Numerical Solution of de St. Venant Equations with Controlled Global Boundaries Between Unsteady Subcritical States

Aldrin P. Mendoza, Adrian Roy L. Valdez, and Carlene P. Arceo

***Abstract*—This paper aims to verify numerical existence of boundary controls that steer the de St. Venant system in finite time, from a given unsteady subcritical state to another. The method of characteristics is used in obtaining the numerical solution. The problem is divided into two parts: first, an unsteady subcritical flow is steered towards a steady one; then such flow is steered towards another unsteady subcritical state.**

***Index Terms*—global controllability; first order quasilinear hyperbolic partial differential equation; de St. Venant equations.**

I. Introduction

FLUID flow in a slightly inclined rectangular open channel with friction is modeled by the de St. Venant equations. If the open channel is defined lengthwise by $x \in [0, L]$, the average velocity of flow by $v(x,t)$, depth of flow by $h(x,t)$, θ as the slope of the open channel, s_f for the coefficient of friction, and g for the gravitational acceleration, then the de St. Venant equations can be written in the following form:

$$\begin{cases} v\partial_x h + h\partial_x v + \partial_t h &= 0, \\ g\partial_x h + v\partial_x v + \partial_t v &= g(\theta - s_f); \end{cases} \tag{1}$$

Equations in (1) are also known as continuity and momentum equations, respectively. The continuity equation is established by equating the net inflow and the rate of change of fluid in the control volume. On the other hand, the momentum equation is derived by applying the second law of motion on the fluid inside the control volume. The control volume is a fixed volume in space wherein fluid may flow in and out. It is taken as volume from infinitesimal length of the channel dx over a small period of time dt.

The de St. Venant equations can be written in a variety of form. It can be written in matrix form as

$$\partial_t \begin{pmatrix} h \\ v \end{pmatrix} + \begin{pmatrix} v & h \\ g & v \end{pmatrix} \partial_x \begin{pmatrix} h \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ g(\theta - s_f) \end{pmatrix}. \tag{2}$$

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The partial differential equations can be reduced into ordinary differential equations known as the characteristic form of the de St. Venant equations, written as follows:

$$\frac{d(v \pm 2c)}{dt} = \frac{\partial(v \pm 2c)}{\partial x}\frac{dx}{dt} + \frac{\partial(v \pm 2c)}{\partial t} = g(\theta - s_f), \tag{3}$$

for $\frac{dx}{dt} = v \pm c$, where $c(t,x) = (gh(x,t))^{\frac{1}{2}}$ is the wave celerity.

They can also be written as a first order quasilinear hyperbolic system of diagonal form

$$\partial_t \begin{pmatrix} R^+ \\ R^- \end{pmatrix} + A\left(R^+, R^-\right) \partial_x \begin{pmatrix} R^+ \\ R^- \end{pmatrix} = \begin{pmatrix} g(\theta - s_f) \\ g(\theta - s_f) \end{pmatrix}, \tag{4}$$

where $R^+ = v + 2c$, $R^- = v - 2c$ are the Riemann invariants and $A\left(R^+, R^-\right)$ is the diagonal matrix

$$\begin{aligned} A\left(R^+, R^-\right) &= \begin{pmatrix} \frac{3}{4}R^+ + \frac{1}{4}R^- & 0 \\ 0 & \frac{1}{4}R^+ + \frac{3}{4}R^- \end{pmatrix} \\ &= \begin{pmatrix} v + c & 0 \\ 0 & v - c \end{pmatrix}. \end{aligned}$$

Numerous studies concerning boundary controllability for quasilinear hyperbolic system were established [1], [2], [9], [6], [5], [7], [8]. Gugat and Leugering [4] were able to prove that one can control globally the flow of fluid from one steady subcritical state to another (see also [3]). Particularly, [4] considered fluid flow through a frictionless horizontal channel that is described by the quasilinear hyperbolic form of the de St. Venant equations

$$\partial_t \begin{pmatrix} R^+ \\ R^- \end{pmatrix} + A\left(R^+, R^-\right) \partial_x \begin{pmatrix} R^+ \\ R^- \end{pmatrix} = 0. \tag{5}$$

Recently, Mendoza, Valdez, and Arceo [10] extended the main result in [4]. They were able to show that the de St. Venant system (5) can be steered from a given unsteady subcritical state to another in finite time by applying nonlinear boundary controls in such a way that the solution is continuously differentiable. Likewise, the derivatives of the solution and boundary controls are also bounded.

