Harmonic Balance Solution of Mulholland Equation

M. Saifur Rahman, M. Alal Hosen and M. Shamsul Alam

***Abstract*—Following a new harmonic balance method, approximate solutions of van der Pol's equation have been determined near the limit cycle. The method is extendable to higher order nonlinear differential system having a limit cycle. In this paper a second approximate solution of Mulholland equation (a third order differential equation) is found. The solution shows a good agreement with the numerical solution.**

Index Terms— Harmonic balance method, Limit cycle, Nonlinear oscillation, Periodic solution.

I. Introduction

Self-excited systems (SES) have a long history in the field of mechanics [1-2]. A self-excited oscillator is a system, which has some source of energy upon which it can draw. One of its most prominent features is the existence of stable limit cycle in phase space, emerging from a balance between energy gain and dissipation. The limit cycle topology is independent by the initial conditions. Recently, SESs have been proposed as fundamental tools for the control and reduction of friction [3-4]. The possible influence of self-excitation dynamics on friction force is based on the idea that when a limit cycle is established, then limited change of external conditions cannot destroy it and system persists on its frictionless oscillating motion.

In SES the damping is a function of position and the most general picture can be described by Lienard's differential equation, $\ddot{x}+\mu f(x)\dot{x}+g(x)=0$.

Equations of this kind arise directly in various mechanical applications. One of the studied equations within this class is the van der Pol's equation $\ddot{x}+x+\varepsilon(x^2-1)\dot{x}=0$, which posses an unique limit cycle.

Many analytical approaches have been developed for approximating periodic solutions of Eq. (1). The most widely used methods are the perturbation methods, whereby the solution is expanded in power series of a small parameter, ε . The LP method [5], KBM method [6] and multi-time expansion method [7] are important among them.

The harmonic balance (HB) method [8-17] is another technique for determining periodic solutions of nonlinear differential equations by using the truncated Fourier series. Since the derivation of higher approximation is complicated, the first and second approximate solutions are usually determined. The advantage of HB method is that the solution gives desired result though nonlinearities become significant.

Recently, Shamsul [18] has presented a new technique for obtaining higher approximation of some strongly nonlinear differential systems. His [18] technique is easier than the existing HB method and the solutions cover the general initial value problem (*i.e.* [$x(0)=\alpha$, $\dot{x}(0)=\beta$]). Usually, it is required to solve a set of nonlinear algebraic (or algebraic-transcendental) equations according to HB method. The numerical solution of those equations gives excellent results; but many authors (see [9-17] for details) modify the HB solution to determine all unknown coefficients in analytical approach.

Some authors extended the perturbation methods to tackle nonlinear oscillations described by a third order differential equations. Gottlieb [21-22] used the HBM to investigate limit cycles of both second- and third order nonlinear problems. The aim of the articles [21-22] was qualitative type studies; but the authors did not clearly determine the approximate solutions of those problems. The aim of the present article is to find the approximate solutions of the third order nonlinear problems (especially Mulholland equation) based on the new HBM (presented [18-19]). The solution shows a good agreement with numerical solution though the nonlinear term becomes significant.

II. The Method

Let us consider a nonlinear differential equation

$$\dddot{x}+\ddot{x}+\dot{x}+x=\varepsilon f(x,\dot{x},\ddot{x}), \qquad (2)$$

where ε is a constant. In general Eq. (2) has damped solution; but in some of the cases it has periodic solution (*e.g.*, near limit cycle of Mulholland equation).

A periodic solution of Eq. (2) is chosen in the form

$$x=a\cos\varphi+a^3(c_3\cos3\varphi+d_3\sin3\varphi)+\cdots\cdots\cdots \qquad (3)$$

where a and $\dot{\varphi}$ are constants. In general the unknown functions, $c_j(a)$ and $d_j(a)$, $j=3,5,\cdots$ are determined together with a and the initial phase, φ_0.

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F. A. Author is with the Department of Mathematics, Rajshahi Universityof Engineering and Technology (RUET), Rajshahi 6204, Bangladesh (corresponding author to provide phone: +8801719754965; e-mail: msr_math_1980@yahoo.com).

S. B. Author is with the Department of Mathematics, Rajshahi Universityof Engineering and Technology (RUET), Rajshahi 6204, Bangladesh (corresponding author to provide phone: +8801717288827; e-mail: alal_ruet@yahoo.com).

