Proceedings of the International MultiConference of Engineers and Computer Scientists 2012 Vol II,
IMECS 2012, March 14 - 16, 2012, Hong Kong

New Results for Synchronized Triomineering

Tao Cao, Alessandro Cincotti and Hiroyuki Iida

***Abstract*—In synchronized games players make their moves simultaneously rather than alternately. Synchronized Triomineering is the synchronized version of Triomineering, a classic two-player combinatorial game. Experimental results for small $\mathbf{m \times n}$ boards with $\mathbf{m + n \leq 14}$ and theoretical results for the $\mathbf{n \times 7}$ and $\mathbf{n \times 8}$ boards are presented.**

***Index Terms*—combinatorial game, synchronized game, Synchronized Triomineering.**

I. Introduction

THE game of Triomineering is a two-player game with perfect information, proposed in 2004 by Blanco and Fraenkel [1]. In Triomineering two players, usually denoted by Vertical and Horizontal, take turns in placing ”straight” triominoes (3×1 tile) on a checkerboard.

Vertical is only allowed to place its triominoes vertically and Horizontal is only allowed to place its triominoes horizontally on the board. Triominoes are not allowed to overlap and the first player that cannot find a place for one of its triominoes loses. After a time the remaining space may separate into several disconnected regions, and each player must choose into which region to place a triomino.

In the game of Synchronized Triomineering [2], a general instance and the legal moves for Vertical and Horizontal are defined exactly in the same way as defined for the game of Triomineering.

There is only one difference: Vertical and Horizontal make their legal moves simultaneously, therefore, triominoes are allowed to overlap if they have a 1×1 tile in common. We note that 1×1 overlap is only possible within a simultaneous move. At the end, if both players cannot make a move, then the game ends in a draw, else if only one player can still make a move, then he/she is the winner.

In Synchronized Triomineering, for each player there exist three possible outcomes:

- The player has a *winning strategy* (ws) independently of the opponent's strategy, or
- The player has a *drawing strategy* (ds), i.e., he/she can always get a draw in the worst case, or
- The player has a *losing strategy* (ls), i.e., he/she does not have a strategy for winning or for drawing.

Table I shows all the possible cases. It is clear that if one player has a winning strategy, then the other player has neither a winning strategy nor a drawing strategy. Therefore, the cases $ws - ws$, $ws - ds$, and $ds - ws$ never happen. As a consequence, if G is an instance of Synchronized Triomineering, then we have six possible legal cases:

- $G = D$ if both players have a drawing strategy, and the game will always end in a draw under perfect play, or
- $G = V$ if Vertical has a winning strategy, or

The authors are with the School of Information Science, Japan Advanced Institute of Science and Technology, 1-1 Asahidai, Nomi, Ishikawa 923-1292 Japan (phone: +81-761-51-1293; e-mail: cincotti@jaist.ac.jp).

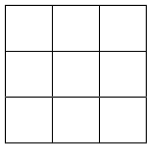
TABLE I
The possible outcomes in Synchronized Triomineering

	Horizontal ls	Horizontal ds	Horizontal ws
Vertical ls	$G = VHD$	$G = HD$	$G = H$
Vertical ds	$G = VD$	$G = D$	–
Vertical ws	$G = V$	–	–

- $G = H$ if Horizontal has a winning strategy, or
- $G = VD$ if Vertical can always get a draw in the worst case, but he/she could be able to win if Horizontal makes a wrong move, or
- $G = HD$ if Horizontal can always get a draw in the worst case, but he/she could be able to win if Vertical makes a wrong move, or
- $G = VHD$ if both players have a losing strategy and the outcome is totally unpredictable.

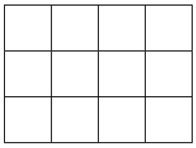
II. Examples of Synchronized Triomineering

The game



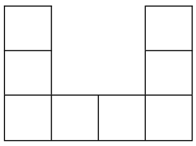
always ends in a draw, therefore $G = D$.

In the game

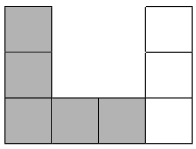
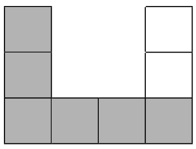


Vertical has a winning strategy moving in the second (or in the third) column, therefore $G = V$.

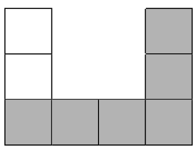
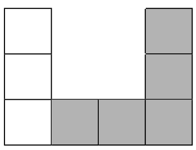
In the game



if Vertical moves in the first column we have two possibilities


or


therefore, either Vertical wins or the game ends in a draw. Symmetrically, if Vertical moves in the third column we have two possibilities


or


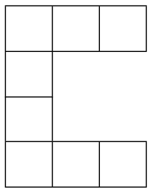
ISBN: 978-988-19251-9-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

TABLE II
OUTCOMES FOR RECTANGLES IN SYNCHRONIZED TRIOMINEERING

	3	4	5	6	7	8	9	10	11
3	D	V	V	D	V	V	D	V	V
4	H	D	V	H	H	HD	H	H	
5	H	H	D	H	H	H	H		
6	D	V	V	D	V	V			
7	H	V	V	H	?				
8	H	VD	V	H					
9	D	V	V						
10	H	V							
11	H								

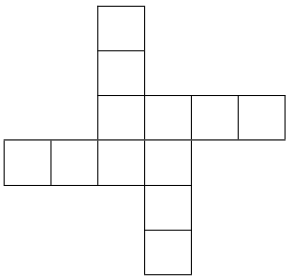
therefore, either Vertical wins or the game ends in a draw. It follows $G = VD$.

