfect matching of $Q_{n+1} \setminus \{x, y\}$. Now suppose that $(a, b) \notin PM_0'$. We may let $\{(a, c), (b, d)\} \subseteq PM_0'$. By Theorem 1, $\{(a', c'), (b', d')\}$ can be extended to a perfect matching of $Q_n^{i,1}$, say PM_1, as Figure 6.(a). Then, $(PM_0' \cup PM_1 \cup \{(a, b), (a', b'), (c, c'), (d, d')\}) \setminus \{(a, c), (b, d), (a', c'), (b', d')\}$ is a perfect matching of $Q_{n+1} \setminus \{x, y\}$ as Figure 6.(b).

Case 3: x is in $Q_n^{i,0}$ and y is in $Q_n^{i,1}$.
Case 3.1: $m_0 \leq n-2$.
In Q_{n+1}, since $n \geq 3$, there exists an edge (c, c') between $Q_n^{i,0}$ and $Q_n^{i,1}$ such that c is an unmatched vertex of $Q_n^{i,0}$, $c \neq x$, $dist(c, x)$ is odd, c' is an unmatched vertex of $Q_n^{i,1}$, $c' \neq y$, and $dist(c', y)$ is odd. See Figure 7.(a). By inductive hypothesis, M_0 can be extended to a perfect matching of $Q_n^{i,0} \setminus \{x, c\}$, say PM_0', and M_1 can be extended to a perfect matching of $Q_n^{i,1} \setminus \{y, c'\}$, say PM_1'. Therefore, $PM_0' \cup PM_1' \cup \{(c, c')\}$ is a perfect matching

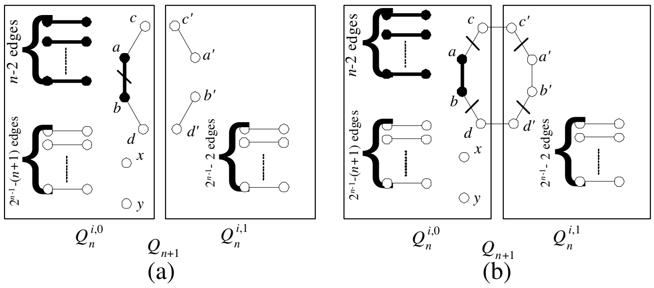

Figure 6: Case 2.2: $m_0 = n-1$.

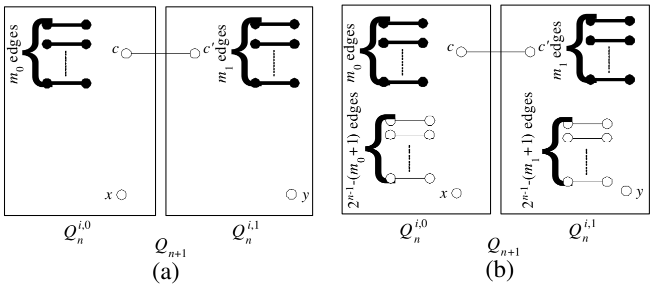

Figure 7: Case 3.1: $m_0 \leq n-2$.

of $Q_{n+1} \setminus \{x, y\}$ as Figure 7.(b).

Case 3.2: $m_0 = n-1$.
By Theorem 1, M_0 can be extended to a perfect matching of $Q_n^{i,0}$, say PM_0. Let (x, c) be an edge of PM_0. By inductive hypothesis, $Q_n^{i,1} \setminus \{c', y\}$ has a perfect matching, say PM_1', since $dist(c', y)$ is odd. Therefore, $(PM_0 \cup PM_1' \cup \{(c, c')\}) \setminus \{(c, x)\}$ is a perfect matching of $Q_{n+1} \setminus \{x, y\}$. $\diamondsuit$

In the following theorem, we shall show that for any arbitrary non-forbidden matching with $m \leq n$ edges in the hypercube Q_n, M can be extended to a perfect matching of Q_n, except M is a forbidden matching of Q_n.