T. C. Author is with the Department of Mathematics, Rajshahi Universityof Engineering and Technology (RUET), Rajshahi 6204, Bangladesh (corresponding author to provide phone: +8801713228563; e-mail: msalam1964@yahoo.com).

According to the KBM method an approximate solution is chosen in powers of small parameter ε, namely $x = a\cos\varphi + \varepsilon u_1(a,\varphi) + \varepsilon^2 u_2(a,\varphi) + \varepsilon^3 \cdots\cdots$ where both $a(t)$ and $\varphi(t)$ are time dependent functions satisfying two first order differential equations $\dot{a} = \varepsilon A_1(a) + \varepsilon^2 \cdots$, $\dot{\varphi} = \omega + \varepsilon B_1(a) + \varepsilon^2 \cdots$ and the unknown function $A_1, A_2, \cdots B_1, B_2, \cdots$ are determined subject to the condition that $u_1, u_2, \cdots$ exclude the first harmonics. The constant ω is the unperturbed frequency of the oscillation (for Mulholland equation, $\omega = 1$). In general KBM method is used to discuss transient. However the method is used to investigate periodic solution in which $\dot{a}$ vanishes and $\dot{\varphi}$ becomes constant (see [18]). Clearly, the approximate solution Eq. (3) is chosen in a form of the KBM method; but the determination of the phase $\varphi(t)$, and unknown functions $c_j(a)$, $d_j(a)$, $j = 3, 5, \cdots$ are different from the KBM method.

Now substituting Eq. (3) into Eq. (2) and expanding the function $f(x,\dot{x},\ddot{x})$ in a Fourier series, we obtain

$$\begin{aligned}
&a(1-\dot{\varphi}^2)\cos\varphi + a^3(c_3 - 9\dot{\varphi}^2 c_3 + 3\dot{\varphi} d_3 \\
&-27\dot{\varphi}^3 d_3)\cos 3\varphi - a\dot{\varphi}(1-\dot{\varphi}^2)\sin\varphi \\
&+a^3(-3\dot{\varphi} c_3 + 27\dot{\varphi}^3 c_3 + d_3 - 9\dot{\varphi}^2 d_3)\sin 3\varphi + \cdots \\
&= \varepsilon\big[F_1(a,\dot{\varphi},c_3,d_3,\cdots)\cos\varphi + F_3(a,\dot{\varphi},c_3,d_3,\cdots)\cos 3\varphi \\
&+\cdots + G_1(a,\dot{\varphi},c_3,d_3,\cdots)\sin\varphi \\
&+G_3(a,\dot{\varphi},c_3,d_3,\cdots)\sin 3\varphi + \cdots\big]
\end{aligned} \tag{4}$$

By comparing the coefficients of equal harmonic, we obtain

$$\begin{aligned}
&a(1-\dot{\varphi}^2) = \varepsilon F_1,\ \ a^3(c_3 - 9\dot{\varphi}^2 c_3 + 3\dot{\varphi} d_3 \\
&-27\dot{\varphi}^3 d_3) = \varepsilon F_3,\ \cdots, -a\dot{\varphi}(1-\dot{\varphi}^2) = \varepsilon G_1, \\
&a^3(-3\dot{\varphi} c_3 + 27\dot{\varphi}^3 c_3 + d_3 - 9\dot{\varphi}^2 d_3) = \varepsilon G_3,\ \cdots
\end{aligned} \tag{5}$$

When $\omega \neq 0$, we use a new parameter $\mu(\varepsilon) << 1$, with $\varepsilon = O(1)$ and solve all the equations of Eq. (5) in powers of μ as

$$\begin{aligned}
&\dot{\varphi} = k_0 + k_1\mu + k_2\mu^2 + k_3\mu^3 + \cdots, \\
&a = l_0 + l_1\mu + l_2\mu^2 + l_3\mu^3 + \cdots, \\
&c_j = c_{j,1}\mu + c_{j,2}\mu^2 + c_{j,3}\mu^3 + \cdots, \\
&d_j = d_{j,1}\mu + d_{j,2}\mu^2 + d_{j,3}\mu^3 + \cdots, \quad j = 3, 5, \cdots
\end{aligned} \tag{6}$$

Differentiating Eq. (3) twice with respect to t and substituting $t = 0$, we obtain the initial conditions equations

$$\begin{aligned}
&x(0) = a\cos\varphi_0 + a^3 c_3\cos 3\varphi_0 + a^3 d_3\sin 3\varphi_0 + \cdots \\
&\dot{x}(0) = -a\sin\varphi_0 - 3a^3 c_3\sin 3\varphi_0 + 3a^3 d_3\cos 3\varphi_0 + \cdots \\
&\ddot{x}(0) = -a\cos\varphi_0 - 9a^3 c_3\cos 3\varphi_0 - 9a^3 d_3\sin 3\varphi_0 - \cdots, \\
&\varphi(0) = \varphi_0.
\end{aligned} \tag{7}$$

Thus from Eqs. (6)-(7), we measure the values of $x(0)$, $\dot{x}(0)$ and $\ddot{x}(0)$.