In this paper, we consider the numerical implementation as a verification for the existence of a global boundary control for the de St. Venant systems (5) from a given unsteady subcritical state to another in finite time.

II. Results

In this section we state the following results proved in [10] concerning the de St. Venant equations steered between unsteady subcritical states. For the definitions and notations please refer to [10]. We start by stating the main theorem as follows:

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Theorem 1. *Consider the de St. Venant system (5) with boundary conditions of the form* $R^+(0,t) = g_1(t)$ *and* $R^-(L,t) = g_2(t)$*. Let a nonempty compact rectangular set* $\Omega = [a_+, b_+] \times [a_-, b_-] \subset \mathbb{R}^2$ *be given such that for all* $(d,e) \in \Omega$*, we have* $\frac{3}{4}d + \frac{1}{4}e > 0$ *and* $\frac{1}{4}d + \frac{3}{4}e < 0$*, i.e.,* Ω *contains only subcritical states.*

Given an $\epsilon > 0$*, let* $\Omega_\epsilon = [a_+ + \frac{\epsilon}{2}, b_+ - \frac{\epsilon}{2}] \times [a_- + \frac{\epsilon}{2}, b_- - \frac{\epsilon}{2}]$*. Then we can find boundary controls* g_1*,* g_2 *that steer the de St. Venant system with boundary conditions* $R^+(0,t) = g_1(t)$ *and* $R^-(L,t) = g_2(t)$ *in finite time* T *from any unsteady initial state* $\Phi \in C^1([0,L], \mathbb{R}^2)$*,* $\Phi[0,L] \subseteq C^1(\Omega_\epsilon)$ *to an unsteady state* $\Psi \in C^1([0,L], \Omega)$ *in such a way that the corresponding solution is continuously differentiable.*

Moreover, this can be achieved in such a way that the absolute values of the derivatives of the solution and of the controls g_1*,* g_2 *remain smaller than any given upper bound.*

For the outline of proof of Theorem 1, two lemmas are used. Lemma 1 is concerned with applying boundary controls which steer the de St. Venant system from unsteady subcritical initial state Φ which is continuously differentiable Ω_ϵ-valued function on $[0,L]$ to a constant terminal subcritical state $\Phi_1 \in \Omega$ in finite time T_1 provided the norm of the difference of the initial and final states is bounded by $\alpha > 0$. Moreover, during this transition the corresponding solution is continuously differentiable. Furthermore, the absolute values of the derivatives of the solution and the controls are bounded. On the other hand, Lemma 2 is concerned with applying boundary controls which steer the de St. Venant system from a steady subcritical state $\Phi_1 \in \Omega$ to an unsteady subcritical state Ψ which is continuously differentiable Ω-valued function on $[0,L]$ in finite time T_2. Also, during this transition the solution is continuously differentiable. Likewise, the absolute values of the derivatives of the solution and the controls are bounded. Thus the theorem is proved by combining the two lemmas.

III. Numerical Solution

In this section we utilize the method of characteristics in determining the numerical solution of the de St. Venant system steered from a given state to another (unsteady to steady or vice versa) in finite time. Numerically, we consider only the extended length of the open channel $L_e = L + 2\delta$, where L is the length of the channel and $\delta > 0$. The extended length of the open channel is subdivided into N Δx intervals for $N \in \mathbb{N}$. Initially at time $t = 0$, for every point in the extended length of the open channel, velocity, height and celerity which are all continuously differentiable functions are given initial values. Likewise, the following constants are used, gravitational acceleration $g = 9.81$, slope of the channel $\theta = 0$ and coefficient of friction $s_f = 0$.

