A Mixed Quadrature Rule by Blending Clenshaw-Curtis and Gauss-Legendre Quadrature Rules for Approximation of Real Definite Integrals in Adaptive Environment

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***Abstract* - A mixed quadrature rule blending Clenshaw-Curtis five point rule and Gauss-Legendre three point rule is formed. The mixed rule has been tested in adaptive environment and it is found to be more effective than that of Clenshaw-Curtis five point rule.**

***Key Words* - Clenshaw-Curtis quadrature rule, Gauss-Legendre 3-point rule, mixed quadrature rule. Adaptive quadrature method.**

I. INTRODUCTION

Real definite integrals of the type

$$I(f)=\int_a^b f(t)\,dt \tag{1.1}$$

have been successfully approximated by several Authors [1], [2], [3] by applying the mixed quadrature rule. The method involves construction of a symmetric quadrature rule of higher precision as a linear combination of two other rules of equal lower precision.

If we consider a Gauss- Legendre rule and a Clenshaw-Curtis rule having same precision, Clenshaw-Curtis rule is better than Gauss-Legendre rule. An n-point Gaussian rule is of precision $2n$-1, while the precision of an n-point Clenshaw-Curtis rule is n. In general, Gauss type rule is of higher precision than that of Clenshaw-Curtis type rule when same number of abscissae are used.

In this paper, taking the advantage of the fact that Gauss-Legendre 3-point rule and Clenshaw-Curtis 5-point rule are of same precision (i.e. precision 5), we formed a mixed quadrature rule of higher precision (i.e. precision7) taking linear combination of these rules. The mixed rule so formed has been tested on different definite integrals giving better results than Clenshaw-Curtis quadrature rule in adaptive environment.

II. THE CLENSHAW-CURTIS QUADRATURE RULE

The Clenshaw-Curtis method [4] essentially approximates a function $f(t)$ over any interval $[\alpha-h,\alpha+h]$ using the Chebyshev polynomials $T_r(x)$ of degree n

$$f(t)=F(x)=\sum_{r=0}^{n}{}' a_r T_r(x) \qquad (-1\le x\le 1) \tag{2.1}$$

where, a_r are the expansion coefficients and $\sum'$ denotes a finite sum whose first term is to be halved before begining to sum. That is,

$$F(x)=\frac{1}{2}a_0T_0(x)+a_1T_1(x)+a_2T_2(x)+\dots+a_nT_n(x) \tag{2.2}$$

Collocating with $f(\alpha+hx)$ at the $n+1$ points

$$x_i=\cos\left(\frac{i\pi}{n}\right), \qquad (i=0,1,\dots,n) \tag{2.3}$$

one can evaluate the expansion coefficients a_r.

The Chebyshev polynomials $T_r(x_i)$ can be expressed as

$$T_r(x_i)=\cos\left(r\cos^{-1}(x_i)\right) \qquad r\ge 0 \tag{2.4}$$

$$=\cos\left(\frac{ri\pi}{n}\right)$$

Then $$\sum_{i=0}^{n}{}'' f(\alpha+hx_i)T_r(x_i)=\sum_{i=0}^{n}{}''\sum_{k=0}^{n}{}' a_kT_k(x_i)T_r(x_i)$$

$$=\sum_{k=0}^{n}{}' a_k\sum_{i=0}^{n}{}''\cos\left(\frac{ki\pi}{n}\right)\cdot\cos\left(\frac{ri\pi}{n}\right) \tag{2.5}$$

The notation $\sum''$ means that the first and last terms are to be halved before summation begins.

