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Approximate Riccati Equation and Its Application to Optimal Control in Discrete-Time Systems

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***Abstract*—We present a method of solving an linear quadratic regulator problem approximately in the case of discrete-time systems. This method leaves parameters as symbols in the evaluation function. We introduce the concept of the approximate linear quadratic regulator problem. We also propose a computation method to solve the problem. A numerical example of the approximate linear quadratic regulator problem is also presented.**

KEYWORDS

Linear System Theory, Optimal Control, Discrete-Time System, Symbolic Computation.

I. INTRODUCTION

Optimal control problem is one of the most essential problems in the control theory [1], [2], [3]. However, the solution of an optimal control problem for a linear system is generally based on the maximization or minimization of an evaluation function. However, it is not straightforwardly clear what sort of influence the evaluation function has to the input-output relation[4]. Hence, it is necessary to do a control design with a trial evaluation function in order to obtain the appropriate evaluation function we need.

In this paper, we present a method of solving an optimal control problem approximately in the case of discrete-time linear systems[2]. Our method leaves parameters as symbols in the evaluation function, that is, postpones the determination of the evaluation function until obtaining the optimal input. Thereby, the relation between the evaluation function and the input-output relation becomes explicitly clear.

We consider the linear quadratic method and propose the notion of approximate linear quadratic method. In this paper, we especially consider the linear quadratic regulator problem. In the classical case of this problem, we need to solve the Riccati equation of matrices. To include parameters, we will introduce the notion of the approximate Riccati equation.

II. APPROXIMATE LINEAR QUADRATIC (LQ) REGULATOR PROBLEM

First, we review the classical linear quadratic (LQ) regulator problem[2]. Consider the state space matrix equation and the output matrix equation as follows:

$$\mathbf{x}(k+1) = A_d\mathbf{x}(k) + B_d\mathbf{u}(k), \quad (1)$$

$$\mathbf{y}(k) = C_d\mathbf{x}(k) + D_d\mathbf{u}(k), \quad (2)$$

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respectively, where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{y}_d \in \mathbb{R}^l$. Let $Q_d \in \mathbb{R}^{n\times n}$ and $R_d \in \mathbb{R}^{m\times m}$ are positive semidefinite and positive definite matrices, respectively, which are weighting matrices. The evaluation function J_d is

$$J_d = \sum_{i=0}^{\infty}(\mathbf{x}(i)^t Q_d \mathbf{x}(i) + \mathbf{u}(i)^t R_d \mathbf{u}(i)).$$

Then, it is known that the feedback matrix F_d in which J_d is minimized can be obtained by solving the Riccati equation

$$P_d = Q_d + A_d^t P_d A_d - A_d^t P_d B_d (R_d + B_d^t P_d B_d)^{-1} B_d^t P_d A_d \quad (3)$$

and as follows:

$$F_d = (R_d + B_d^t P_d B_d)^{-1} B_d^t P_d A_d$$

(see, for example, [2]). The Riccati equation of (3) can be solved by numerical iteration.

Based on this classical LQ regulator problem, we propose the notion of *approximate* LQ regulator problem. Consider again the state space matrix equation and the output matrix equation of (1) and (2), respectively. Suppose here that we have v parameters $a_1, a_2, \cdots, a_v$. Let $\widehat{Q}_d \in \mathbb{R}[a_1,\ldots,a_v]^{n\times n}$ and $\widehat{R}_d \in \mathbb{R}[a_1,\ldots,a_v]^{m\times m}$. We assume, without loss of generality, that when all parameters a_i's are set to 0, then the matrices $\widehat{Q}_d$ and $\widehat{R}_d$ are positive semidefinite and positive definite, respectively.

