On a Subclass of Meromorphic Multivalent Function Associated With Salagean Operator

Abdolreza Tehranchi

***Abstract*—In this paper, we have discussed the meromorphic p-valent functions that satisfy the differential subordinations**

$$\frac{z(\mathcal{I}_p(r,\lambda)f(z))^{(j+1)}}{(\mathcal{I}_p(r,\lambda)f(z))^{(j)}} \prec (1+\frac{A-B}{a}\frac{\beta z}{1+Bz})(p+j), z\in\Delta^*,$$

where β is complex number and $I_p(r,\lambda)$ is salagean Operator. Also we study coefficient inequalities and hadamard product (convolution) and found radius of starlikeness and convexity. We investigate some interesting properties on $\mathcal{A}_p^*(\lambda,r,j,\beta,a,A,B)$; too.

***Index Terms*—Meromorphic multivalent function, Differential subordinations, multiplier transformation, differential operator, complex order.**

I. Introduction

LET $\mathcal{A}$ be the class of function analytic in $\Delta=\{z\in\mathbb{C}:|z|<1\}$ and $\mathcal{A}_p$ the subclass of $\mathcal{A}$ such that included of functions $f(z)$ of the form

$$f(z)=ez^{-p}-\sum_{n=p-1}^{2p-1}t_{n-p+1}z^{n-p+1}+\ {}_2F_1(a,b;c;z),$$

$$(n\geq p; p\in\mathcal{N}=\{1,2,3,\cdots\}, e>0) \qquad (1.1)$$

which are analytic and meromorphic p-valent in the annulus $\Delta^*=\{z:0<|z|<1, z\in\mathbb{C}\}=\Delta-\{0\}$. Also

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)n!}z^n$$

$$(a,n)=\frac{\Gamma(a+n)}{\Gamma(a)}=a(a+1,n-1),\quad c>b>0, c>a+b$$

and

$$t_{n-p+1}=\frac{(a,n-p+1)(b,n-p+1)}{(c,n-p+1)(n-p+1)!}.$$

These functions are analytic in the unit disk Δ. For more details on hypergeometric functions ${}_2F_1(a,b;c.z)$ see [1], [4].

Definition 1 : A function $f\in\mathcal{A}_p$ is said to be in the class $S_p^*(\alpha)$ of meromorphic p-valently starlike functions of order α, if it satisfies $Re\left\{\frac{-zf'(z)}{f(z)}\right\}>\alpha,\quad (0\leq\alpha<p,\ z\in\Delta^*)$. We write $S_p^*(0)=S_p^*$, the class of meromorphic p-valently starlike functions in Δ^*.

A function $f\in\mathcal{A}_p$ is said to be in the class $\mathcal{C}_p(\alpha)$ of meromorphic p-valently convex of order α, if it satisfies $Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha,\quad (0\leq\alpha<p, z\in\Delta^*)$.

Manuscript received Jan 27, 2011; revised Feb 12, 2011. This work was supported in part by Department of Mathematics, Islamic Azad University, South Tehran Branch.

Abdolreza Tehranchi is with the Department of Mathematics, Islamic Azad University, South Tehran Branch, Tehran, Iran, e-mail: Tehranchi@azad.ac.ir,Tehranchiab@gmail.com

Definition 2 : For two functions f and g, analytic in Δ^* we say f is subordinate to g denoted by $f\prec g$ if there exists a Schwarz function $w(z)$, analytic yuin Δ with $w(0)=1$ and $|w(z)|<1$, such that $f(z)=g(w(z)), z\in\Delta$. Also, we say that g is superordinate to f [5].

Definition 3 : Motivated by the multiplier transformation on $\mathcal{A}_p$, we define the operator $\mathcal{I}_p(r,\lambda)$ by the following infinite series when $f(z)=z^{-p}+\sum\limits_{n=p+1}^{\infty}a_nz^n$ then

$$\mathcal{I}_p(r,\lambda)f(z)=z^{-p}+\sum_{n=1+p}^{\infty}\left(\frac{n+\lambda}{p+\lambda}\right)^r a_nz^n,\quad (\lambda\geq 0) \qquad (1.2)$$

Sălăgean derivative operators [6] is closely related to the operators $\mathcal{I}_p(r,\lambda)$. Also Uralegaddi and Somanatha [7] studied $\mathcal{I}_1(r,1)=\mathcal{I}_r$. The operator $\mathcal{I}_1(r,\lambda)=\mathcal{I}_r^\lambda$ was studied recently by Cho and Srivastava [3] and Cho and Kim [2].