The orthogonality of the cosine function [5] with respect to the points $x_i=\cos\left(\frac{i\pi}{n}\right)$ is expressed by

$$\sum_{i=0}^{n}{}''\cos\left(\frac{ki\pi}{n}\right)\cdot\cos\left(\frac{ri\pi}{n}\right)=\begin{cases} n & r=k=0 \text{ or } n\\ \frac{n}{2} & 0<r=k<n\\ 0 & r\ne k\end{cases} \tag{2.6}$$

Substituting Eq(2.6) into Eq(2.5), gives

$$\sum_{i=0}^{n}{}'' f(\alpha+hx_i)T_r(x_i)=\begin{cases}\frac{n}{2}a_r & 0\le r=k<n\\ n\ a_r & r=k=n\\ 0 & r\ne k\end{cases}$$

Hence

$$a_r=\begin{cases}\frac{2}{n}\sum_{i=0}^{n}{}'' f(\alpha+hx_i)T_r(x_i) & (r=0,1,\dots,n-1)\\ \frac{1}{n}\sum_{i=0}^{n}{}'' f(\alpha+hx_i)T_r(x_i) & (r=n)\end{cases} \tag{2.7}$$

Denoting the integral of $f(t)$ over the interval $[\alpha-h,\alpha+h]$ by I, and replacing t by $\alpha+hx$, we get

$$I=h\int_{-1}^{1}f(\alpha+hx)\,dx$$

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Assuming $I \approx I_n$, we write

$$I_n = h\int_{-1}^{1}\sum_{r=0}^{n}{}' a_r T_r(x)dx$$

$$= h\sum_{r=0}^{n}{}' a_r \int_{-1}^{1} T_r(x)dx$$

Substituting the values of a_r (as given in Eq 2.7), we get

$$= h\sum_{r=0}^{n}{}'' \frac{2}{n}\sum_{i=0}^{n}{}'' f(\alpha + hx_i)T_r(x_i)\int_{-1}^{1} T_r(x)dx$$

Since $\int_{-1}^{1} T_r(x)dx = \frac{-2}{r^2-1} \quad (r = \text{even})$, (2.8)

we get $I_n = h\sum_{i=0}^{n}{}'' w_i f(\alpha + hx_i)$ (2.9 (a))

where $w_i = -\frac{4}{n}\sum_{\substack{r=0\\ r=\text{even}}}^{n}{}'' \frac{1}{r^2-1}T_r(x_i) \quad (i = 0,1,\ldots,n)$ (2.9 (b))

With $n = 4$,

$$I_4 = \frac{h}{15}\left[f(\alpha+h) + 8f\left(\alpha + \frac{h}{\sqrt{2}}\right) + 12f(\alpha) + 8f\left(\alpha - \frac{h}{\sqrt{2}}\right) + f(\alpha - h)\right] \quad (2.10)$$

III. CONSTRUCTION OF THE MIXED QUADRATURE RULE OF PRECISION SEVEN

We choose Clenshaw-Curtis five point rule

$$I(f) = \int_{-1}^{1} f(x)dx \approx R_{CC_5}(f)$$

$$= \frac{1}{15}\left[f(1) + 8f\left(\frac{1}{\sqrt{2}}\right) + 12f(0) + 8f\left(-\frac{1}{\sqrt{2}}\right) + f(-1)\right] \quad (3.1)$$

and the Gauss-Legendre three point rule

$$I(f) = \int_{-1}^{1} f(x)dx \approx R_{GL_3}(f)$$

$$= \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right] \quad (3.2)$$

Each of the rules(3.1) and (3.2) is of precision five. Let $E_{CC_5}(f)$ and $E_{GL_3}(f)$ denote the errors in approximating the integral $I(f)$ by the rules (3.1) and (3.2) respectively. Using Maclaurin's expansion of functions in Eqs (3.1) and (3.2), we get

$$I(f) = R_{CC_5}(f) + E_{CC_5}(f) \quad (3.3)$$

and $I(f) = R_{GL_3}(f) + E_{GL_3}(f)$ (3.4)

where

$$E_{CC_5}(f) = \frac{1}{315\times 5!}f^{(6)}(0) + \frac{1}{360\times 7!}f^{(8)}(0) + \ldots$$

$$E_{GL_3}(f) = \frac{1}{175\times 90}f^{(6)}(0) + \frac{88}{1125\times 8!}f^{(8)}(0) + \ldots$$