Theorem 2 *Assume that $n \geq 1$ is an integer. Let M be an arbitrary matching with at most n edges in the hypercube Q_n. Then, the matching M can be extended to a perfect matching of Q_n, except M is a forbidden matching of Q_n.*

Proof. We prove it by induction on n. Suppose that $n = 1, 2$, it is clear that for any matching with at most n edges, it can be extended to a perfect matching of Q_n. Suppose that $n = 3$, by Lemma 1 and 2, for any matching with at most three edges, it can be extended to a perfect matching of Q_3, except it is a forbidden matching of Q_3.

Assume that the theorem is true for n, which means that for any arbitrary matching M with at most n edges, M can be extended to a perfect matching of Q_n, except M

is a forbidden matching of Q_n. Now, we shall show that in the hypercube Q_{n+1}, let M' be an arbitrary matching with $m \le n+1$ edges in Q_{n+1}, the matching M' can be extended to a perfect matching of Q_{n+1}, except M' is a forbidden matching of Q_{n+1}. By Theorem 1, M' can be extended to a perfect matching of Q_{n+1} if $m \le n$. Hence, we may only consider that $m = n+1$ in the following proof of this theorem.

By the definition of hypercubes, the labels of two end-point vertices of each edge in Q_{n+1} is exactly different with one bit. Since M' contains $n+1$ edges, there exists one dimension $i \in \{1, 2, \cdots, n+1\}$ of $Q_{n+1} = Q_n^{i,0} \bigoplus Q_n^{i,1}$ such that at most one edge of M' is distributed between $Q_n^{i,0}$ and $Q_n^{i,1}$. Let $M_0 = M' \cap Q_n^{i,0}$, $M_1 = M' \cap Q_n^{i,1}$, and $M_c = M' \setminus (M_0 \cup M_1)$. That is that there are $m_0 = |M_0|$ edges in $Q_n^{i,0}$, $m_1 = |M_1|$ edges in $Q_n^{i,1}$, $m_c = |M_c| = 0$ or 1 edge between $Q_n^{i,0}$ and $Q_n^{i,1}$, and $m_0 + m_1 + m_c = m$. We may without loss of generality assume that $m_0 \ge m_1$. Hence, $m_1 \le \lfloor (n+1)/2 \rfloor \le n-1$ for $n \ge 3$. We divide the proof into the following three cases.

Case 1: $m_0 = n+1$**.**
Assume that $M_0 = \{(a_i, b_i) | i = 1, 2, \cdots, n+1\}$. We let $M_1' = \{(a_i', b_i') | i = 1, 2, \cdots, n+1\}$ and $P = \{(c, c') | c \in (V(Q_n^{i,0}) \setminus \{a_i, b_i | i = 1, 2, \cdots, n+1\})\}$. Then, $M_0 \cup M_1' \cup P$ is a perfect matching of Q_{n+1}.

Case 2: $m_0 = n$**.**
Firstly, suppose that $m_c = 1$. Assume that $M_0 = \{(a_i, b_i) | i = 1, 2, \cdots, n\}$. We let $M_1' = \{(a_i', b_i') | i = 1, 2, \cdots, n\}$ and $P = \{(c, c') | c \in (V(Q_n^{i,0}) \setminus \{a_i, b_i | i = 1, 2, \cdots, n\})\}$. Then, $M_0 \cup M_1' \cup P$ is a perfect matching of Q_{n+1}. Now, suppose that $m_1 = 1$. We divide this case into two subcases.

Case 2.1: M_0 **is a non-forbidden matching of** $Q_n^{i,0}$**.**
By inductive hypothesis, M_0 can be extended to a perfect matching of $Q_n^{i,0}$, say PM_0. By another, since $m_1 = 1$ and $n \ge 3$, M_1 can be extended to a perfect matching of $Q_n^{i,1}$, say PM_1. Therefore, $PM_0 \cup PM_1$ is a perfect matching of Q_{n+1}.