The solution for a particular point P in the advanced time $t + \Delta t_1$ from a previous time t is numerically the intersection of the forward characteristic $\frac{d(v+2c)}{dt}$ for $\frac{dx}{dt} = v + c$ and backward characteristic $\frac{d(v-2c)}{dt}$ for $\frac{dx}{dt} = v - c$ emanating from points Q and R at time t. The region bounded by the forward and backward characteristics from points Q and R through point P is referred to as the domain of dependence of point P. When determining the value of the solution at a particular point P, the information coming from the domain of dependence of point P is sufficient to achieve stability and accuracy of the numerical solution at point P. Thus stability and accuracy of the numerical solution depend on the values of Δt and Δx. The forward characteristic from point Q to point P has slope $\frac{dt}{dx} = \frac{1}{v+c}$ and the backward characteristic from point R to point P has slope $\frac{dt}{dx} = \frac{1}{v-c}$. To remain within the domain of dependence of point P, $\frac{dx}{dt} > (v + c)$ must be satisfied. This stability criteria is referred to as the Courant-Friedrichs-Lewy (CFL) condition. Hence, to ensure stability of numerical calculation for every point in the extended length of the channel for a particular period of time, we have to choose the time step Δt_1 such that $\Delta t_1 \le \Delta t = \frac{\Delta x}{\max(v_i + c_i)}$, where v_i is the initial velocity, c_i is the initial celerity at initial time $t = 0$ for $i = 1, \cdots, N+1$.

We now formulate an algorithm in steering the de St. Venant system from a given state to another state in finite time T by the following pseudocode below.

Step 1. Set the values of the following:
- time T;
- length L;
- $\delta > 0$;
- $L_e = L + 2\delta$;
- $N \in \mathbb{N}$;
- $\Delta x = \frac{L_e}{N}$;
- $x = [0 : N] * \Delta x$;
- $g = 9.81$;
- slope of the open channel θ;
- coefficient of friction s_f;
- initial states: velocity v_i, height h_i, celerity c_i;
- $\Delta t = \frac{\Delta x}{\max(v_i + c_i)}$;
- $\frac{T}{\Delta t_1} \in \mathbb{N}$ s.t. $\Delta t_1 < \Delta t$ (CFL condition);
- $T = [0 : \frac{T}{\Delta t_1} = T_N] * \Delta t_1$.

Step 2. Define the initial condition on $[-\delta, L + \delta]$.
for $n = 1, ..., N + 1$
- $v_i(x_n, 0)$;
- $c_i(x_n, 0)$;
- $h_i(x_n, 0)$.

Let

$$\begin{aligned} V &= [v_i(x_1,0) \cdots v_i(x_{N+1},0)]\,; \\ C &= [c_i(x_1,0) \cdots c_i(x_{N+1},0)]\,; \\ H &= [h_i(x_1,0) \cdots h_i(x_{N+1},0)]\,. \end{aligned}$$

Step 3. For $n = 1, ..., N + 1$ determine $v(x_n, t + 1)$ and $c(x_n, t + 1)$ from a previous time t.

for $t = 1, ..., T_N$
  for $n = 1, ..., N$
    set the values of v_m, c_m as

$$v_m(x_n,t) = \frac{v_i(x_n,t) + v_i(x_{n+1},t)}{2};$$
$$c_m(x_n,t) = \frac{c_i(x_n,t) + c_i(x_{n+1},t)}{2};$$

  for $n = 2, ..., N$

$$\begin{aligned} v(x_n,t+1) &= \tfrac{1}{2}v_m(x_{n-1},t) + \tfrac{1}{2}v_m(x_n,t) \\ &\quad + c_m(x_{n-1},t) - c_m(x_n,t) \\ &\quad + \Delta t * g(\theta - s_f); \\ c(x_n,t+1) &= \tfrac{1}{4}v_m(x_{n-1},t) - \tfrac{1}{4}v_m(x_n,t) \\ &\quad + \tfrac{1}{2}c_m(x_{n-1},t) + \tfrac{1}{2}c_m(x_n,t); \\ h(x_n,t+1) &= \frac{c^2(x_n,t+1)}{g}; \end{aligned}$$