Now multiplying the equations (3.3) and (3.4) by $\frac{1}{5}$ and $-\frac{1}{12}$ respectively, and then adding the resulting equations, we obtain

$$I(f) = \frac{1}{7}\left[12R_{CC_5}(f) - 5R_{GL_3}(f)\right] + \frac{1}{7}\left[12E_{CC_5}(f) - 5E_{GL_3}(f)\right]$$

or $I(f) = R_{CC_5GL_3}(f) + E_{CC_5GL_3}(f)$ (3.5)

where $R_{CC_5GL_3}(f) = \frac{1}{7}\left[12R_{CC_5}(f) - 5R_{GL_3}(f)\right]$ (3.6)

This is the desired mixed quadrature rule of precision seven for the approximate evaluation of $I(f)$. The truncation error generated in this approximation is given by

$$E_{CC_5GL_3}(f) = \frac{1}{7}\left[12E_{CC_5}(f) - 5E_{GL_3}(f)\right]$$

$$= -\frac{1}{450\times 7!}f^{(8)}(0) + \cdots \quad (3.7)$$

The rule (3.6) may be called a mixed type rule as it is constructed from two different types of rules of the same precision. (i.e, precision 5)

IV. ERROR ANALYSIS

An asymptotic error estimate and an error bound of the rule (3.6) are given in theorems 4.1(a) and 4.1(b), respectively.

Theorem 4.1(a)

Let $f(x)$ be a sufficiently differentiable function in the closed interval [-1, 1]. Then the error $E_{CC_5GL_3}(f)$ associated with the rule $R_{CC_5GL_3}(f)$ is given by

$$\left|E_{CC_5GL_3}(f)\right| \approx \frac{1}{450\times 7!}\left|f^{(8)}(0)\right|$$

Proof

From Eq(3.5),

$$I(f) = R_{CC_5GL_3}(f) + E_{CC_5GL_3}(f)$$

where $R_{CC_5GL_3}(f) = \frac{1}{7}\left[12R_{CC_5}(f) - 5R_{GL_3}(f)\right]$

and $E_{CC_5GL_3}(f) = \frac{1}{7}\left[12E_{CC_5}(f) - 5E_{GL_3}(f)\right]$

Hence $E_{CC_5GL_3(f)} = -\frac{1}{450\times 7!}f^{(8)}(0) + \cdots$

So $\left|E_{CC_5GL_3}(f)\right| \approx \frac{1}{450\times 7!}\left|f^{(8)}(0)\right|$

Theorem 4.1(b)

The bound for the truncation error

$$E_{CC_5GL_3}(f) = I(f) - R_{CC_5GL_3}(f)$$

is given by

$$\left|E_{CC_5GL_3}(f)\right| \le \frac{M}{22050}\left|(\eta_2 - \eta_1)\right| \qquad \eta_1, \eta_2 \in [-1,1]$$

where $M = \max_{-1\le x\le 1}\left|f^{(7)}(x)\right|$

Proof

We have $E_{CC_5}(f) \approx \frac{1}{315\times 5!}f^{(6)}(\eta_2) \quad \eta_2 \in [-1,1]$

$$E_{GL_3}(f) \approx \frac{1}{175\times 90}f^{(6)}(\eta_1) \quad \eta_1 \in [-1,1]$$

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Hence $E_{CC_5GL_3} = \frac{1}{7}\left[12E_{CC_5}(f) - 5E_{GL_3}(f)\right]$

$$= \frac{1}{22050}\left[f^{(6)}(\eta_2) - f^{(6)}(\eta_1)\right]$$

$$= \frac{1}{22050}\int_{\eta_1}^{\eta_2} f^{(7)}(x)dx \text{ (assuming } \eta_1 < \eta_2\text{)}$$

From this we obtain

$$\left|E_{CC_5GL_3}(f)\right| = \left|\frac{1}{22050}\int_{\eta_1}^{\eta_2} f^{(7)}(x)dx\right| \le \frac{1}{22050}\int_{\eta_1}^{\eta_2}\left|f^{(7)}(x)\right|dx$$

So $\left|E_{CC_5GL_3}(f)\right| \le \frac{M}{22050}\left|(\eta_2 - \eta_1)\right|$

which gives only a theoretical error bound, as η_1, η_2 are unknown points in $[-1,1]$. It shows that the error in the approximation will be less if the points η_1, η_2 are close to each other.