Now we introduce *r-th approximate LQ regulator problem*, where r is a non-negative integer. To state it, we use the expression $\pmod{a_1^{r+1},\ldots,a_v^{r+1}}$ as an abbreviation of $\pmod{a_1^{r+1}}$ $\pmod{a_2^{r+1}}$ $\cdots$ $\pmod{a_v^{r+1}}$. For example,

$$\alpha = \beta \pmod{a_1^{r+1},\ldots,a_v^{r+1}}$$

is equivalent to

$$\alpha = \beta \pmod{a_1^{r+1}} \pmod{a_2^{r+1}} \cdots \pmod{a_v^{r+1}}.$$

The r-th approximate LQ regulator problem is to find the matrix $\widehat{F}_d$ such that the following matrix equations hold:

$$\begin{aligned}\widehat{P}_d = {} & Q_d + A_d^t \widehat{P}_d A_d \\ & - A_d^t \widehat{P}_d B_d (\widehat{R}_d + B_d^t \widehat{P}_d B_d)^{-1} B_d^t \widehat{P}_d A_d \\ & \pmod{a_1^{r+1},\ldots,a_v^{r+1}}, \end{aligned} \quad (4)$$

$$\begin{aligned}\widehat{F}_d = {} & (\widehat{R}_d + B_d^t \widehat{P}_d B_d)^{-1} B_d^t \widehat{P}_d A_d \\ & \pmod{a_1^{r+1},\ldots,a_v^{r+1}}. \end{aligned} \quad (5)$$

When r is obvious, we may omit "r-th" and use "approximate LQ regulator problem."

The definition of the r-th approximate LQ regulator problem requires the equality modulo $a_1^{r+1},\cdots,a_v^{r+1}$. This

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and the sense of "approximate" follow the definitions of the approximate eigenvalue and eigenvector proposed by Kitamoto[5] as well as the approximate factorization proposed by Sasaki *et al.*[6].

We note that the r-th approximate LQ regulator problem defined above is possible to compute naively. For example, we can apply Taylor's expansion. However, this is too ineffective to employ it because this uses derivatives many times in the rational functions, which is much ineffective.

III. Computing the LQ method

In this section, we present a computing method of the r-th approximate LQ regulator problem. First, we consider $\widehat{P}_d$ of (4). Here, observe that in (4), matrices A_d and B_d are matrices over $\mathbb{R}$. Weighting matrices Q_d and R_d are matrices over $\mathbb{R}[a_1, \ldots, a_v]$. Further P_d is a matrix over $\mathbb{R}[a_1, \ldots, a_v]$ such that degree of every entry of P_d is less than or equal to r with respect to each a_i $(1 \le i \le r)$.

Observe further that, in (4), we have a matrix inverse $(\widehat{R}_d + B_d^t \widehat{P}_d B_d)^{-1}$. If we do not use "(mod $a_1^{r+1}, \ldots, a_v^{r+1}$)," this will be a matrix over the formal power series $\mathbb{R}[[a_1, \ldots, a_v]]$. This fact suggests us the possible computation of the matrix inverse $(\widehat{R}_d + B_d^t \widehat{P}_d B_d)^{-1}$.

If all a_i's are set to zero, the equation (4) is equal to (3). This means that the matrix $(\widehat{R}_d + B_d^t \widehat{P}_d B_d)$ with all a_i's being zero has its inverse. We denote by Φ_0 the inverse of the matrix $(\widehat{R}_d + B_d^t \widehat{P}_d B_d)$ with all a_i's being zero. This Φ_0 is a matrix over $\mathbb{R}$.

Consider the matrix $(\widehat{R}_d + B_d^t \widehat{P}_d B_d)\Phi_0$. If all a_i's are zero, this matrix is an identity matrix. Thus, denote by $I - \Psi$ the matrix $(\widehat{R}_d + B_d^t \widehat{P}_d B_d)\Phi_0$. Then, its inverse can be expressed as

$$I + \Psi + \Psi^2 + \Psi^3 + \cdots$$

in a formal power series. More precisely, by letting $\Psi = I - (\widehat{R}_d + B_d^t \widehat{P}_d B_d)\Phi_0$, we have

$$(\widehat{R}_d + B_d^t \widehat{P}_d B_d)^{-1} = \Phi_0(I + \Psi + \Psi^2 + \Psi^3 + \cdots)$$

and further

$$(\widehat{R}_d + B_d^t \widehat{P}_d B_d)^{-1} = \Phi_0(I + \Psi + \Psi^2 + \Psi^3 + \cdots) \quad (6)$$
$$(\text{mod } a_1^{r+1}, \ldots, a_v^{r+1}).$$