Definition 4 : Differential operator, for each $f(z)=z^{-p}+\sum\limits_{n=p+1}^{\infty}a_nz^n$ we have

$$f^{(j)}(z)=\frac{(-1)^j(p+j-1)!}{(p-1)!}z^{-(p+j)}+\sum_{n=1+p}^{\infty}\frac{n!}{(n-j)!}a_nz^{n-j} \qquad (1.3)$$

where $n,p\in N, p>j$, and $j\in N_0=\{0\}\cup N$. For $j=0$ we have $f^{(0)}(z)=f(z)$.

Definition 5 : A function $f\in\mathcal{A}_p$ is said to be in the class $\mathcal{A}_p(r,j;h)$ if it satisfies the following subordination

$$\frac{z(\mathcal{I}_p(r,\lambda)f(z))^{(j+1)}}{(p+j)(\mathcal{I}_p(r,\lambda)f(z))^{(j)}}\prec h(z) \qquad (1.4)$$

where in this paper we choose

$$h(z)=1+\frac{A-B}{a}\frac{\beta z}{1+Bz},\quad z\in\Delta,$$

where $-1\leq B<A\leq 1, 0<B<1, a>0$ and $\beta(\neq 0)$ is a complex number, so we denote $\mathcal{A}_p^*(\lambda,r,j;h)=\mathcal{A}_p^*(\lambda,r,j,\beta,a,A,B)$.

We say that $f(z)$ is superordinate to $h(z)$ if $f(z)$ satisfies the following

$$h(z)\prec\frac{z(\mathcal{I}_p(r,\lambda)f(z))^{(j+1)}}{(p+j)(\mathcal{I}_p(r,\lambda)f(z))^{(j)}}$$

where $h(z)$ is analytic in Δ and $h(0)=1$.

II. Main Results

In the following theorem we obtain coefficient bound for this class.

Theorem 2.1 : Let the function $f(z)$ of the form (1.1), be in $\mathcal{A}_p$. Then the function $f(z)$ belongs to the class

ISBN: 978-988-18210-3-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$\mathcal{A}_p^*(\lambda, r, j, \beta, a, A, B)$ if and only if $\sum_{m=p+1}^{\infty} \gamma_\lambda^r(m,p)(a\delta(m,j+1)(1-B) - \delta(m,j)$ $(p+j)(a(1-B)-(A-B)\beta))k_m < e(B-A)\beta\eta(p,j)$ (2.1) where j is any odd number we show $j \in 2N_0+1$, and $p \in N, -1 \leq B < A \leq 1, 0 < B < 1, 0 < a, 0 < e$, and β is a nonzero complex number. The result is sharp for the function $f(z)$ given by complex number
$f(z) = ez^{-p} + \frac{e\beta(B-A)\eta(p,j)}{\gamma_\lambda^r(m,p)(a\delta(m,j+1)(1-B)-\delta(m,j)(p+j)(a(1-B)-(A-B)\beta)} z^q$, $q \geq 1+p$.

Proof : The function $f(z)$ in the theorem can be expressed in the form

$$f(z) = ez^{-p} + \sum_{n=2p}^{\infty} k_{n-p+1} z^{n-p+1},$$

or

$$f(z) = ez^{-p} + \sum_{m=p+1}^{\infty} k_m z^m \qquad (2.2)$$

where $m = n-p+1$ and $k_m = \frac{(a,m)(b,m)}{(c,m)m!}$, $k_m \geq 0$ and also we have for all $r, j \in \mathcal{N}_0$
$(\mathcal{I}_p(r,\lambda)(f(z)))^{(j)} = e\frac{(-1)^j(p+j-1)!}{(p-1)!} z^{-(p+j)} + \sum_{p+1}^{\infty} \left(\frac{m+\lambda}{p+\lambda}\right)^r \left(\frac{m!}{(m-j)!}\right) k_m z^{m-j}$
$= e\eta(p, j - 1) z^{-(p+j)} + \sum_{m=p+1}^{\infty} \gamma_\lambda^r(m,p)\delta(m,j) k_m z^{m-j}$ (2.3)
Now,assume that the condition (2.1) is true. We must show that $f \in \mathcal{A}_p^*(\lambda, r, j, \beta, a, A, B)$, or equivalently we prove that

