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Comparisons of Power of Parametric and Nonparametric Test for Testing Means of Several Weibull Populations

Lakhana Watthanacheewakul

***Abstract*—Power of the analysis of variance (ANOVA) and the Kruskal-Wallis test for comparing the several Weibull population means is investigated. Since Weibull data sets are nonnormal distributed, they must be transformed to normal distribution with constant variance before ANOVA is applied whereas the Kruskal-Wallis test does not need the normality assumption. The power of them was compared in a number of different situations and different sample sizes. The results depended on the coefficient of variation of population means. It seems that the power of ANOVA is higher than the power of the Kruskal-Wallis test a little in every case.**

***Index Terms*—Weibull Data, Power, The Coefficient of Variation, Population Means**

I. Introduction

THERE are many methods for comparing of more than two population means such as the analysis of variance (ANOVA) and the likelihood ratio test (LRT). It is easier to use the ANOVA if the data are normally distributed. In case of nonnormal data, we usually apply the LRT to scale, location or shape parameter. Nagarsenker [1] derived the exact distribution of the likelihood ratio statistic for testing the equality of parameters of k exponential populations. The exact significant points for moderate sample sizes were checked using Box's chi-squared approximation and the beta approximation. When computed numerically, the beta approximation was shown to be better than Box's chi-squared approximation. Singh [2] derived the likelihood ratio test for testing the equality of location parameters of $k(\geq 2)$ two-parameter exponential distributions based on a type-II censored sample, assuming that the scale parameters were unknown but equal. Gill [3] proposed a likelihood-based test for comparing the means of two or more log-normal distributions with unequal variances when the sample sizes were small. The performance of the proposed procedure was compared with the F-ratio test using Monte Carlo simulation. The likelihood-based test had a better power than the F-test when the population dispersion parameters were heterogeneous.

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L. Watthanacheewakul is with the Faculty of Science, Maejo University, Chiang Mai, Thailand (phone: 66-53-873-551; fax: 66-53-878-225; e-mail: lakhana@mju.ac.th).

However, it is difficult to find the exact distribution of the likelihood ratio statistic, Λ. For large samples the statistic $-2\ln\Lambda$ is approximately distributed as chi-squared with k-1 degrees of freedom, where k is a number of the population. For small samples the approximated chi-square may be inaccurate. Alternatively, the Kruskal-Wallis non-parametric test can be used. ANOVA can be applied if and only if the nonnormal data are transformed to fit the required assumption of it. In this paper, the power of the ANOVA and the Kruskal-Wallis test for comparing the several Weibull population means is investigated.

II. The Weibull Distribution

The Weibull distribution is a continuous probability distribution. It is named after Waloddi Weibull who described it in detail in 1951. The probability density function of a two parameter Weibull random variable X is

$$f(x)=\begin{cases}\dfrac{\alpha}{\gamma}\left(\dfrac{x}{\gamma}\right)^{\alpha-1}e^{-\left(\frac{x}{\gamma}\right)^{\alpha}}, & x\geq 0\ ;\ \alpha,\ \gamma>0\\ 0, & x<0\end{cases} \tag{1}$$

where α is the shape parameter and γ is the scale parameter. The mean is $\gamma\cdot\Gamma\left(\frac{1}{\alpha}+1\right)$. It's useful in many fields such as survival analysis, extreme value theory, weather forecasting, reliability engineering and failure analysis. Moreover, it is used to describe wind speed distribution, the particle size distribution, and so on. Furthermore, it is related to the other probability distribution such as the exponential distribution when α=1 [4].

III. Tests for Comparisons of Several Weibull Population Means

A. The ANOVA

Usually, a Box-Cox transformation is used to transform data to normality. For the ANOVA, the Box-Cox transformation is in the form

$$Y_{ij}=\begin{cases}\dfrac{X_{ij}^{\lambda}-1}{\lambda}, & \lambda\neq 0\\ \ln X_{ij}, & \lambda=0\end{cases} \tag{2}$$