Because Ψ does not have nonzero constant entry, (6) can be rewritten as

$$(\widehat{R}_d + B_d^t \widehat{P}_d B_d)^{-1}$$
$$= \Phi_0(I + \Psi + \Psi^2 + \Psi^3 + \cdots + \Psi^r) \quad (7)$$
$$(\text{mod } a_1^{r+1}, \ldots, a_v^{r+1}).$$

Now, we can use the right hand side of (7) instead of $(\widehat{R}_d + B_d^t \widehat{P}_d B_d)^{-1}$ in (4). By this relation, (4) can be rewritten as

$$\widehat{P}_d = Q_d + A_d^t \widehat{P}_d A_d$$
$$-A_d^t \widehat{P}_d B_d \Phi_0(I + \Psi + \Psi^2 + \Psi^3 + \cdots + \Psi^r) B_d^t \widehat{P}_d A_d$$
$$(\text{mod } a_1^{r+1}, \ldots, a_v^{r+1}) \quad (8)$$

Also (5) can be rewritten as

$$\widehat{F}_d = \Phi_0(I + \Psi + \Psi^2 + \Psi^3 + \cdots + \Psi^r) B_d^t \widehat{P}_d A_d$$
$$(\text{mod } a_1^{r+1}, \ldots, a_v^{r+1}).$$

Let $\widehat{P}_{d0}$, $\widehat{Q}_{d0}$, and $\widehat{R}_{d0}$ be the $\widehat{P}_d$, $\widehat{Q}_d$, and $\widehat{R}_d$, respectively, with all a_i's being zero. Further let $\widehat{P}_{d+}$, $\widehat{Q}_{d+}$, and $\widehat{R}_{d+}$ be the $\widehat{P}_d - \widehat{P}_{d0}$, $\widehat{Q}_d - \widehat{Q}_{d0}$, and $\widehat{R}_d - \widehat{R}_{d0}$, respectively. Then $\widehat{P}_{d0}$ can be obtained by solving the following matrix equation:

$$\widehat{P}_{d0} = Q_{d0} + A_d^t \widehat{P}_{d0} A_d \quad (9)$$
$$-A_d^t \widehat{P}_{d0} B_d (\widehat{R}_{d0} + B_d^t \widehat{P}_{d0} B_d)^{-1} B_d^t \widehat{P}_{d0} A_d.$$

This is done by the computation same as in (3).

Then the matrix $(\widehat{R}_d + B_d^t \widehat{P}_d B_d)$ can be rewritten as

$$\widehat{R}_d + B_d^t \widehat{P}_d B_d = \widehat{R}_{d0} + \widehat{R}_{d+} + B_d^t(\widehat{P}_{d0} + \widehat{P}_{d+})B_d$$
$$= \widehat{R}_{d0} + \widehat{R}_{d+} + B_d^t \widehat{P}_{d0} B_d + B_d^t \widehat{P}_{d+} B_d.$$

Thus, we have

$$\Phi_0 = \widehat{R}_{d0} + B_d^t \widehat{P}_{d0} B_d. \quad (10)$$

Now we have

$$\Psi = I - (\widehat{R}_d + B_d^t \widehat{P}_d B_d)\Phi_0$$
$$(\text{mod } a_1^{r+1}, \ldots, a_v^{r+1}), \quad (11)$$
$$\widehat{P}_d = Q_d + A_d^t \widehat{P}_d A_d - A_d^t \widehat{P}_d B_d \Phi_0 (I + \sum_{i=1}^{r} \Psi^r) B_d^t \widehat{P}_d A_d$$
$$(\text{mod } a_1^{r+1}, \ldots, a_v^{r+1}). \quad (12)$$

Now we can write the procedure to obtain $\widehat{P}_d$ of (4) and $\widehat{F}_d$ of (5):