$$\left|\frac{az(\mathcal{I}_p(r,\lambda)f(z))^{(j+1)} - a(p+j)(\mathcal{I}_p(r,\lambda)f(z)^{(j)}}{(p+j)R(\mathcal{I}_p(r,\lambda)f(z))^{(j)} - Baz(\mathcal{I}_p(r,\lambda)f(z))^{(j+1)}}\right| < 1. \qquad (2.4)$$

where $\mathcal{R} = aB + (A-B)\beta$. But we have
$\left|\frac{az(\mathcal{I}_p(r,\lambda)f(z))^{(j+1)} - a(p+j)(\mathcal{I}_p(r,\lambda)f(z))^{(j)}}{(p+j)R(I_p(r,\lambda)f(z))^{(j)} - Baz(\mathcal{I}_p(r,\lambda)f(z))^{(j+1)}}\right| =$
$\left|\frac{\sum_{m=p+1}^{\infty} a\gamma_\lambda^r(m,p)(\delta(m,j+1)-a(p-j)\delta(m,j))k_m z^{m+p}}{e\eta(p,j)(R-Ba)+\sum_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)k_m((p+j)R\delta(m,j)-Ba\delta(m,j+1))}\right|$
$\leq$
$\left\{\frac{\sum_{m=p+1}^{\infty} a\gamma_\lambda^r(m,p)k_m(\delta(m,j+1)-a(p-j)\delta(m,j))}{e\eta(p,j)(B-A)\beta - \sum_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)k_m((p+j)R\delta(m,j)-Ba\delta(m,j+1))}\right\}$
< 1.
The last inequality by (2.1) is true.

Conversely, assume that $f(z) \in \mathcal{A}_p^*(\lambda, r, j, a, \beta, A, B)$. We must show that the condition (2.1) holds true. We have

$$\left|\frac{az(\mathcal{I}_p(r,\lambda)f(z))^{(j+1)} - a(p+j)(\mathcal{I}_p(r,\lambda)f(z))^{(j)}}{(p+j)R(\mathcal{I}_p(r,\lambda)f(z))^{(j)} - Baz(\mathcal{I}_p(r,\lambda)f(z))^{(j+1)}}\right| < 1$$

where $\mathcal{R} = aB + (A-B)\beta$. So by 2.3 we have
$\left|\sum_{m=p+1}^{\infty} a\gamma_\lambda^r(m,p)k_m z^{m-j}(\delta(m,j+1) - (p+j)\delta(m,j))\right|$
$\left|(e\eta(p,j)(R-Ba) + \sum_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)k_m z^{m-j}((p+j)R\right|$
$|\delta(m,\delta(m,j)) - Ba\delta(m,j+1))| < 1$.
Since $Re(z) < |z|$, so we have
$Re[(\sum_{m=p+1}^{\infty} a\gamma_\lambda^r(m,p)k_m z^{m+p}(\delta(m,j+1) - (p+j)\delta(m,j)))/(e(B-A)\beta\eta(p,j) + \sum_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)k_m z^{m+p}((p+j)R\delta(m,j)-Ba\delta(m,j+1)))]$ < 1.
We choose the values of z on the real axis and letting $z \to 1^-$ then we have
$[(\sum_{m=p+1}^{\infty} a\gamma_\lambda^r(m,p)(\delta(m,j+1)-(p+j)\delta(m,j))k_m)/(e(B-A)\beta\eta(p,j) - \sum_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)((p+j)R\delta(m,j) - Ba\delta(m,j+1))k_m)] < 1$,
and that $\sum_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)(a\delta(m,j+1)(1-B) - \delta(m,j)(p+j)(a(1-B)-(A-B)\beta))k_m < e(B-A)\beta\eta(p,j)$ and the proof is complete. □

Corollary 2.1 ; Let $f(z) \in \mathcal{A}_p^*(r, j, \beta, a, A, B)$ then we have $k_m \leq \frac{e(B-A)\eta(p,j)\beta}{\gamma_\lambda^r(m,p)(a\delta(m,j+1)(1-B)-\delta(m,j)(p+j)(a(1-B)-(A-B)\beta))}$,

$m \geq p+1$.