Example

Consider the Mulholland equation

$$\dddot{x} + \ddot{x} + \dot{x} + x = \varepsilon(1 - x^2 - \dot{x}^2 - \ddot{x}^2)(\ddot{x} + \dot{x}). \tag{8}$$

Let us consider a truncated form of Eq. (3) as

$$x = a\cos\varphi + a^3(c_3\cos 3\varphi + d_3\sin 3\varphi). \tag{9}$$

From Eq. (8) and Eq. (9), we easily obtain

$$\begin{aligned}
&a(1-\dot{\varphi}^2)\cos\varphi + a^3(c_3 - 9\dot{\varphi}^2 c_3 + 3\dot{\varphi} d_3 \\
&-27\dot{\varphi}^3 d_3)\cos 3\varphi - a\dot{\varphi}(1-\dot{\varphi}^2)\sin\varphi \\
&+a^3(-3\dot{\varphi} c_3 + 27\dot{\varphi}^3 c_3 + d_3 - 9\dot{\varphi}^2 d_3)\sin 3\varphi \\
&= \varepsilon[(-a\dot{\varphi}^2 + a^3(3\dot{\varphi}^2 + \dot{\varphi}^4 + 3\dot{\varphi}^6 + 11a^2\dot{\varphi}^2 c_3 \\
&-3a^2\dot{\varphi}^4 c_3 + 27a^2\dot{\varphi}^6 c_3 + 38a^4\dot{\varphi}^2 c_3^2 + 18a^4\dot{\varphi}^4 c_3^2 \\
&+286a^4\dot{\varphi}^6 c_3^2 - a^2\dot{\varphi} d_3 + 9a^2\dot{\varphi}^2 d_3 \\
&+15a^2\dot{\varphi}^5 d_3 + 38a^4\dot{\varphi}^2 d_3^2 + 18a^4\dot{\varphi}^4 d_3^2 \\
&+286a^4\dot{\varphi}^6 d_3^2))\cos\varphi/4 + (-9a^3\dot{\varphi}^2 c_3 + 3a^3\dot{\varphi} d_3 \\
&+a^3(\dot{\varphi}^2 - \dot{\varphi}^4 + \dot{\varphi}^6 + 22a^2\dot{\varphi}^2 c_3 + 18a^2\dot{\varphi}^4 c_3 \\
&+54a^2\dot{\varphi}^6 c_3 + 27a^6\dot{\varphi}^2 c_3^3 + 81a^6\dot{\varphi}^4 c_3^3 \\
&+2187a^6\dot{\varphi}^6 c_3^3 - 6a^2\dot{\varphi} d_3 - 18a^2\dot{\varphi}^3 d_3 \\
&-6a^2\dot{\varphi}^5 d_3 - 3a^6\dot{\varphi} c_3^2 d_3 - 81a^6\dot{\varphi}^3 c_3^2 d_3 \\
&-243a^6\dot{\varphi}^5 c_3^2 d_3 + 27a^6\dot{\varphi}^2 c_3 d_3^2 + 81a^6\dot{\varphi}^4 c_3 d_3^2 \\
&+2187a^6\dot{\varphi}^6 c_3 d_3^2 - 3a^6\dot{\varphi} d_3^3 - 81a^6\dot{\varphi}^3 d_3^3 \\
&-243a^6\dot{\varphi}^5 d_3^3))\cos 3\varphi/4 + (a\dot{\varphi} + a^3(\dot{\varphi} + 3\dot{\varphi}^3 \\
&+\dot{\varphi}^5 + a^2\dot{\varphi} c_3 - 9a^2\dot{\varphi}^3 c_3 - 15a^2\dot{\varphi}^5 c_3 \\
&+2a^4\dot{\varphi} c_3^2 + 54a^4\dot{\varphi}^3 c_3^2 + 162a^4\dot{\varphi}^5 c_3^2 \\
&+11a^2\dot{\varphi}^2 d_3 - 3a^2\dot{\varphi}^4 d_3 + 27a^2\dot{\varphi}^6 d_3 + 2a^4\dot{\varphi} d_3^2 \\
&+54a^4\dot{\varphi}^3 d_3^2 + 162a^4\dot{\varphi}^5 d_3^2))\sin\varphi/4 \\
&+(-3a^3\dot{\varphi} c_3 - 9a^3\dot{\varphi}^2 d_3 + a^3(\dot{\varphi} - \dot{\varphi}^3 \\
&+\dot{\varphi}^5 + 6a^2\dot{\varphi} c_3 + 18a^2\dot{\varphi}^3 c_3 + 6a^2\dot{\varphi}^5 c_3 \\
&+3a^6\dot{\varphi} c_3^3 + 81a^6\dot{\varphi}^3 c_3^3 + 243a^6\dot{\varphi}^5 c_3^3 + 22a^2\dot{\varphi}^2 d_3 \\
&+18a^2\dot{\varphi}^4 d_3 + 54a^2\dot{\varphi}^6 d_3 + 27a^6\dot{\varphi}^2 c_3^2 d_3 \\
&+81a^6\dot{\varphi}^4 c_3^2 d_3 + 2187a^6\dot{\varphi}^6 c_3^2 d_3 + 3a^6\dot{\varphi} c_3 d_3^2 \\
&+81a^6\dot{\varphi}^3 c_3 d_3^2 + 243a^6\dot{\varphi}^5 c_3 d_3^2 + 27a^6\dot{\varphi}^2 d_3^3 \\
&+81a^6\dot{\varphi}^4 d_3^3 + 2187a^6\dot{\varphi}^6 d_3^3))\sin 3\varphi/4] + \text{HOH},
\end{aligned}$$