1: Obtain P_{d0} from (9).
2: Let Φ_0 be $\widehat{R}_{d0} + B_d^t \widehat{P}_{d0} B_d$.
3: Let $\widehat{P}_d = \widehat{P}_{d0}$.
4: Let $\Psi = I - (\widehat{R}_d + B_d^t \widehat{P}_d B_d)\Phi_0$ $(\text{mod } a_1^{r+1}, \ldots, a_v^{r+1})$.
5: Let $\widehat{P}_{d\text{new}} = Q_d + A_d^t \widehat{P}_d A_d - A_d^t \widehat{P}_d B_d \Phi_0 (I + \sum_{i=1}^{r} \Psi^i) B_d^t \widehat{P}_d A_d$ $(\text{mod } a_1^{r+1}, \ldots, a_v^{r+1})$.
6: If $\widehat{P}_{d\text{new}}$ is sufficiently equal to $\widehat{P}_d$, then let $\widehat{P}_d = \widehat{P}_{d\text{new}}$ and go to 9.
7: Let $\widehat{P}_d = \widehat{P}_{d\text{new}}$.
8: Go to 4.
9: Let $\widehat{F}_d = \Phi_0 (I + \sum_{i=1}^{r} \Psi^i) B_d^t \widehat{P}_d A_d$ $(\text{mod } a_1^{r+1}, \ldots, a_v^{r+1})$.

10: Return $\widehat{P}_d$ and $\widehat{F}_d$.

In the following section, we will present an example of the computation of the r-th approximate LQ regulator problem.

IV. Example

In this section, we present an example of our result.

We employ, as the example, an inverted pendulum on a cart, shown in Figure 1. The following symbols are used:

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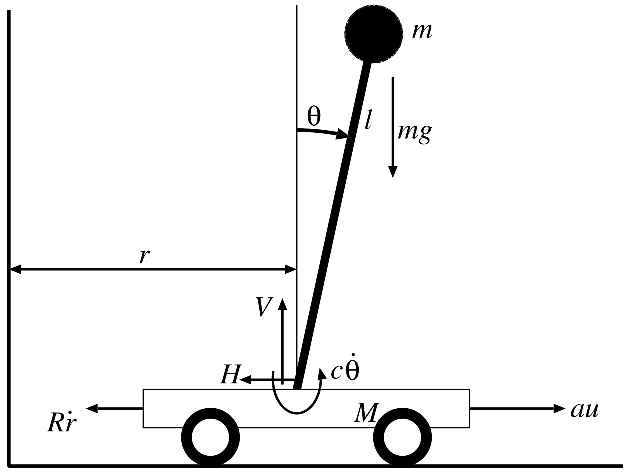

Fig. 1. An inverted pendulum on a cart.

M: Mass of the cart
r: Location of the cart
u: Voltage to the motor of the cart
a: Gain of the voltage to the force
R: Coefficient of viscosity for the motor
H: Horizontal motion of center of gravity of the pendulum rod
J: Moment of inertia of the rod about its center of gravity
θ: Angle of the rod from the vertical line
V: Vertical motion of center of gravity of the pendulum rod
l: Length of the pendulum rod
c: Coefficient of viscosity for the rod
m: Mass of the rod of the pendulum
g: Gravitational acceleration.

This physical model can be described as follows:

$$M\ddot{r} + R\dot{r} = au - H, \quad (13)$$
$$J\ddot{\theta} = Vl\sin(\theta) - Hl\cos(\theta) - c\dot{\theta}, \quad (14)$$
$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(r + l\sin(\theta)) = H, \quad (15)$$
$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(l\cos(\theta)) = V - mg. \quad (16)$$

Deleting V and H from the equations above, we have

$$(M+m)\ddot{r} + ml\cos(\theta)\ddot{\theta} + R\dot{r} - ml\sin(\theta)\dot{\theta}^2 = au, \quad (17)$$
$$ml\cos(\theta)\ddot{r} + (J+ml^2)\ddot{\theta} + c\dot{\theta} - mlg\sin(\theta) = 0. \quad (18)$$