Corollary 2.2 : Let $0 \leq r_2 < r_1$ then

$$\mathcal{A}_p^*(\lambda, r_2, j, \beta, a, A, B) \subseteq \mathcal{A}_p^*(\lambda, r_1, j, \beta, a, A, B).$$

Proof : Suppose that $f \in \mathcal{A}_p^*(\lambda, r_2, j, \beta, a, A, B)$ then $\sum_{m=p+1}^{\infty}(\gamma_\lambda^{r_2}(m+p)(a\delta(m,j+1)(1-B) - \delta(m,j)(p+j)(a(1-B)-(A-B)\beta)))/(e(B-A)\beta\eta(p,j))k_m$ < 1.
We must prove
$\sum_{m=p+1}^{\infty}(\gamma_\lambda^{r_1}(m,p)(a\delta(m,j+1)(1-B) - \delta(m,j)(p+j)(a(1-B)-(A-B)\beta)))/(e(B-A)\beta\eta(p,j))k_m$ < 1.
But last inequality holds true if $\gamma_\lambda^{r_2}(m,p) < \gamma_\lambda^{r_1}(m,p)$.

In view of hypothesis the preceding inequality definitely holds true.

Corollary 2.3 : Let $0 \leq a_2 < a_1$ then

$$\mathcal{A}_p^*(\lambda, r, j, \beta, a_2, A, B) \subseteq \mathcal{A}_p^*(\lambda, r, j, b, a_1, A, B).$$

Theorem 2.2 : Let the function $f(z)$ defined by (2.2) being the class $\mathcal{A}_p^*(\lambda, r, j, \beta, a, A, B)$. Then

(i) $(\mathcal{I}_p(r,\lambda)f(z))$ is meromorphically p-valent starlike of order $\mu(0 \leq \mu < p)$ in the disk $|z| < r_1$, where
$r_1 = r_1(\lambda, r, j, \beta, a, A, B, \mu) = \inf_{m \geq p+1}$
$\left\{\frac{(a\delta(m,j+1)(1-B)-\delta(m,j)(p+j)(a(1-B)-(A-B)\beta))}{(B-A)\beta\eta(p,j)}\right\}^{\frac{1}{m+p}}$
$\times \left\{\left(\frac{p-\mu}{m+\mu}\right)\right\}^{\frac{1}{m+p}}$.

(2.5)

(ii) $(\mathcal{I}_p(r,\lambda)f(z))$ is meromorphically p-valent convex of order $\mu(0 \leq \mu < p)$ in the disk $|z| < r_2$, where
$r_2 = r_2(\lambda, r, j, \beta, a, A, B, \mu) = \inf_{m \geq p+1}$
$\left\{\frac{(a\delta(m,j+1)(1-B)-\delta(m,j)(p+j)(a(1-B)-(A-B)\beta))}{e(B-A)\beta\eta(p,j)}\frac{ep(p-\mu)}{m(m+\mu)}\right\}^{\frac{1}{m+p}}$
(2.6)

Proof: For showing $(\mathcal{I}_p(r,\lambda)f(z))$ is meromorphically p-valent starlike of order μ we must show $Re\left\{\frac{-z(\mathcal{I}_p(r,\lambda)f(z))'}{\mathcal{I}_p(r,\lambda)f(z)}\right\} > \mu$ or equivalently

$$\left|\frac{z(\mathcal{I}_p(r,\lambda)f(z))' + p(\mathcal{I}_p(r,\lambda)f(z))}{z(\mathcal{I}_p(r,\lambda(f(z))' + (2\mu-p)(\mathcal{I}_p(r,\lambda)f(z))}\right| < 1$$

or we can write

ISBN: 978-988-18210-3-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$\left|\frac{z(\mathcal{I}_p(r,\lambda)f(z))'+p(\mathcal{I}_p(r,\lambda)f(z))'}{z(I_p(r,\lambda(f(z))+(2\mu-p)(I_p(r,\lambda)f(z))}\right| \leq$

$\frac{\sum\limits_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)(p+m)k_m|z|^{m+p}}{2e(p-\mu)-\sum\limits_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)(m+2\mu-p)k_m|z|^{m+p}} < 1$
(2.7)
$(|z| < r_1, 0 \leq \mu < p)$.