Corollary

The error bound for the truncation error $E_{CC_5GL_3}(f)$ is given by

$$\left|E_{CC_5GL_3}(f)\right| \le \left(\frac{M}{11025}\right)\left|(\eta_2 - \eta_1)\right|$$

Proof

We know from the theorem 4.1 (b) that

$$\left|E_{CC_5GL_3}(f)\right| \le \frac{M}{22050}\left|(\eta_2 - \eta_1)\right| \qquad \eta_1, \eta_2 \in [-1,1]$$

where $M = \max_{-1 \le x \le 1}\left|f^{(7)}(x)\right|$

Choosing $\left|(\eta_1 - \eta_2)\right| \le 2$, we have

$$\left|E_{CC_5GL_3}(f)\right| \le \frac{M}{11025}$$

V. NUMERICAL VERIFICATION

TABLE -1

Comparison of the mixed quadrature rule with Clenshaw-Curtis 5- point rule in approximation of some real definite integrals by adaptive quadrature method.

Integrals	**Exact value**	**Approximate value** **Clenshaw-Curtis 5-point quadrature rule $(R_{CC_5}(f))$ by adaptive quadrature method**	**No. of Inter-vals divided**	**Mixed quadrature rule $(R_{CC_5GL_3}(f))$ by adaptive quadrature method**	**No. of Intervals divided**	**Maximum admissible absolute error(ε)**
$I_1 = \int_0^{\pi/2} \frac{dx}{1+\cos x}$	1	0.999999824050772	2	1.000000097429282	1	$\varepsilon_1 = 0.000001$
$I_2 = \int_0^{\pi} \frac{dx}{5+4\cos x}$	1.047197551196503	1.047197543767417	6	1.047197549459474	3	$\varepsilon_2 = 0.000001$
$I_3 = \int_0^{1} \frac{dx}{1+25x^2}$	0.274680153389003	0.274680188936363	6	0.274680147293959	3	$\varepsilon_3 = 0.000001$
$I_4 = \int_0^{\pi/2} \cos^3 x \; dx$	0.666666666666...	0.666666655780439	6	0.666666666387270	3	$\varepsilon_4 = 0.000001$
$I_5 = \int_0^{\pi/4} \frac{1}{1+\sin x} dx$	0.5857864372626905	0.585786436967197	3	0.585786438175938	1	$\varepsilon_5 = 0.00000001$
$I_6 = \int_0^{1} \frac{1}{1+x} dx$	0.693147180559945	0.693147151845538	2	0.693147191045928	1	$\varepsilon_6 = 0.0000001$
$I_7 = \int_0^{1} \frac{1}{1-0.5x^4} dx$	1.143667254069416	1.143666134639847	3	1.143669272509230	2	$\varepsilon_7 = 0.00001$
$I_8 = \int_0^{1} \frac{1}{1+100x^2} dx$	0.147112767430373	0.147112762552179	8	0.147112867618549	4	$\varepsilon_8 = 0.000001$
$I_9 = \int_1^{2} \frac{\ln x}{x} dx$	0.240226506959101	0.240226563432822	2	0.240226480745529	1	$\varepsilon_9 = 0.000001$
$I_{10} = \int_1^{2} \frac{1}{e^x - 1} dx$	0.313261687518223	0.313261687317049	6	0.313261688035760	2	$\varepsilon_{10} = 0.000000001$

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VI. CONCLUSION

Above ten examples give a clear picture about the effectiveness of imposing mixed quadrature rule in adaptive environment. The mixed quadrature rule $\left(R_{CC_5GL_3}(f)\right)$ reduces the number of steps required to approximate an integral in adaptive quadrature mehtod in comparison to its constitute Clenshaw-Curtis quadrature rule.

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