Now we assume that both θ and $\dot{\theta}$ are nearly equal to 0. Then we have $\sin(\theta) = \theta$, $\cos(\theta) = 1$, $\sin(\theta)\dot{\theta} = 0$. Now the equations can be modified as

$$(M+m)\ddot{r} + ml\ddot{\theta} + R\dot{r} = au, \quad (19)$$
$$ml\ddot{r} + (J+ml^2)\ddot{\theta} + c\dot{\theta} - ml\theta g = 0, \quad (20)$$

which can be further modified as

$$\begin{bmatrix} \ddot{r} \\ \ddot{\theta} \end{bmatrix} = \frac{1}{\alpha_0} \begin{bmatrix} 0 & 0 \\ -m^2l^2g & mlg(M+m) \\ -R(J+ml^2) & mlR \\ mlc & -c(M+m) \\ a(J+ml^2) & -mla \end{bmatrix}^t \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \\ u \end{bmatrix}, \quad (21)$$

where $\alpha_0 = J(M+m) + mMl^2$.

This can be described as in the state space matrix equation and the output matrix equation as follows:

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u}, \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u}, \end{aligned}$$

where

$$\mathbf{x} = \begin{bmatrix} r \\ \theta \\ \dot{r} \\ \dot{\theta} \end{bmatrix}, \quad \mathbf{u} = [u], \quad \mathbf{y} = \begin{bmatrix} r \\ \theta \end{bmatrix},$$

$$A = \frac{1}{\alpha_0} \times \begin{bmatrix} 0 & 0 & \alpha_0 & 0 \\ 0 & 0 & 0 & \alpha_0 \\ 0 & -m^2l^2g & -R(J+ml^2) & mlc \\ 0 & mlg(M+m) & mlR & -c(M+m) \end{bmatrix},$$

$$B = \frac{1}{\alpha_0} \begin{bmatrix} 0 \\ 0 \\ a(J+ml^2) \\ -mla \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = O.$$

We here use the following values:

$$\begin{aligned} M &= 2.52 \text{ [kg]} \\ a &= 88.0 \text{ [N/V]} \\ R &= 88.89 \text{ [kg/s]} \\ l &= 0.115 \text{ [m]} \\ c &= 185.95 * 10^{-6} \text{ [kg} \cdot \text{m}^2\text{/s]} \\ m &= 0.1 \text{ [kg]} \\ g &= 9.8 \text{ [m/s}^2\text{]}. \end{aligned}$$

Now we modify this continuous-time system to a discrete-time system with the sampling period $T_s = 6.0 \times 10^{-3}$ [sec]. This is described as in the state space matrix equation and the output matrix equation as follows:

$$\begin{aligned} \mathbf{x}(k+1) &= A_d\mathbf{x}(k) + B_d\mathbf{u}(k), \\ \mathbf{y}(k) &= C_d\mathbf{x}(k) + D_d\mathbf{u}(k), \end{aligned}$$

where

$$A_d = \begin{bmatrix} 1 & -0.000012155 & 0.0053894 & -4.6894 \times 10^{-9} \\ 0 & 1.0030 & 0.0095869 & 0.0060010 \\ 0 & -0.0039102 & 0.80361 & -5.7031 \times 10^{-6} \\ 0 & 0.98480 & 3.0841 & 1.0013 \end{bmatrix},$$

$$B_d = [0.00060449 \quad -0.0094909 \quad 0.19442 \quad -3.0532]^t,$$

$$C_d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$D_d = [0 \quad 0]^t.$$

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We consider the following weight matrices:

$$Q_d = \begin{bmatrix} 1+a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$R_d = [\,0.01\,].$$

Then, from (9), P_{d0} can be computed as:

$$P_{d0} = \begin{bmatrix} 246.35 & 187.01 & 97.138 & 6.6726 \\ 187.01 & 976.62 & 268.88 & 19.886 \\ 97.138 & 268.88 & 124.67 & 8.6264 \\ 6.6726 & 19.886 & 8.6264 & 1.6057 \end{bmatrix}.$$

In the following, we consider the case $r = 5$ and present the result of the 5-th approximate LQ regulator problem.