The last inequality (2.7) holds true if $\sum_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)\frac{(m+\mu)}{e(p-\mu)}k_m|z|^{m+p} < 1$.

In view of (2.1), the last inequality holds true if
$\left(\gamma_\lambda^r(m,p)\frac{m+\mu}{e(p-\mu)}\right)|z|^{m+p}$
$\leq \frac{\gamma_\lambda^r(m,p)(a\delta(m,j+1)(1-B)-\delta(m,j)(p+j)(a(1-B)-(A-B)\beta))}{e(B-A)\beta\eta(p,j)}$
$(m \geq p+1, p \in \mathcal{N})$
which when solved for $|z|$, yields (2.5). (ii) For convexity, we will show that

$$Re\left\{-1-\frac{z(\mathcal{I}_p(r,\lambda)f(z))''}{(\mathcal{I}_p(r,\lambda)f(z))'}\right\} > \mu$$

equivalently
$\left|\frac{z(\mathcal{I}_p(r,\lambda)f(z))''+(p+1)(\mathcal{I}_p(r,\lambda)f(z))'}{z(\mathcal{I}_p(r,\lambda(f(z))''+(2\mu-p)(\mathcal{I}_p(r,\lambda)f(z))'}\right| < 1$
or we can write
$\left|\frac{z(\mathcal{I}_p(r,\lambda)f(z))''+(p+1)(\mathcal{I}_p(r,\lambda)f(z))'}{z(\mathcal{I}_p(r,\lambda(f(z))''+(2\mu-p)(\mathcal{I}_p(r,\lambda)f(z))'}\right| \leq$

$\frac{\sum\limits_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)m(m+p)k_m|z|^{m+p}}{2ep(p-\mu)-\sum\limits_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)m(m-p+2\mu)k_m|z|^{m+p}}$
≤ 1
(2.8) $(|z| < r_2,\ 0 \leq \mu \leq p)$.

The last inequality (2.8) holds true if

$$\sum_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)\frac{m(m+\mu)}{ep(p-\mu)}k_m|z|^{m+p} \leq 1.$$

(2.9)
According to Theorem 2.1 the inequality (2.9) is true if
$\gamma_\lambda^r(m,p)\frac{m(m+\mu)}{ep(p-\mu)}|z|^{m+p}$
$\leq \frac{\gamma_\lambda^r(m,p)(a\delta(m,j+1)(1-B)-\delta(m,j)(p+j)(a(1-B)-(A-B)\beta))}{e(B-A)\beta\eta(p,j)}$
$(m \geq p+1, p \in \mathcal{N})$
or if
$|z| \leq$
$\left\{\frac{(a\delta(m,j+1)(1-B)-\delta(m,j)(p+j)(a(1-B)-(A-B)\beta))}{e(B-A)\beta\eta(p,j)}\right\}^{\frac{1}{m+p}}$
$\left\{\frac{ep(p-\mu)}{m(m+\mu)}\right\}^{\frac{1}{m+p}}$
$(m \geq p+1, p \in \mathcal{N})$. The proof is completed.

III. Important Properties

Theorem 3.1 : Suppose $f(z)$ and $g(z)$ belong to $\mathcal{A}_p^*(\lambda,r,j,\beta,a,A,B)$ such that
$f(z) = ez^{-p}+\sum_{m=p+1}^{\infty}k_mz^m$,
$g(z) = ez^{-p}+\sum_{m=p+1}^{\infty}S_mz^m$. (3.1)
Then $T(z) = ez^{-p}+\sum_{m=p+1}^{\infty}(k_m^2+S_m^2)z^m$ is the class $\mathcal{A}_p^*(\lambda,r,j,\beta,a,A_1,B_1)$ such that $A_1 \geq (1-B_1)\mu^2+B_1$ and $B_1 \leq 1$ where
$\mu =$
$\frac{\sqrt{ae\beta\eta(1,j)\delta(p+1,j+1)}(B-A)}{\sqrt{\gamma_\lambda^r(p+1,1)}(a\delta(p+1,j+1)(1-B)-\delta(p+1,j)(p+j)(a(1-B)-(B-A)\beta)}$.