where HOH stands for the higher order harmonics.

Comparing the coefficients of equal harmonics, we obtain

$$\begin{aligned}
&1-\dot{\varphi}^2 = \varepsilon(-\dot{\varphi}^2 + a^2(3\dot{\varphi}^2 + \dot{\varphi}^4 \\
&+3\dot{\varphi}^6 11a^2\dot{\varphi}^2 c_3 - 3a^2\dot{\varphi}^4 c_3 + 27a^2\dot{\varphi}^6 c_3 \\
&+38a^4\dot{\varphi}^2 c_3^2 + 18a^4\dot{\varphi}^4 c_3^2 + 286a^4\dot{\varphi}^6 c_3^2 \\
&-a^2\dot{\varphi} d_3 + 9a^2\dot{\varphi}^2 d_3 + 15a^2\dot{\varphi}^5 d_3 + 38a^4\dot{\varphi}^2 d_3^2 \\
&+18a^4\dot{\varphi}^4 d_3^2 + 286a^4\dot{\varphi}^6 d_3^2)/4)
\end{aligned}$$

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$$
\begin{aligned}
c_3 &= 9\dot{\varphi}^2 c_3 - 3\dot{\varphi} d_3 + 27\dot{\varphi}^3 d_3 \\
&+ \varepsilon(-9\dot{\varphi}^2 c_3 + 3\dot{\varphi} d_3 + (\dot{\varphi}^2 \\
&- \dot{\varphi}^4 + \dot{\varphi}^6 + 22a^2\dot{\varphi}^2 c_3 + 18a^2\dot{\varphi}^4 c_3 \\
&+ 54a^2\dot{\varphi}^6 c_3 + 27a^6\dot{\varphi}^2 c_3^3 + 81a^6\dot{\varphi}^4 c_3^3 \\
&+ 2187a^6\dot{\varphi}^6 c_3^3 - 6a^2\dot{\varphi} d_3 - 18a^2\dot{\varphi}^3 d_3 \qquad (10) \\
&- 6a^2\dot{\varphi}^5 d_3 - 3a^6\dot{\varphi} c_3^2 d_3 - 81a^6\dot{\varphi}^3 c_3^2 d_3 \\
&- 243a^6\dot{\varphi}^5 c_3^2 d_3 + 27a^6\dot{\varphi}^2 c_3 d_3^2 + 81a^6\dot{\varphi}^4 c_3 d_3^2 \\
&+ 2187a^6\dot{\varphi}^6 c_3 d_3^2 - 3a^6\dot{\varphi} d_3^3 - 81a^6\dot{\varphi}^3 d_3^3 \\
&- 243a^6\dot{\varphi}^5 d_3^3)/4) \\
-\dot{\varphi}(1-\dot{\varphi}^2) &= \varepsilon(\dot{\varphi} + a^2(\dot{\varphi} + 3\dot{\varphi}^3 + \dot{\varphi}^5 \\
&+ a^2\dot{\varphi} c_3 - 9a^2\dot{\varphi}^3 c_3 - 15a^2\dot{\varphi}^5 c_3 \\
&+ 2a^4\dot{\varphi} c_3^2 + 54a^4\dot{\varphi}^3 c_3^2 + 162a^4\dot{\varphi}^5 c_3^2 \\
&+ 11a^2\dot{\varphi}^2 d_3 - 3a^2\dot{\varphi}^4 d_3 + 27a^2\dot{\varphi}^6 d_3 \\
&+ 2a^4\dot{\varphi} d_3^2 + 54a^4\dot{\varphi}^3 d_3^2 + 162a^4\dot{\varphi}^5 d_3^2)/4) \\
d_3 &= 3\dot{\varphi} c_3 - 27\dot{\varphi}^3 c_3 + 9\dot{\varphi}^2 d_3 \\