Case 2.2: M_0 **is a forbidden matching of** $Q_n^{i,0}$**.**
Let $M_0 = \{(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)\}$ and c be the common neighbor of $a_1, a_2, \cdots, a_n$ in $Q_n^{i,0}$. Since $m_1 = 1$ and $n \ge 3$, among the $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$, there exists a matching element, say (a_n, b_n), and a neighbor unmatched vertex of b_n, say d, in $Q_n^{i,0}$ such that d' is an unmatched vertex. Here, it is clear that c' is also an unmatched vertex, otherwise M is a forbidden matching of Q_{n+1}. By inductive hypothesis, since $(M_0 \cup \{(b_n, d)\}) \setminus \{(a_n, b_n)\}$ is a non-forbidden matching of $Q_n^{i,0}$ with n edges, it can be extended to a perfect matching of $Q_n^{i,0}$, say PM_0. By Lemma 4, M_1 can be extended to a perfect matching of $Q_n^{i,1} \setminus \{c', d'\}$, say PM_1'. See Figure 8.(a). Therefore, $(PM_0 \cup PM_1' \cup \{(a_n, b_n), (c, c'), (d, d')\}) \setminus \{(a_n, c), (b_n, d)\}$ forms a perfect matching of Q_{n+1} as Figure 8.(b).

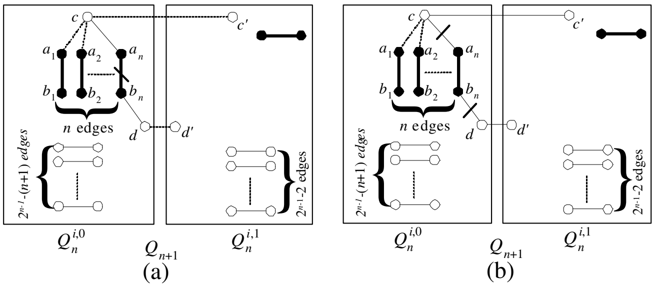

Figure 8: Case 2.2: M_0 is a forbidden matching of $Q_n^{i,0}$.

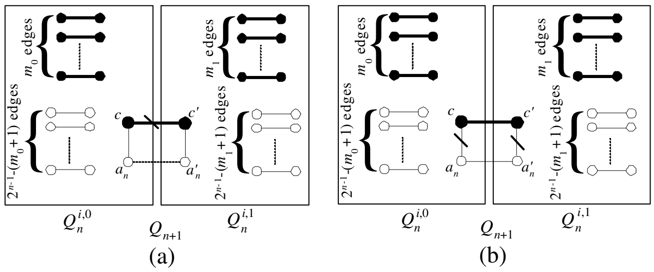

Figure 9: Case 3.1: a_n and a_n' are unmatched vertices.

Case 3: $m_0 \le n-1$**.**
Firstly, suppose that $m_c = 0$. M_0 can be extended to a perfect matching of $Q_n^{i,0}$, say PM_0. By another, since $m_1 \le n-1$ for $n \ge 3$, M_1 can be extended to a perfect matching of $Q_n^{i,1}$, say PM_1. Therefore, $PM_0 \cup PM_1$ is a perfect matching of Q_{n+1}. Now, suppose that $m_c = 1$. Let $M_c = \{(c, c')\}$ where c is in $Q_n^{i,0}$. Then, in $Q_n^{i,0}$, among the n neighbors of c, denoted by $a_1, a_2, \cdots, a_n$, if there exists at least an unmatched vertex, say a_n, such that $M_0 \cup \{(c, a_n)\}$ is a non-forbidden matching of $Q_n^{i,0}$ and a_n' is also an unmatched vertex as Figure 9.(a), this shall be discussed in the following Case 3.1. Otherwise, a_i or a_i' is matched in Q_{n+1} for $i = 1, 2, \cdots, n$ as Figure 10.(a), this shall be discussed in Case 3.2.