Proof : Since $f,g \in \mathcal{A}_p^*(\lambda,r,j,\beta,a,A,B)$ so Theorem 2.1 yields

$\sum_{m=p+1}^{\infty}$
$\left[\gamma_\lambda^r(m,p)\frac{a\delta(m,j+1)(1-B)-\delta(m,j)(p+j)(a(1-B)-(A-B)\beta)}{e(B-A)\beta\eta(p,j)}k_m\right]^2$
≤ 1
$\sum_{m=p+1}^{\infty}$
$\left[\gamma_\lambda^r(m,p)\frac{a\delta(m,j+1)(1-B)-\delta(m,j)(p+j)(a(1-B)-(A-B)\beta)}{e(B-A)\beta\eta(p,j)}S_m\right]^2$
≤ 1.
We find from two last inequalities
$\sum_{m=p+1}^{\infty}\frac{1}{2}$
$\left[\gamma_\lambda^r(m,p)\frac{a\delta(m,j+1)(1-B)-\delta(m,j)(p+j)(a(1-B)-(A-B)\beta)}{e(B-A)\beta\eta(p,j)}\right]^2$
$(k_m^2+S_m^2) < 1$ (3.2)
But $T(z) \in \mathcal{A}_p^*(\lambda,r,j,\beta,A,B)$ if and only if
$\sum_{m=p+1}^{\infty}\gamma_\lambda^r(m,p)$
$\frac{a\delta(m,j+1)(1-B_1)-\delta(m,j)(p+j)(a(1-B_1)-(A_1-B_1)\beta)}{e(B_1-A_1)\beta\eta(p,j)}$
$(k_m^2+S_m^2) < 1$ (3.3) where
$-1 < B_1 < A_1 \leq 1, 0 < B_1 < 1$, however, (3.2) implies (3.3) if
$\frac{a\delta(m,j+1)(1-B_1)-\delta(m,j)(p+j)(a(1-B_1)-(A_1-B_1)\beta)}{(B_1-A_1)}$
$< \frac{\gamma_\lambda^r(m,p)}{2e\beta\eta(p,j)}\xi^2$
where
$\xi = \frac{a\delta(m,j+1)(1-B)-\delta(m,j)(p+j)(a(1-B)-(A-B)\beta)}{(B-A)}$.
In other words

$$\frac{a\delta(m,j+1)(1-B_1)}{(B_1-A_1)} < \frac{\gamma_\lambda^r(m,1)}{e\beta\eta(1,j)}\xi^2$$

which is equivalent to
$\frac{B_1-A_1}{1-B_1} > \frac{ae\beta\eta(1,j)\delta(m,j+1)}{\gamma_\lambda^r(m,1)\xi^2}$.
Since $A-B > B-A$, so we can write
$\frac{A_1-B_1}{1-B_1} >$
$\frac{ae\beta\eta(1,j)\delta(p+1,j+1)(B-A)^2}{\gamma_\lambda^r(p+1,1)[(a\delta(p+1,j+1)(1-B)-\delta(p+1,j)(p+j)(a(1-B)-(A-B)\beta)]^2}$
$= \mu^2$(3.4)
Now keeping B_1 fixed in (3.4) we get $A_1 \geq (1-B_1)\mu^2+B_1$ and since $A_1 \leq 1$ then $B_1 \leq 1$.

□

Theorem 3.2 : Suppose $f(z)$ and $g(z)$ of the form (3.1) belong to $\mathcal{A}_p^*(\lambda,r,j,\beta,a,A,B)$. Then convolution or Hadamard product, two functions f and g belong to the class that is $(f*g)(z) \in \mathcal{A}_p^*(\lambda,r,j,\beta,a,A_1,B_1)$ where $A_1 \geq (1-B_1)v^2+B_1, B_1 \leq 1$ and

$$v = \frac{e\beta a\delta(p+1,j+1)\eta(1,j)}{\gamma_\lambda^r(p+1,1)\tau^2}$$

and
$\tau = \frac{a\delta(m,j+1)(1-B)-\delta(m,j)(p+j)(a(1-B_1)-(A_1-B_1)\beta)}{B-A}$.