&+ \varepsilon(-3\dot{\varphi} c_3 - 9\dot{\varphi}^2 d_3 + (\dot{\varphi} - \dot{\varphi}^3 + \dot{\varphi}^5 \\
&+ 6a^2\dot{\varphi} c_3 + 18a^2\dot{\varphi}^3 c_3 + 6a^2\dot{\varphi}^5 c_3 \\
&+ 3a^6\dot{\varphi} c_3^3 + 81a^6\dot{\varphi}^3 c_3^3 + 243a^6\dot{\varphi}^5 c_3^3 \\
&+ 22a^2\dot{\varphi}^2 d_3 + 18a^2\dot{\varphi}^4 d_3 + 54a^2\dot{\varphi}^6 d_3 \\
&+ 27a^6\dot{\varphi}^2 c_3^2 d_3 + 81a^6\dot{\varphi}^4 c_3^2 d_3 + 2187a^6\dot{\varphi}^6 c_3^2 d_3 \\
&+ 3a^6\dot{\varphi} c_3 d_3^2 + 81a^6\dot{\varphi}^3 c_3 d_3^2 + 243a^6\dot{\varphi}^5 c_3 d_3^2 \\
&+ 27a^6\dot{\varphi}^2 d_3^3 + 81a^6\dot{\varphi}^4 d_3^3 + 2187a^6\dot{\varphi}^6 d_3^3)/4)
\end{aligned}
$$

Herein four unknown quantities $\dot{\varphi}, a, c_3, d_3$ will be calculated from four nonlinear equations of Eq. (10). According to the KBM method, two nonlinear algebraic-transcendental equations are usually solved to calculate a and φ_0 for the same initial conditions whatever the order of the approximate solution is. But we have to solve four equations for the harmonic balance solution according to the proposed method. It is not difficult to solve numerically the system described by Eq. (10). But we solve the said nonlinear algebraic-transcendental system with less effort according to the principle of [18]. According to [18], we shall be able to find an approximate solution of equation (10) in the form of Eq. (6) as

$$
\begin{aligned}
a &= .8165 + 2.8305\mu + 21.6837\mu^2 - 28.22\mu^3 + O(\mu^4) \\
\dot{\varphi} &= 1 - 2.6667\mu - 15.2893\mu^2 + 96.5018\mu^3 + O(\mu^4) \\
c_3 &= 0.02\mu - 3.54714\mu^2 - 12.68122\mu^3 + O(\mu^4) \qquad (11) \\
d_3 &= -0.04\mu - 1.1735\mu^2 + 12.4879\mu^3 + O(\mu^4), \\
\mu &= \varepsilon/32.
\end{aligned}
$$

From Eq. (11) we obtain $a, \dot{\varphi}, c_3, d_3$ and then φ.

Finally, substituting the values of a, c_3, d_3 and φ into Eq. (7)

We obtain the values of $x(0), \dot{x}(0)$ and $\ddot{x}(0)$, which represents the initial values of x, $\dot{x}$ and $\ddot{x}$ for the steady-state solution.