Case 3.1: a_n **and** a_n' **are unmatched vertices.**
Then, by inductive hypothesis, $M_0 \cup \{(c, a_n)\}$ can be extended to a perfect matching of $Q_n^{i,0}$, say PM_0. Since $m_0 \ge m_1$ and $m_c = 1$, $m_1 \le \lfloor \frac{(n+1)-1}{2} \rfloor \le n-2$ for $n \ge 3$. Hence, $M_1 \cup \{(c', a_n')\}$ can be extended to a perfect matching of $Q_n^{i,1}$, say PM_1. See Figure 9.(a). Therefore, $(PM_0 \cup PM_1 \cup \{(c, c'), (a_n, a_n')\}) \setminus \{(c, a_n), (c', a_n')\}$ is a perfect matching of Q_{n+1} as Figure 9.(b).

Case 3.2: a_i **or** a_i' **is matched in** Q_{n+1} **for** $i = 1, 2, \cdots, n$**.**
Since $m_0 \ge m_1$, $m_0 \le n-1$, and $m_0 + m_1 = n$, we may without loss of generality assume that $M_0 = \{(a_1, b_1), (a_2, b_2), \cdots, (a_i, b_i)\}$ and $M_1 = \{(a_{i+1}', d_{i+1}'), (a_{i+2}', d_{i+2}'), \cdots, (a_n', d_n')\}$ where $i \in \{2, 3, \cdots, n-1\}$. Let $\{(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)\}$

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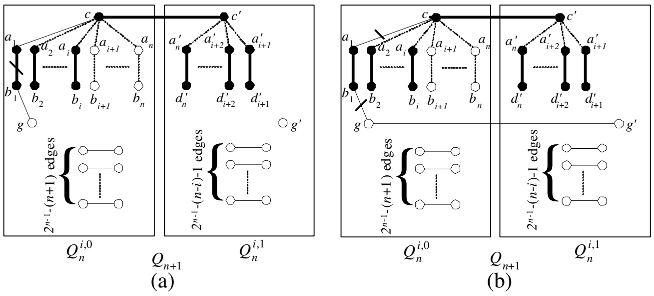

Figure 10: Case 3.2: a_i or a'_i is matched in Q_{n+1} for $i = 1, 2, \cdots, n$.

be a forbidden matching of $Q_n^{i,0}$, note the c is the common neighbor of $a_1, a_2, \cdots, a_n$. Then, $\{(c, a_1), (a_2, b_2), (a_3, b_3), \cdots, (a_n, b_n)\}$ is a non-forbidden matching of $Q_n^{i,0}$ with n edges, and it can be extended to a perfect matching of $Q_n^{i,0}$, say PM_0. Let $(b_1, g) \in PM_0$. On account of $dist(c, g) = 3$ and in $Q_n^{i,1}$, the distance of c' and each of the matched vertices of $Q_n^{i,1}$ is at most 2, so g' is unmatched. By Lemma 4, M_1 can be extended to a perfect matching of $Q_n^{i,1} \setminus \{c', g'\}$, say PM'_1, since $m_1 \leq n-2$ and $dist(c', g')$ is odd. See Figure 10.(a). Therefore, $(PM_0 \cup PM'_1 \cup \{(a_1, b_1), (c, c'), (g, g')\}) \setminus \{(a_1, c), (b_1, g)\}$ is a perfect matching of Q_{n+1}. See Figure 10.(b). $\diamondsuit$

It has shown in [7] that every perfect matching of the n-dimensional hypercube with $n \geq 2$ can be extended to a Hamiltonian cycle. Consequently, we have the following corollary.

Corollary 1 *Assume that $n \geq 2$ is an integer. Let M be an arbitrary matching with at most n edges in the hypercube Q_n. Then, the matching M can be extended to a Hamiltonian cycle of Q_n, except M is a forbidden matching of Q_n.*

3 Conclusion Remarks

Perfect matchings can be applied to network disclosure attacks, such as the Perfect Matching Disclosure Attack. Perfect matching problem is also applicable to connectivity of networks. In this paper, we assign an arbitrary non-forbidden matching with at most n edges to form a perfect matching or a Hamiltonian cycle on the hypercube Q_n for $n \geq 2$.

An open problem is that on the hypercube Q_n, if we assign a matching with more than n edges, how can we restrict the prescribed matching to obtain a perfect matching or a Hamiltonian cycle.

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