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On the Complexity of n-player Toppling Dominoes

Alessandro Cincotti *

Abstract— **Why are n-player games much more complex than two-player games? Is it much more difficult to cooperate or to compete? n-player Toppling Dominoes is an n-player version of Toppling Dominoes, a two-player combinatorial game. Because of queer games, i.e., games where no player has a winning strategy, cooperation is a key-factor in n-player games and, as a consequence, n-player Toppling Dominoes played on a set of rows of dominoes is $\mathcal{PSPACE}$-complete.**

Keywords: Combinatorial games, Toppling Dominoes, n-player games, $\mathcal{PSPACE}$-complete

1 Introduction

Toppling Dominoes is a combinatorial game defined in [1] and played on a row of black or white dominoes. Two players, called Black and White, move alternately. Black chooses a black domino and topples it either left or right, every domino in that direction also topples and is removed from the game. White chooses a white domino and topples it either left or right, every domino in that direction also topples and is removed from the game. The first player unable to move because there is no domino of his/her color is the loser. An example of Toppling Dominoes is shown in Fig. 1. n-player Toppling Dominoes is the n-player version of Toppling Dominoes played on a row of dominoes. Every domino is labeled by an integer $j \in \{1, 2, \ldots, n\}$. The first player chooses a domino labeled 1 and topples it either left or right, every domino in that direction also topples and is removed from the game. The second player chooses a domino labeled 2 and topples it either left or right, every domino in that direction also topples and is removed from the game. The other players move in similar way.

Players take turns making legal moves in cyclic fashion (1-st, 2-nd, ..., n-th, 1-st, 2-nd, ...). When one of the players is unable to move, that player leaves the game and the remaining $n-1$ players continue playing in the same mutual order as before. The remaining player is the winner.

*A. Cincotti is with the School of Information Science, Japan Advanced Institute of Science and Technology, 1-1 Asahidai, Nomi, Ishikawa 923-1292 Japan (phone: +81-761-51-1293; fax: +81-761-51-1149; e-mail: cincotti@jaist.ac.jp).

We briefly recall the definition of *queer* game introduced by Propp [2]:

Definition 1. *A position in a three-player combinatorial game is called queer if no player can force a win.*

Such a definition is easily generalizable to n players:

Definition 2. *A position in an n-player combinatorial game is called queer if no player can force a win.*

In the game of n-player Toppling Dominoes, it is not always possible to determine the winner because of queer games, as shown in Fig. 2. In this case, no player has a winning strategy because if the first player topples left his/her domino, then the third player wins but if the first player topples right his/her domino, then the second player wins.

In two player games [3, 4] players are in conflict to each other and coalitions are not allowed but in n-player games [5, 6, 7, 8], when the game is queer, only cooperation between players can guarantee a winning strategy, i.e., one player of the coalition is always able to make the last move. As a consequence, to establish whether or not a coalition has a winning strategy is a crucial point.

In previous works, we analyzed the complexity of three-player Hackenbush played on strings [9], three-player Col played on trees [10], and three-player Snort played on complete graphs [11]. In this paper we show that, in Toppling Dominoes, cooperation between a group of players can be much more difficult than competition and, as a consequence, n-player Toppling Dominoes is $\mathcal{PSPACE}$-complete.

2 The Complexity of n-player Toppling Dominoes

In this section we show that the $\mathcal{PSPACE}$-complete problem of *Quantified Boolean Formulas* [12], QBF for short, can be reduced by a polynomial time reduction to n-player Toppling Dominoes.

Let $\varphi \equiv \exists x_1 \forall x_2 \exists x_3 \ldots Q x_n \psi$ be an instance of QBF, where Q is $\exists$ for n odd and $\forall$ otherwise, and ψ is a quantifier-free Boolean formula in conjunctive normal

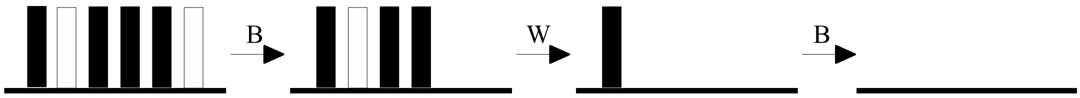

Figure 1: An example of Toppling Dominoes.

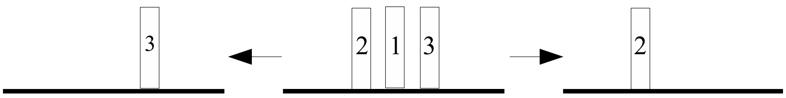

Figure 2: An example of queer game in 3-player Toppling Dominoes.

form where every clause has 3 distinct literals. We recall that QBF asks if there exists an assignment to the variables $x_1, x_3, \ldots, x_{2\lceil n/2\rceil-1}$ such that the formula evaluates to true.

If n is the number of variables and k is the number of clauses in ψ, then the instance of n-player Toppling Dominoes will have $n+k+2$ players and $n+2$ rows, organized as follows:

- For each variable x_i, with $1 \leq i \leq n$, we add a new row containing a domino labeled i sandwiched between two groups of dominoes. In the group on the right side, there is a domino for each clause that contains x_i. These dominoes are labeled j, with $n+1 \leq j \leq n+k$, and arranged in increasing order from right to left. In the group on the left side, there is a domino for each clause that contains $\overline{x_i}$. These dominoes are labeled j, with $n+1 \leq j \leq n+k$, and arranged in increasing order from left to right.
- The $n+1$-th row contains four dominoes labeled $n+k+1$.
- The last row contains k dominoes labeled j, with $n+1 \leq j \leq n+k$ sandwiched between 3 dominoes labeled $n+k+2$ and one domino labeled $n+k+2$.