Based on Section III, we can compute $\widehat{P}_d$ and $\widehat{F}_d$. Let $\widehat{P}_d = (\hat{p}_{dij})$ $\widehat{F}_d = (\hat{f}_{dij})$. Then we have:

$$\begin{aligned}
\hat{p}_{d11} &= 246.35 + 161.42a - 21.877a^2 + 10.612a^3 \\ &\quad -6.5893a^4, \\
\hat{p}_{d12} &= 187.01 + 96.242a - 22.751a^2 + 11.349a^3 \\ &\quad -7.0914a^4, \\
\hat{p}_{d13} &= 97.138 + 56.044a - 10.632a^2 + 5.1617a^3 \\ &\quad -3.1967a^4, \\
\hat{p}_{d14} &= 6.6726 + 3.8126a - 0.73782a^2 + 0.35910a^3 \\ &\quad -0.22257a^4, \\
\hat{p}_{d21} &= 187.01 + 96.242a - 22.751a^2 + 11.349a^3 \\ &\quad -7.0914a^4, \\
\hat{p}_{d22} &= 976.62 + 81.458a - 23.816a^2 + 12.146a^3 \\ &\quad -7.6329a^4, \\
\hat{p}_{d23} &= 268.88 + 42.694a - 11.075a^2 + 5.5198a^3 \\ &\quad -3.4401a^4, \\
\hat{p}_{d24} &= 19.886 + 2.9327a - 0.76895a^2 + 0.38404a^3 \\ &\quad -0.23953a^4, \\
\hat{p}_{d31} &= 97.138 + 56.044a - 10.632a^2 + 5.1617a^3 \\ &\quad -3.1967a^4, \\
\hat{p}_{d32} &= 268.88 + 42.694a - 11.075a^2 + 5.5198a^3 \\ &\quad -3.4401a^4, \\
\hat{p}_{d33} &= 124.67 + 23.349a - 5.1708a^2 + 2.5108a^3 \\ &\quad -1.5510a^4, \\
\hat{p}_{d34} &= 8.6264 + 1.5979a - 0.35888a^2 + 0.17467a^3 \\ &\quad -0.10798a^4, \\
\hat{p}_{d41} &= 6.6726 + 3.8126a - 0.73782a^2 + 0.35910a^3 \\ &\quad -0.22257a^4, \\
\hat{p}_{d42} &= 19.886 + 2.9327a - 0.76895a^2 + 0.38404a^3 \\ &\quad -0.23953a^4, \\
\hat{p}_{d43} &= 8.6264 + 1.5979a - 0.35888a^2 + 0.17467a^3 \\ &\quad -0.10798a^4, \\
\hat{p}_{d44} &= 1.6057 + 0.10939a - 0.024909a^2 + 0.012152a^3 \\ &\quad -0.0075185a^4
\end{aligned}$$

and

$$\begin{aligned}
\hat{f}_{d11} &= -0.32118 - 0.160208a + 0.0402236a^2 \\ &\quad -0.0201227a^3 + 0.0125831a^4, \\
\hat{f}_{d21} &= -2.16539 - 0.141132a + 0.0421333a^2 \\ &\quad -0.0215371a^3 + 0.0135442a^4, \\
\hat{f}_{d31} &= -1.48484 - 0.0731351a + 0.0195841a^2 \\ &\quad -0.00978713a^3 + 0.00610421a^4, \\
\hat{f}_{d41} &= -0.363138 - 0.00502926a + 0.00135979a^2 \\ &\quad -0.000680941a^3 + 0.000425018a^4.
\end{aligned}$$

We compare this result and the classical numerical result. The error e_i of $\hat{f}_{di1}$ for ($i = 1$ to 4) is given as

$$e_i(a) = \frac{|f_{di1} - \hat{f}_{di1}|}{|f_{di1}|}.$$

Figures 2 to 5 show these errors. From these figures, we have obtained that when a is between -0.94 and 2.48, all errors are less than 0.5. This shows that the result of the r-th approximate LQ regulator problem gives almost exact result when the parameter is nearly equal to zero.