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for $x_{ij} > 0$, where X_{ij} is a random variable in the j*th* trial from the i*th* population, Y_{ij} the transformed variable of X_{ij} and λ a transformation parameter, but the condition of observation is that the value of it is greater than zero. In the sets of Weibull data, the some observations may be zero. In order to cope with this problem, Watthanacheewakul [5] presented the alternative transformation for any sets of Weibull data to normality with constant variance in this form

$$Y_{ij} = \begin{cases} \dfrac{\left[X_{ij} + 0.01c_i\right]^{\lambda} - 1}{\lambda}, & \lambda \neq 0 \\ \ln\left[X_{ij} + 0.01c_i\right], & \lambda = 0 \end{cases} \tag{3}$$

where X_{ij} is a random variable in the j*th* trial from the i*th* Weibull distribution,
Y_{ij} the transformed variable of X_{ij},
c_i the range of the i*th* Weibull distribution, and
λ a transformation parameter.

Furthermore, the power of the ANOVA increases as sample size increases. When the differences among the population means are larger, higher powers of the tests are obtained.

B. *The Kruskal-Wallis Test*

This is a non-parametric analogue, based on rank, of one-way analysis of variance. The test statistic is

$$H = \frac{12}{n(n+1)} \sum_{i=1}^{k} n_i \left(\overline{R}_i - \frac{n+1}{2} \right)^2 \tag{4}$$

where $\overline{R}_i$ is the average rank of the member of the i*th* sample obtained after ranking all of the $n = \sum_{i=1}^{k} n_i$ observations. Kruskal [6] proves that if H_0 is true, the statistic H has a limiting chi-square distribution with k–1 degrees of freedom as $n_i \to \infty$ simultaneously.

IV. Power Functions of Tests

In order to compare the power of the ANOVA with its of the Kruskal-Wallis test, the power functions are taken into consideration. Patnaik [7] and Pearson and Hartley [8] showed that the power function of ANOVA is in the form

$$\beta(\delta) = \sum_{t=0}^{\infty} \frac{e^{-\frac{1}{2}\delta}\left(\frac{1}{2}\delta\right)^t}{t!B\left(\frac{1}{2}(k-1)+t, \frac{1}{2}(n-k)\right)} \times \int_{F_\alpha}^{\infty} \left(\frac{(k-1)}{(n-k)}\right)^{\frac{1}{2}(k-1)+t} F'^{\frac{1}{2}(k-1)-1+t} \left(1 + \frac{(k-1)}{(n-k)}F'\right)^{-\frac{1}{2}(n-1)-t} dF' \tag{5}$$

where B is a beta function, F' is a distribution function of a non-central F, with a non-centrality parameter, δ. Andrew [9] showed that the power function of the Kruskal-Wallis test has an approximate non-central χ^2 distribution with k-1 degrees of freedom and the non-centrality parameter is defined as

$$\delta = 12 \left[\int_{-\infty}^{+\infty} F'(x) dF(x) \right]^2 \sum_{i=1}^{k} s_i (\xi_i - \overline{\xi})^2 \tag{6}$$

where ξ_i 's are not all equal and $s_i = \dfrac{n_i}{n}$.

The power function is given by

$$\beta(\delta) = \int_{\chi^2_\alpha}^{\infty} \frac{e^{-\frac{1}{2}\chi'^2} e^{-\frac{1}{2}\delta}}{2^{\frac{1}{2}n}} \sum_{t=0}^{\infty} \frac{(\chi'^2)^{\frac{1}{2}n+t-1} \delta^t}{\Gamma\left(\frac{1}{2}n + t\right) 2^{2t}.t!} d\chi'^2 . \tag{7}$$

V. Simulation Method

Since the power functions of the two tests are in different forms, they cannot be compared directly. To draw a conclusion, the power of the tests is studied numerically for particular cases by simulation method.

A. *Simulation Method*

A power comparison of several tests was suggested in two steps [10].

1) To find the critical value for rejection of the null hypothesis, Weibull populations of size $N_i = 4{,}000$ are generated for $\alpha_1 = \alpha_2 = ... = \alpha_k$ and $\gamma_1 = \gamma_2 = ... = \gamma_k$, $i = 1, 2, .., k$. From each generated population, 1,000 random samples, each of size n_i, $i = 1, 2, ..., k$, are drawn. The test statistics of the two tests are calculated from each of the 1,000 samples. For each test, the 1,000 values of the test statistics are arranged in an increasing order and the 95*th* percentile is identified. This gives the critical values at $\alpha = 0.05$ for the two tests.