Proof: Since $f,g \in \mathcal{A}_p^*(\lambda,r,j,\beta,a,A,B)$ so by applying Cauchy-Schwarz inequality and Theorem 2.1, we obtain

$$\sum_{m=p+1}^{\infty}\xi\sqrt{k_mS_m} \leq (\sum_{m=p+1}^{\infty}\xi k_m)^{1/2}(\sum_{m=p+1}^{\infty}\xi S_m)^{1/2} \leq 1 \quad (3.5)$$

where
$\xi =$
$\gamma_\lambda^r(m,p)\frac{a\delta(m,j+1)(1-B)-\delta(m,j)(p+j)(a(1-B)-(A-B)\beta)}{e(B-A)\beta\eta(p,j)}$.

We must find the values of A_1, B_1 so that

$$\sum_{m=p+1}^{\infty}\mu k_mS_m < 1 \quad (3.6)$$

ISBN: 978-988-18210-3-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

where
$\mu = \gamma_\lambda^r(m,p)\frac{a\delta(m,j+1)(1-B_1)-\delta(m,j)(p+j)(a(1-B_1)-(A_1-B_1)\beta)}{e(B_1-A_1)\beta\eta(p,j)}$

Therefore by (3.5), (3.6) holds true if
$\sqrt{k_m S_m} \le \frac{\xi}{\mu}, m \ge p+1, k_m \ne 0, S_m \ne 0,$
since by (3.5) we have $\sqrt{k_m S_m} < \frac{1}{\xi}$, therefore (3.7) holds true if $\mu \le \xi^2$, which is equivalent by

$\gamma_\lambda^r(m,p)\frac{a\delta(m,j+1)(1-B_1)-\delta(m,j)(p+j)(a(1-B_1)+(A_1-B_1)\beta)}{e(A_1-B_1)\beta\eta(p,j)} < \xi^2$
or $\frac{a\delta(m,j+1)(1-B_1)-\delta(m,j)(p+j)(a(1-B_1)-(A_1-B_1)\beta)}{B_1-A_1}$
$< \frac{\gamma_\lambda^r(m,p)}{e\beta\eta(p,j)}\tau^2$
where
$\tau = \frac{a\delta(m,j+1)(1-B)-\delta(m,j)(p+j)(a(1-B_1)-(A_1-B_1)\beta)}{B-A}$

or we can write
$\frac{a\delta(m,j+1)(1-B_1)}{B_1-A_1} < \frac{\gamma_\lambda^r(m,1)}{e\beta\eta(1,j)}\tau^2.$
In other words

$$\frac{1-B_1}{B_1-A_1} < \frac{\gamma_\lambda^r(m,1)}{e\beta\eta(1,j)a\delta(m,j+1)}\tau^2$$

then we have

$$\frac{B_1-A_1}{1-B_1} > \frac{e\beta a\delta(p+1,j+1)\eta(1,j)}{\gamma_\lambda^r(p+1,1)\tau^2} = v$$

Since $A_1 - B_1 > B_1 - A_1$ and keeping B_1 fixed, we get $A_1 > v(1-B_1) + B_1$.

REFERENCES

[1] H. Kopka and P. W. Daly, *A Guide to LaTeX*, 3rd ed. Harlow, England: Addison-Wesley, 1999.
[2] N. Meghanathan and G. W. Skelton, "Risk Notification Message Dissemination Protocol for Energy Efficient Broadcast in Vehicular Ad hoc Networks," *IAENG International Journal of Computer Science*, vol. 37, no. 1, pp. 1-10, Jul. 2010.
[3] E. H. Miller, "A note on reflector arrays (Periodical style-Accepted for publication)," *Engineering Letters*, to be published.
[4] J. Wang, "Fundamentals of erbium-doped fiber amplifiers arrays (Periodical style-Submitted for publication)," *IAENG International Journal of Applied Mathematics*, submitted for publication.
[5] N. Sohaee and C. V. Rorst, "Bounded Diameter Clustering Scheme For Protein Interaction Networks," in *Lecture Notes in Engineering and Computer Science: World Congress on Engineering and Computer Science 2009*, pp. 1-7.
[6] J. M. Merigo, "Using the Probabilistic Weight Average in Decision Making with Distsance Measures," in *Lecture Notes in Engineering and Computer Science: World Congress on Engineering 2010*, pp. 1-4.
[7] T. Gonsalves and K. Itoh, "Multi-Objective Optimization for Software Development Projects," in *Lecture Notes in Engineering and Computer Science: International Multiconference of Engineers and Computer Scientist 2010*, pp. 1-6.

ISBN: 978-988-18210-3-4
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)