  for $n = 1$

$$v(x_1,t+1) = v_m(x_1,t);$$
$$c(x_1,t+1) = c_m(x_1,t);$$

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$h(x_1, t+1) = \frac{c^2(x_1,t+1)}{g}$;
for $n = N+1$
$v(x_{N+1}, t+1) = v_m(x_N, t)$;
$c(x_{N+1}, t+1) = c_m(x_N, t)$;
$h(x_{N+1}, t+1) = \frac{c^2(x_{N+1},t+1)}{g}$;

$$V(x_1 : x_{N+1}, t+1) = \left[V; v(x_1 : x_{N+1}, t+1)^T\right];$$

$$C(x_1 : x_{N+1}, t+1) = \left[C; c(x_1 : x_{N+1}, t+1)^T\right];$$

$$H(x_1 : x_{N+1}, t+1) = \left[H; h(x_1 : x_{N+1}, t+1)^T\right];$$

$$v_i(x_1 : x_{N+1}, t) = v(x_1 : x_{N+1}, t+1);$$

$$c_i(x_1 : x_{N+1}, t) = c(x_1 : x_{N+1}, t+1).$$

Step 4. Plot the graph of the height, velocity or celerity of the open channel.

IV. Numerical Examples

Executing our code in Scilab version 4.1.1, we establish th following numerical examples.

We first numerically verify the existence of boundary controls that steer the de St. Venant system from a given unsteady subcritical state to a steady subcritical state. We define an initial state Φ which is a continuously differentiable function satisfying conditions of Theorem 1 in [10].

Example 1. We want the unsteady subcritical state $\Phi(x) = (\Phi_1(x), \Phi_2(x)) \in \Omega_\epsilon$, where $\Phi_1(x) = v(x) + 2c(x)$ and $\Phi_2(x) = v(x) - 2c(x)$ steered to a constant subcritical state $(d_2, e_2) \in \Omega$. To do this, first let $\delta > 0$ be given. To work with the Cauchy problem the initial state Φ must be defined in an infinite domain. Thus we define explicitly the function $\widehat{\Phi} = (\widehat{\Phi}_1(x), \widehat{\Phi}_2(x))$ as follows:

$$\widehat{\Phi}_1(x) = \begin{cases} d_2, & x \in (-\infty, -\delta), \\ d_2 + f_1(x), & x \in [-\delta, 0), \\ \Phi_1(x), & x \in [0, L], \\ d_2 + f_1(L+\delta-x), & x \in (L, L+\delta], \\ d_2, & x \in (L+\delta, \infty], \end{cases}$$

and

$$\widehat{\Phi}_2(x) = \begin{cases} e_2, & x \in (-\infty, -\delta), \\ e_2 + f_2(x), & x \in [-\delta, 0), \\ \Phi_2(x), & x \in [0, L], \\ e_2 + f_2(L+\delta-x), & x \in (L, L+\delta], \\ e_2, & x \in (L+\delta, \infty], \end{cases}$$

where

$$f_i(x) = \alpha_i(x)\left(\frac{2x^3}{\delta^3} - \frac{x^4}{\delta^4}\right) \text{ for } i \in \{1, 2\},$$

with

$$\alpha_1(x) = \Phi_1(x) - d_2 \text{ and } \alpha_2(x) = \Phi_2(x) - e_2.$$

Initially for the height of the fluid we define a nonconstant C^1-function as

$$\widehat{h}_i(x) = \begin{cases} 0.5, & x \in (-\infty, -\delta), \\ \beta_1(x), & x \in [-\delta, 0), \\ \beta_2(x), & x \in [0, L], \\ \beta_3(x), & x \in (L, L+\delta], \\ 0.5, & x \in (L+\delta, \infty], \end{cases}$$

where

$$\beta_1(x) = \frac{\left(\sqrt{0.2(9.81)} + \frac{3\gamma_1(x)}{\delta^3}\left(5\frac{(x-5)^2}{2} - \frac{(x-5)^3}{3}\right)\right)^2}{9.81},$$
$$\beta_2(x) = (3 + cos(2\pi x)),$$
$$\beta_3(x) = \frac{\left(\sqrt{0.2(9.81)} + \frac{3\gamma_2(x)}{\delta^3}\left(5\frac{(25-x)^2}{2} - \frac{(25-x)^3}{3}\right)\right)^2}{9.81},$$