2) Since the proportion of rejections of the null hypothesis when the alternative is true is needed, Weibull populations of size $N_i = 4{,}000$ are generated for various parameter values α_i and γ_i, $i = 1, 2, .., k$. Since means of Weibull distributions depend on both shape parameter, α_i, and scale parameter, γ_i, $i = 1, 2, .., k$ and there are many values of shape parameter and scale parameter, the difference of means of several Weibull distributions are considered. The difference of means is measured by the coefficient of variation (C.V.),

$$C.V.(\mu) = \frac{S.D.(\mu_1, \mu_2, ..., \mu_k)}{Mean(\mu_1, \mu_2, ..., \mu_k)} . \tag{8}$$

They are set for two cases: the shape parameters are the same and the scale parameters are different and both shape and scale parameters are different. We ignore in case of the shape parameters are different but the scale parameters are the same because the coefficients of variation have a little change in each situation.

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From each generated Weibull(α_i,γ_i) population, 1,000 random samples, each of size n_i, are drawn. The test statistics of the two tests are calculated. If the value of the test statistic is in the rejection area, as defined by the critical values, then the null hypothesis is rejected. The power of the test is the proportion of times that the null hypothesis is rejected.

Let $\hat{\beta}_F$ be the power of the ANOVA, and $\hat{\beta}_H$ the power of the Kruskal -Wallis test. The values of parameters and the significant value are set as follows:

1) k = number of the populations = 3
2) n_i = sample sizes from the i*th* Weibull population, is between 10 and 50, for $i = 1,...,k$
3) α_i = the shape parameter of the i*th* Weibull population, is between 2 and 4
4) γ_i = the scale parameter of the i*th* Weibull population, is between 1,000 and 2,000
5) $\alpha = 0.05$.

B. *Results of the Power Comparison*

For a fixed null hypothesis, the power of the two tests is obtained as the proportion of rejection when the alternative hypothesis is true with different values of the coefficients of variation of population means. When the coefficient of variation of population means is zero, the null hypothesis is true. Hence, the proportion of rejecting the null hypothesis is equal or nearly closed to the level of significance α. When the coefficient of variation of population means is greater than zero, the null hypothesis is false or the alternative hypothesis is true. Thus the proportion of rejection of the null hypothesis is the estimate of the power of the test. The results of the power of tests for different sample sizes and different coefficients of variation of population means are compared when α is set equal to 0.05.

The results of comparison are divided into two cases.

1) Same Shape Parameters but Different Scale Parameters

To make it clear, the examples of the values of shape parameter and scale parameter and the calculation of the coefficient of variation are illustrated in Table I and Table II, respectively.

The results are shown in Table III to Table V.

TABLE I
THE VALUES OF SAME SHAPE PARAMETERS AND DIFFERENT SCALE PARAMETERS

No.	α_i and γ_i Values of Parameters
1	$\alpha_1 = 2, \alpha_2 = 2, \alpha_3 = 2, \gamma_1 = 1000, \gamma_2 = 1001, \gamma_3 = 1002$
2	$\alpha_1 = 2, \alpha_2 = 2, \alpha_3 = 2, \gamma_1 = 1000, \gamma_2 = 1050, \gamma_3 = 1100$
3	$\alpha_1 = 2, \alpha_2 = 2, \alpha_3 = 2, \gamma_1 = 1000, \gamma_2 = 1100, \gamma_3 = 1200$
4	$\alpha_1 = 2, \alpha_2 = 2, \alpha_3 = 2, \gamma_1 = 1000, \gamma_2 = 1150, \gamma_3 = 1300$
5	$\alpha_1 = 2, \alpha_2 = 2, \alpha_3 = 2, \gamma_1 = 1000, \gamma_2 = 1200, \gamma_3 = 1500$
6	$\alpha_1 = 2, \alpha_2 = 2, \alpha_3 = 2, \gamma_1 = 1000, \gamma_2 = 1500, \gamma_3 = 2000$