Let us suppose that:

- The first coalition is formed by $\lfloor n/2\rfloor+1$ players corresponding to the dominoes labeled 2, 4, …, $2\lfloor n/2\rfloor$, and $n+k+1$,
- The second coalition is formed by the remaining players.

An example is shown in Fig. 3 where

$$\varphi \equiv \exists x_1 \forall x_2 \exists x_3 \forall x_4 (C_5 \wedge C_6 \wedge C_7)$$

and

$$\begin{aligned} C_5 &\equiv (\overline{x_1} \vee x_2 \vee \overline{x_3}) \\ C_6 &\equiv (x_1 \vee \overline{x_2} \vee \overline{x_4}) \\ C_7 &\equiv (x_1 \vee \overline{x_3} \vee x_4) \end{aligned}$$

The problem to determine the winning coalition is strictly connected to the problem of QBF, as shown in the following theorem.

Theorem 1. *Let G be a general instance of n-player Toppling Dominoes played on a set of rows of dominoes. Then, to establish whether or not a given coalition has a winning strategy is a $\mathcal{PSPACE}$-complete problem.*

Proof. We show that it is possible to reduce every instance of QBF to a graph G representing an instance of n-player Toppling Dominoes. Previously we have described how to construct the instance of n-player Toppling Dominoes, therefore we just have to prove that QBF is satisfiable if and only if the second coalition has a winning strategy.

If QBF is satisfiable, then there exists an assignment of x_i such that ψ is true with $i \in \{1, 3, \ldots, 2\lceil n/2\rceil - 1\}$. If x_i is true, then the i-th player topples left his/her domino and removes all the dominoes corresponding to the clauses containing $\overline{x_i}$. If x_i is false, then the i-th player topples right his/her domino and removes all the dominoes corresponding to the clauses containing x_i. Every clause contains at least a true literal, therefore the i-th player with $i \in \{n+1, n+2, \ldots, n+k\}$ can always topple one domino from the row corresponding to that literal. In this way, at the end of the first round, the $n+k+2$-th player can topple the domino on the right side of the last row and, at the end of the game, he/she will be able to make the last move. Therefore, the second coalition has a winning strategy.

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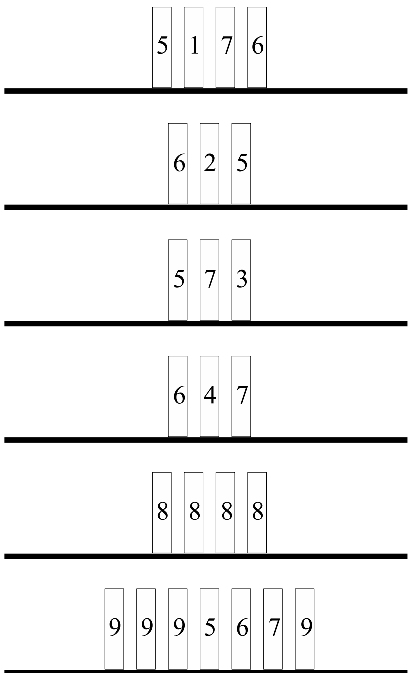

Figure 3: An example of Quantified Boolean Formula reduced to n-player Toppling Dominoes.

Conversely, let us suppose that the second coalition has a winning strategy. We observe that the $n+k+1$-th player is always able to make 4 moves, therefore even the $n+k+2$-th player must be able to make 4 moves in order to assure a winning strategy for the second coalition. As a consequence, the i-th player with $i \in \{n+1, n+2, \ldots, n+k\}$ does not topple any domino in the last row before the $n+k+2$-th player makes his/her first move, i.e., every clause has at least one true literal and QBF is satisfiable.

Therefore, to establish whether or not a coalition has a winning strategy in n-player Toppling Dominoes played on a set of rows of dominoes is $\mathcal{PSPACE}$-hard.

To show that the problem is in $\mathcal{PSPACE}$ we present a polynomial-space recursive algorithm to determine which coalition has a winning strategy. Let us introduce some useful notations:

- $G = (V, E)$ is the graph representing an instance of n-player Toppling Dominoes;
- p_i is the i-th player;
- C_0 is the set of current players belonging to the first coalition;
- C_1 is the set of current players belonging to the second coalition;
- `coalition`(p_i) returns 0 if $p_i \in C_0$ and 1 if $p_i \in C_1$;
- `label`(e) returns the label of the domino e;
- `next`(p_i) returns the player which has to play after p_i;
- `topple`(G,e,d) returns the graph obtained after that the domino e has been toppled in the direction d and all the dominoes in that direction have been removed from G.

Algorithm 1 performs an exhaustive search until a winning strategy is found and its correctness can be easily proved by induction on the depth of the game tree.

Algorithm 1 is clearly in $\mathcal{PSPACE}$ because the number of nested recursive calls is at most $|E|$ and therefore the total space complexity is $O(|E|^2)$.

□

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input : A graph $G = (V, E)$, the two initial coalitions C_0 and C_1 and the player p_i that has to move;
output: 0 if the first coalition wins and 1 if the second coalition wins.

```
Algorithm Check(G, C_0, C_1, p_i)
j ← coalition (p_i);
if ∄e ∈ E : label(e) = i then
    C_j ← C_j \ {p_i};
    if C_j = ∅ then
        return 1 − j;
    else
        return Check(G,C_0,C_1,next(p_i));
    end
else
    forall e ∈ E : label(e) = i do
        G' ← topple(G,e,left);
        G'' ← topple(G,e,right);
        if Check(G',C_0,C_1,next(p_i))= j or
        Check(G'',C_0,C_1,next(p_i))= j then
            return j;
        end
    end
    return 1 − j;
end
```

Algorithm 1: A polynomial-space algorithm for n-player Toppling Dominoes.

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