with

$$\gamma_1(x) = 2\left(\sqrt{9.81(3 + \cos(2\pi(x-5))} - \sqrt{9.81(0.2)}\right),$$

$$\gamma_2(x) = 2\left(\sqrt{9.81(3 + \cos(2\pi(25-x))} - \sqrt{9.81(0.2)}\right),$$

and we define the initial velocity as

$$\widehat{V}_i(x) = \quad 0.9(1.2 + \cos(\pi x)), \quad \text{for all } x \in \mathbb{R} \ .$$

Numerically, we only consider the strip $[-\delta, L+\delta] \times [0, T_1]$, where $L = 10$, $\delta = 5$, and $T_1 > 0$. The interval $[0, L]$ corresponds to the space interval $[10, 20]$, the interval $[-\delta, 0]$ corresponds to $[5, 10]$ and the interval $[L, L+\delta]$ to $[20, 25]$.

Figures 1, 2, 3 correspond to the initial height, initial velocity, and initial celerity of the fluid, which are all continuously differentiable functions, respectively. Note that we satisfy the conditions of Theorem 1 in [10], i.e., we consider only the subcritical state of flow.

Fig. 4 shows the numerical existence of nonlinear boundary controls that steer the de St. Venant system from an initial velocity and an initial height (rightmost) defined above towards a constant height of 0.5 (leftmost) with a constant velocity of 1.98 in finite time T_1.

Second, we verify numerically the existence of boundary controls that steer the de St. Venant system from a given steady subcritical state to an unsteady subcritical state in finite time T_2.

Example 2. Now, consider the steady subcritical state $(d_2, e_2) \in \Omega$. To direct it towards an unsteady subcritical state $\Psi(x) = (\Psi_1(x), \Psi_2(x)) \in \Omega$, where $\Psi_1(x) = v(x) + 2c(x)$, and $\Psi_2(x) = v(x) - 2c(x)$, we work with the Cauchy problem, i.e., we must extend the initial state in an infinite domain. Define explicitly the function $\widehat{\Psi}(x) = \left(\widehat{\Psi}_1(x), \widehat{\Psi}_2(x)\right)$ as follows:

$$\widehat{\Psi}_1(x) = \begin{cases} \Psi_{11}(x), & x \in (-\infty, 0), \\ d_2, & x \in [0, L], \\ \Psi_{12}(x), & x \in (L, \infty], \end{cases}$$

and

$$\widehat{\Psi}_2(x) = \begin{cases} \Psi_{21}(x), & x \in (-\infty, 0), \\ e_2, & x \in [0, L], \\ \Psi_{22}(x), & x \in (L, \infty), \end{cases}$$

where $\Psi_{ij}(x)$ are given for $i, j \in 1, 2$.

Initially for the height of the fluid we define a C^1-function

$$\widehat{h}_i(x) = \begin{cases} 0.5(2 + \cos(\pi x)), & x \in (-\infty, 0), \\ 0.5, & x \in [0, L], \\ 0.5(2 + \cos(\pi x)), & x \in (L, \infty), \end{cases}$$

and we define the initial velocity

$$\widehat{V}_i(x) = 1.96, \text{ for all } x \in \mathbb{R}.$$

Again, numerically, we only consider the strip $[-\delta, L+\delta] \times [0, T_2]$, where $L = 10$, $\delta = 15$, and $T_2 > 0$. The interval

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$[0, L]$ corresponds to the space interval $[15, 25]$, the interval $[-\delta, 0]$ corresponds to $[0, 15]$ and the interval $[L, L + \delta]$ to $[25, 40]$.

Figures 5, 6, 7 correspond to the initial height, initial velocity, initial celerity of the fluid, which are all continuously differentiable functions, respectively. Note also that we satisfy the conditions of Theorem 1 in [10], i.e., we consider only the subcritical state of flow.