TABLE II
THE CALCULATION OF THE COEFFICIENT OF VARIATION OF POPULATION MEANS

No.	μ_1	μ_2	μ_3	C.V. (%)
1	886.23	887.11	888.00	0.10
2	886.23	930.54	974.85	4.76
3	886.23	974.85	1063.47	9.09
4	886.23	1019.16	1152.10	13.04
5	886.23	1063.47	1329.34	20.40
6	886.23	1329.34	1772.45	33.33

TABLE III
THE POWER OF TESTS WHEN THE C.V. OF POPULATION MEANS VARIES WITH EQUAL SAMPLE SIZES OF $n_i = 10$ AND $n_i = 20$

No.	C.V. (%)	$n_i = 10$		$n_i = 20$	
		$\hat{\beta}_F$	$\hat{\beta}_H$	$\hat{\beta}_F$	$\hat{\beta}_H$
1	0.10	0.053	0.050	0.057	0.057
2	4.76	0.058	0.058	0.085	0.077
3	9.09	0.101	0.096	0.111	0.106
4	13.04	0.145	0.131	0.240	0.205
5	20.40	0.288	0.246	0.579	0.508
6	33.33	0.633	0.590	0.941	0.912

TABLE IV
THE POWER OF TESTS WHEN THE C.V. OF POPULATION MEANS VARIES WITH EQUAL SAMPLE SIZES OF $n_i = 30$ AND $n_i = 50$

No.	C.V. (%)	$n_i = 30$		$n_i = 50$	
		$\hat{\beta}_F$	$\hat{\beta}_H$	$\hat{\beta}_F$	$\hat{\beta}_H$
1	0.10	0.059	0.058	0.058	0.055
2	4.76	0.098	0.085	0.100	0.088
3	9.09	0.181	0.156	0.293	0.244
4	13.04	0.358	0.283	0.544	0.453
5	20.40	0.736	0.668	0.938	0.899
6	33.33	0.993	0.987	1.000	1.000

TABLE V
THE POWER OF TESTS WHEN THE C.V. OF POPULATION MEANS VARIES WITH UNEQUAL SAMPLE SIZES OF $n_1 = 10, n_2 = 20, n_3 = 30$ AND $n_1 = 10, n_2 = 30, n_3 = 50$

No.	C.V. (%)	$n_1 = 10, n_2 = 20, n_3 = 30$		$n_1 = 10, n_2 = 30, n_3 = 50$	
		$\hat{\beta}_F$	$\hat{\beta}_H$	$\hat{\beta}_F$	$\hat{\beta}_H$
1	0.10	0.052	0.052	0.057	0.051
2	4.76	0.075	0.068	0.058	0.056
3	9.09	0.100	0.092	0.101	0.097
4	13.04	0.175	0.154	0.233	0.209
5	20.40	0.459	0.417	0.577	0.542
6	33.33	0.838	0.837	0.913	0.906

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In this case, the power of ANOVA is often higher than the power of the Kruskal-Wallis test a little both equal and unequal sample sizes. Furthermore, the power of the two tests increases as sample size increases. When the differences among the population means are larger, higher power of the tests is obtained. Furthermore, they have the same power.

2) Different Shape Parameters and Different Scale Parameters

From the values of shape parameters and scale parameters as Table VI and the values of the coefficient of variation as Table VII, the power of the two tests is shown in Table VIII to Table X.

TABLE VI
THE VALUES OF DIFFERENT SHAPE PARAMETERS AND DIFFERENT SCALE PARAMETERS

No.	α_i and γ_i Values of Parameters
1	$\alpha_1 = 2, \alpha_2 = 2.25, \alpha_3 = 2.4, \gamma_1 = 1000, \gamma_2 = 1001, \gamma_3 = 1002$
2	$\alpha_1 = 2, \alpha_2 = 3, \alpha_3 = 4, \gamma_1 = 1000, \gamma_2 = 1050, \gamma_3 = 1100$
3	$\alpha_1 = 2, \alpha_2 = 3, \alpha_3 = 4, \gamma_1 = 1000, \gamma_2 = 1100, \gamma_3 = 1200$
4	$\alpha_1 = 2, \alpha_2 = 3, \alpha_3 = 4, \gamma_1 = 1000, \gamma_2 = 1150, \gamma_3 = 1300$
5	$\alpha_1 = 2, \alpha_2 = 3, \alpha_3 = 4, \gamma_1 = 1000, \gamma_2 = 1200, \gamma_3 = 1500$
6	$\alpha_1 = 2, \alpha_2 = 3, \alpha_3 = 4, \gamma_1 = 1000, \gamma_2 = 1500, \gamma_3 = 2000$