4 Results and Discussion:

In order to test the accuracy of an approximate solution, some authors [13,16,18] compared analytical solutions to those obtained by the numerical techniques. We have compared such an approximate solution of the Mulholland equation Eq. (8) to the numerical solution for different values of ε.

First of all we plot in **Fig. 1(a)**, the second approximate solution of Eq. (8) for $\varepsilon = 0.1$ with initial conditions $[x(0) = 0.825878,\ \dot{x}(0) = -0.002111,\ \ddot{x}(0) = -0.814538]$ in which the unknown coefficients $a, \dot{\varphi}, c_3, d_3$ are calculated by the Eq. (11) and substituting these values in Eq. (7), we obtain $x(0), \dot{x}(0)$ and $\ddot{x}(0)$. Then corresponding numerical solution has been computed by *Runge-Kutta* (fourth-order) method. In **Fig. 1(b)**, the second approximate perturbation solution (see **Appendix A**) and the corresponding numerical solution have been plotted for the same values of ε with the initial conditions $[x(0) = 0.8102,\ \dot{x}(0) = 0,\ \ddot{x}(0) = -0.799395]$. Comparing the figures, it is clear that the harmonic based solution of Eq. (8) shows a better coincidence with the numerical solution than the perturbation solution (originally presented by Mulholland [20]).

In **Fig. 2(a)**, we have plotted the second approximate solution and the numerical solution when $\varepsilon = 0.5$. In this case we have calculated the initial conditions $[x(0) = 0.867282,\ \dot{x}(0) = -0.01215,\ \ddot{x}(0) = -0.80101]$. The figure indicates that the harmonic based solution again shows a good coincidence with the numerical solution. In **Fig. 2(b)**, we have also compared the perturbation solution to the numerical solution when $\varepsilon = 0.5$. (in this case with initial conditions are $[x(0) = 0.771,\ \dot{x}(0) = 0,\ \ddot{x}(0) = -0.784784]$). In this figure the perturbation solution has greatly deviated from the numerical solution. Thus increasing with the values of ε, the perturbation solution loses it suitability while the harmonic based solution shows a good agreement with the numerical solution.

Appendix A

A second approximate solution (perturbation) of Eq. (8) is [20]

$$
\begin{aligned}
x(t,\varepsilon) &= \alpha_0 \cos\omega t + \varepsilon(\alpha_1 - \alpha_0^3/160)\cos\omega t \\
&+ \varepsilon(3\alpha_0^3/80)\sin\omega t + \varepsilon(\alpha_0^3/160)\cos 3\omega t \qquad (A.1) \\
&- \varepsilon(\alpha_0^3/80)\sin 3\omega t + O(\varepsilon^2)
\end{aligned}
$$

where $\alpha_0 = 0.82,\ \alpha_1 = -0.098$ and $\omega = 1 - 0.084\varepsilon + 0.0024\varepsilon^2 + O(\varepsilon^3)$

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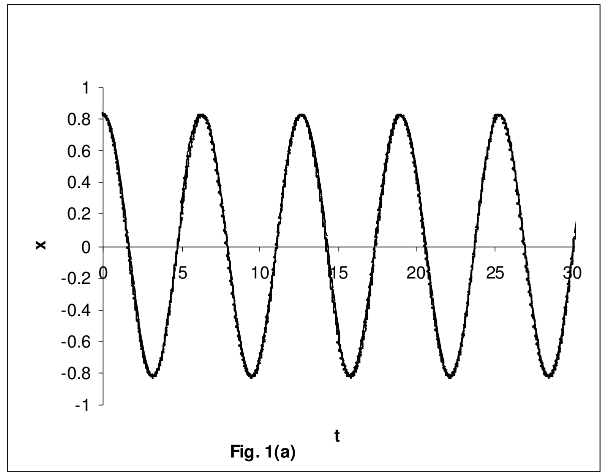

Fig. 1(a): Harmonic balance solution of Eq. (8) is denoted by ($-o-$) and corresponding numerical solutions is denoted by (—). Here a, $\dot{\varphi}$, c_3 and d_3 are calculated by Eq. (11) with initial conditions [$x(0)=0.825878,\ \dot{x}(0)=-0.002111,\ \ddot{x}(0)=-0.814538$] when $\varepsilon=0.1$. The values of unknowns are $a=0.825548$, $\dot{\varphi}=0.991517$, $c_3=0.00059$, $d_3=-0.001261$