Fig. 8 shows the numerical existence of boundary controls that steer the de St. Venant system from a constant height of 0.5 (rightmost) with a constant velocity of 1.98 towards a varying height (leftmost) in finite time T_2.

Combining the two numerical examples above, we have shown numerically the existence of boundary controls that steer the de St. Venant system from a given unsteady state $\Phi \in C^1([0, L], \Omega_\epsilon)$ towards another unsteady state $\Psi \in ([0, L], \Omega)$ in finite time $T \geq T_1 + T_2$.

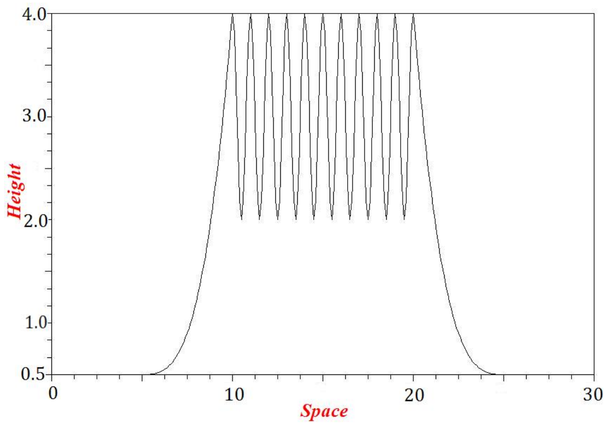

Fig. 1. Fluid initial unsteady height.

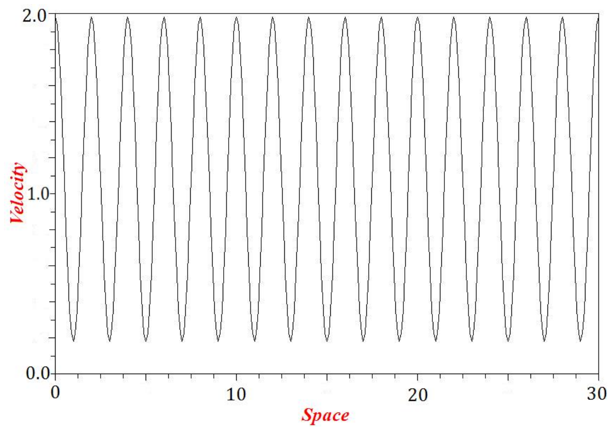

Fig. 2. Fluid initial unsteady velocity.

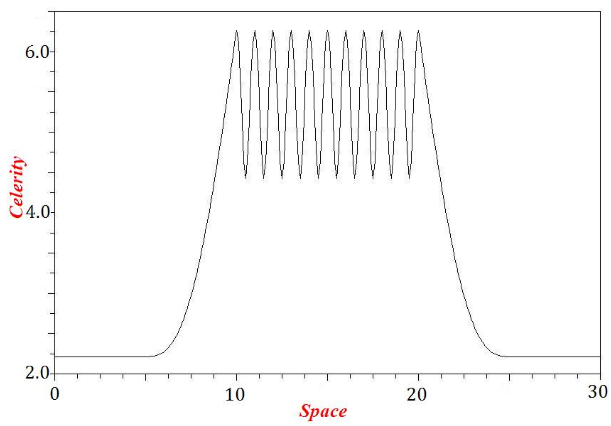

Fig. 3. Fluid initial celerity.

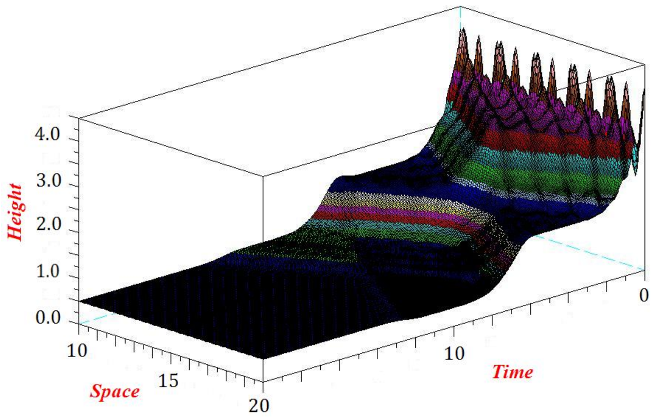

Fig. 4. Fluid height (Unsteady subcritical state to steady state).