TABLE VII
THE CALCULATION OF THE COEFFICIENT OF VARIATION OF POPULATION MEANS

No.	μ_1	μ_2	μ_3	C.V. (%)
1	886.23	893.87	908.22	0.12
2	886.23	937.63	997.04	5.90
3	886.23	982.28	1087.68	10.23
4	886.23	1026.93	1178.32	14.18
5	886.23	1071.58	1359.60	21.57
6	886.23	1339.47	1812.80	34.42

From Table VIII to Table X, the results are as the same in case of same shape parameters but different scale parameters.

TABLE VIII
THE POWER OF TESTS WHEN THE C.V. OF POPULATION MEANS VARIES WITH EQUAL SAMPLE SIZES OF $n_i = 10$ AND $n_i = 20$

No.	C.V. (%)	$n_i = 10$		$n_i = 20$	
		$\hat{\beta}_F$	$\hat{\beta}_H$	$\hat{\beta}_F$	$\hat{\beta}_H$
1	0.12	0.058	0.057	0.061	0.060
2	5.90	0.132	0.121	0.265	0.262
3	10.23	0.142	0.137	0.269	0.266
4	14.18	0.260	0.247	0.495	0.476
5	21.57	0.638	0.608	0.926	0.913
6	34.42	0.957	0.948	1.000	1.000

TABLE IX
THE POWER OF TESTS WHEN THE C.V. OF POPULATION MEANS VARIES WITH EQUAL SAMPLE SIZES OF $n_i = 30$ AND $n_i = 50$

No.	C.V. (%)	$n_i = 30$		$n_i = 50$	
		$\hat{\beta}_F$	$\hat{\beta}_H$	$\hat{\beta}_F$	$\hat{\beta}_H$
1	0.12	0.063	0.061	0.072	0.069
2	5.90	0.341	0.334	0.527	0.517
3	10.23	0.391	0.372	0.623	0.613
4	14.18	0.662	0.639	0.876	0.872
5	21.57	0.990	0.988	1.000	1.000
6	34.42	1.000	1.000	1.000	1.000

TABLE X
THE POWER OF TESTS WHEN THE C.V. OF POPULATION MEANS VARIES WITH UNEQUAL SAMPLE SIZES OF $n_1 = 10, n_2 = 20, n_3 = 30$ AND $n_1 = 10, n_2 = 30, n_3 = 50$

No.	C.V. (%)	$n_1 = 10, n_2 = 20, n_3 = 30$		$n_1 = 10, n_2 = 30, n_3 = 50$	
		$\hat{\beta}_F$	$\hat{\beta}_H$	$\hat{\beta}_F$	$\hat{\beta}_H$
1	0.12	0.066	0.058	0.064	0.060
2	5.90	0.320	0.260	0.409	0.333
3	10.23	0.324	0.254	0.429	0.349
4	14.18	0.481	0.431	0.625	0.565
5	21.57	0.922	0.878	0.984	0.961
6	34.42	0.997	0.995	1.000	0.999

VI. CONCLUSION

In order to compare the power of parametric and nonparametric test for testing means of several Weibull population, both the ANOVA and the Kruskal-Wallis test were investigated. The Weibull data must be transformed to normal distribution with constant variance before ANOVA is applied whereas the Kruskal-Wallis test does not need the normality assumption. Furthermore, the power of the ANOVA is compared to the power of the Kruskal-Wallis test. Since the power functions of the two tests are of different forms and cannot be compared explicitly, a numerical method is then used for comparison purposes. It is found that the power of ANOVA is often higher than the power of the Kruskal-Wallis test a little both equal and unequal sample sizes. Furthermore, the power of the two tests increases as sample size increases. When the differences among the population means are larger, higher power of them is obtained and they have the same power. The results are the same both in case of same shape parameters but different scale parameters and different shape parameters and different scale parameters.

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