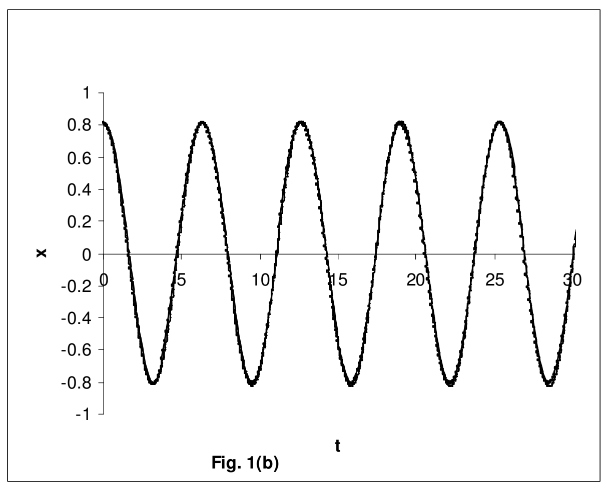

Fig. 1(b): Perturbation solution of Eq. (8) (see [20]) with initial conditions [$x(0)=0.8102,\ \dot{x}(0)=0,\ \ddot{x}(0)=-0.799395$] when $\varepsilon=0.1$ is denoted by (-□-) and corresponding numerical solutions is denoted by (—). The values of unknowns are $\alpha_0=0.82,\ \alpha_1=-0.098$, $\omega=0.991624$.

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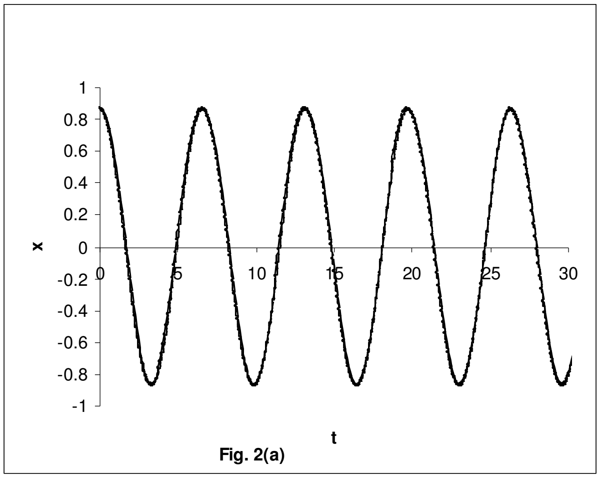

Fig. 2(a): Harmonic balance solution of Eq. (8) is denoted by ($-o-$) and corresponding numerical solutions is denoted by (—). Here a, $\dot{\varphi}$, c_3 and d_3 are calculated by Eq. (11) with initial conditions [$x(0) = 0.867282,\ \dot{x}(0) = -0.01215,\ \ddot{x}(0) = -0.80101$] when $\varepsilon = 0.5$. The values of unknowns are $a = 0.865815$, $\dot{\varphi} = 0.954601$, $c_3 = 0.002259$, $d_3 = -0.00636$.

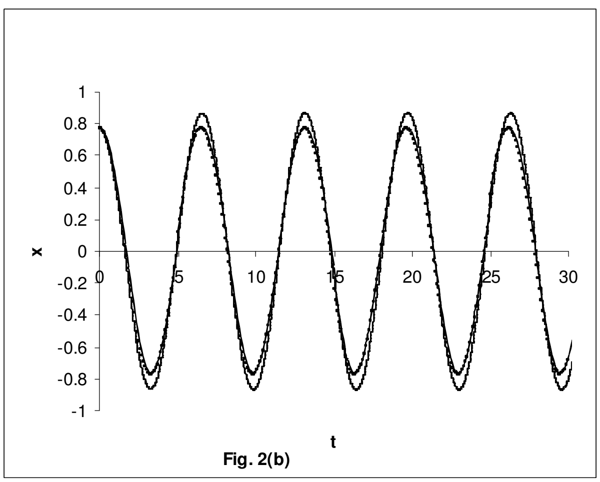

Fig. 2(b): Perturbation solution of Eq. (8) (see [20]) with initial conditions [$x(0) = 0.771,\ \dot{x}(0) = 0,\ \ddot{x}(0) = -0.721149$] when $\varepsilon = 0.5$ is denoted by (-□-) and corresponding numerical solutions is denoted by (—). The values of unknowns are $\alpha_0 = 0.82,\ \ \alpha_1 = -0.098$, $\omega = 0.9586$.

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