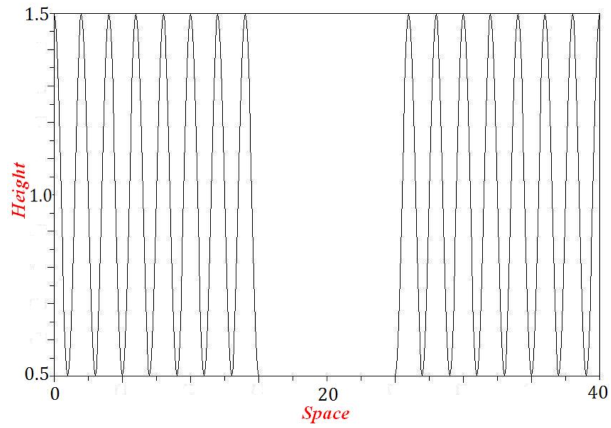

Fig. 5. Fluid initial height.

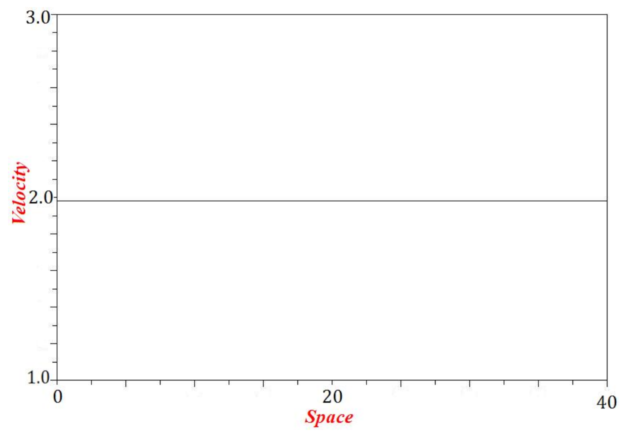

Fig. 6. Fluid initial velocity.

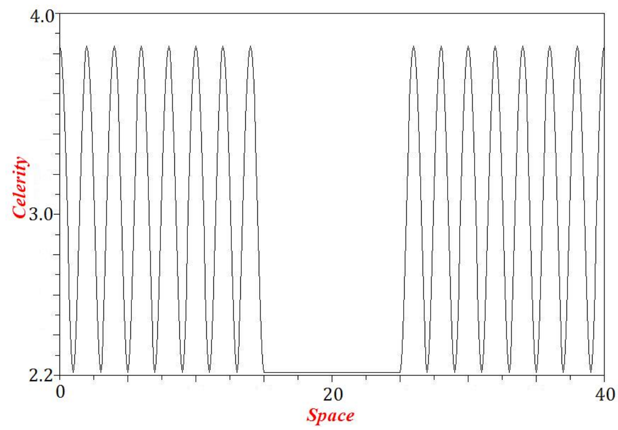

Fig. 7. Fluid initial celerity.

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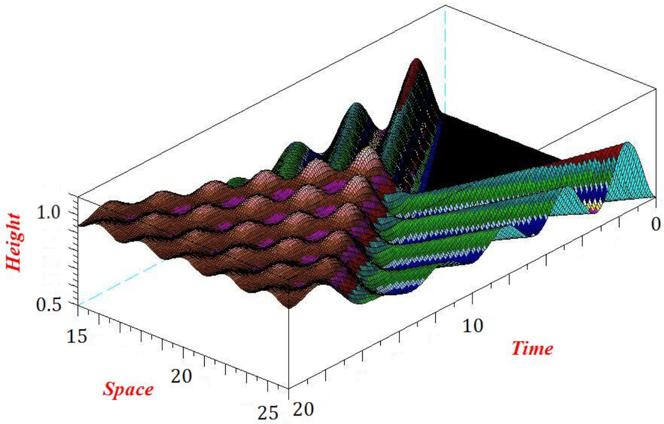

Fig. 8. Fluid height (Steady state to unsteady subcritical state).

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