V. Conclusion

In this paper, we have introduced the notion of the r-th approximate LQ regulator problem, in which the evaluation function can include parameters as symbols. We have presented the computation method to solve the r-th approximate LQ regulator problem in which the evaluation function can include parameters as symbols. Also, a numerical example has been presented, which has shown the effectiveness of the notion of the r-th approximate LQ regulator problem. Detailed evaluation of the computation method we have proposed should be done in the near future.

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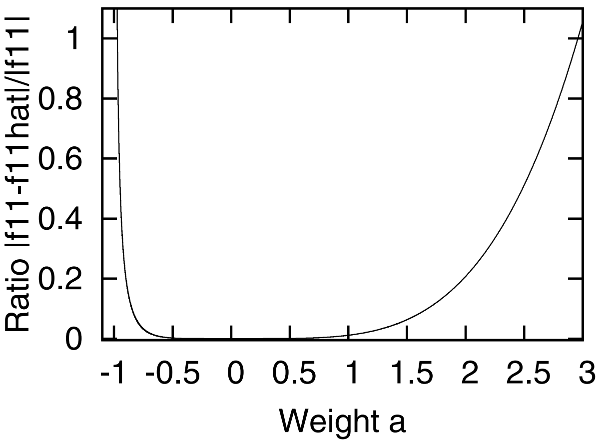

Fig. 2. Error of $\hat{f}_{d11}$.

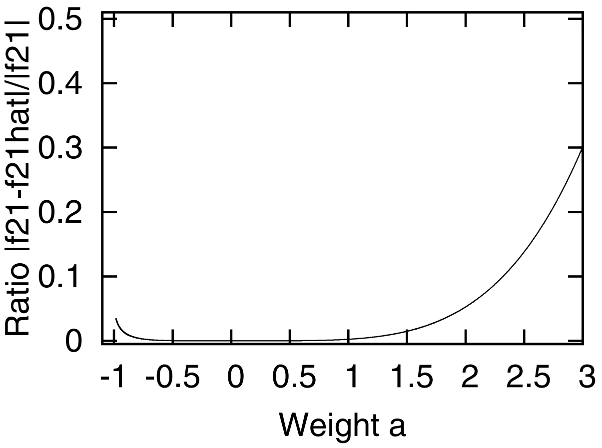

Fig. 3. Error of $\hat{f}_{d21}$.

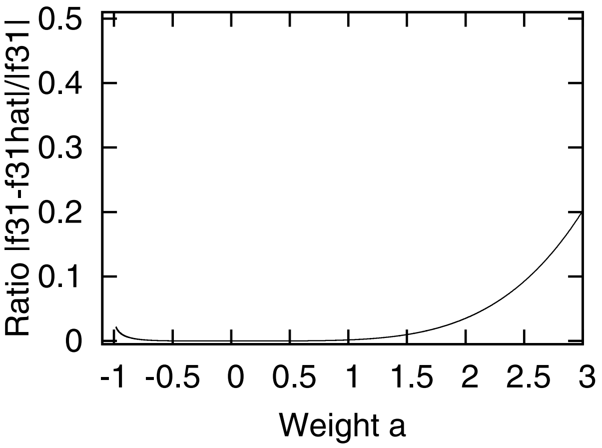

Fig. 4. Error of $\hat{f}_{d31}$.

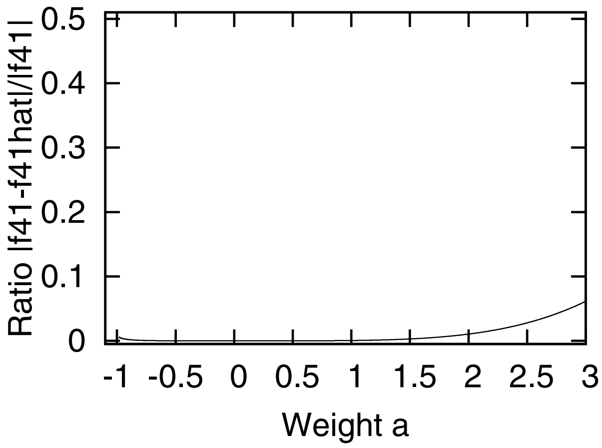

Fig. 5. Error of $\hat{f}_{d41}$.

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