Symmetrically, in the game



either Horizontal wins or the game ends in a draw therefore, $G = HD$.

In the game



each player has 4 possible moves. For every move of Vertical, Horizontal can win or draw (and sometimes lose); likewise, for every move by Horizontal, Vertical can win or draw (and sometimes lose). As a result it follows that $G = VHD$.

III. NEW RESULTS

Table II shows the results obtained using an exhaustive search algorithm for $n \times m$ boards with $n + m \leq 14$. The case 7×7 is still unsolved but, because of the symmetry of the board, either $G = D$ or $G = VHD$.

Theorem 1: Let G be a $n \times 7$ board of Synchronized Triomineering with $n \geq 27$. Then, Vertical has a winning strategy.

Proof: In the beginning, Vertical will always move into the third and the fifth column of the board, i.e., (k, c), $(k+1, c)$, $(k+2, c)$, (k, e), $(k+1, e)$, and $(k+2, e)$, where $k \equiv 1 \pmod 3$, as shown in Fig. 1.

When Vertical cannot move anymore into the third and the fifth column, let us imagine that we divide the main rectangle into 3×7 sub-rectangles starting from the top of the board (by using horizontal cuts). Of course, if $n \not\equiv 0 \pmod 3$, then the last sub-rectangle will be of size either 1×7 or 2×7, and Horizontal will be able to make respectively either two more move or four more moves.

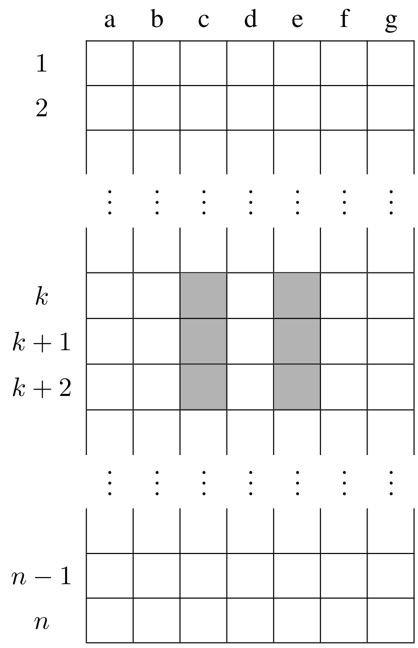


Fig. 1. Vertical strategy on the $n \times 7$ board of Synchronized Triomineering.

We can classify all these sub-rectangles into 13 different classes according to:

- The number of vertical triominoes already placed in the sub-rectangle (vt),
- The number of horizontal triominoes already placed in the sub-rectangle (ht),
- The number of moves that Vertical is able to make in the worst case, in all the sub-rectangles of that class (vm),
- The number of moves that Horizontal is able to make in the best case, in all the sub-rectangles of that class (hm),

as shown in Table III. We denote with $|A|$ the number of sub-rectangles in the A class, with $|B|$ the number of sub-rectangles in the B class, and so on. The value of vm in all the sub-rectangles belonging to the class D, E, H, and I considered as a group is

$$2|D| + \lceil |D|/2 \rceil + \lceil |E|/2 \rceil + 2|H| + \lceil |I|/2 \rceil$$

The last statement is true under the assumption that Vertical moves first into the sub-rectangles of class H as long as they exist, second into the sub-rectangles of class D, E, and I as long as they exist, and finally into the sub-rectangles of the other classes. When Vertical cannot move anymore into the third and the fifth column, both Vertical and Horizontal have placed the same number of triominoes, therefore

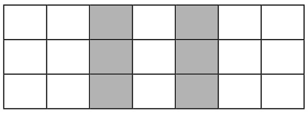
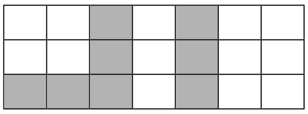
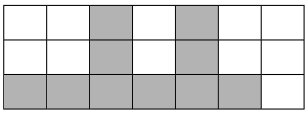
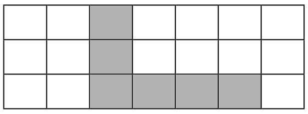
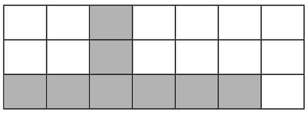
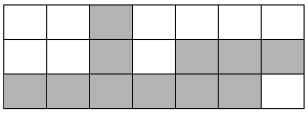
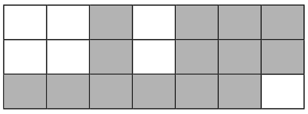
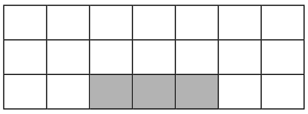
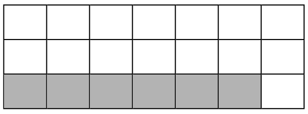
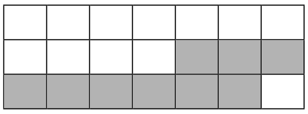
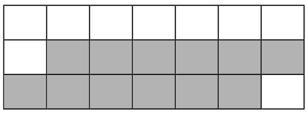
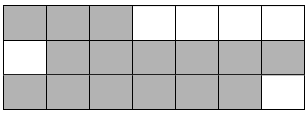
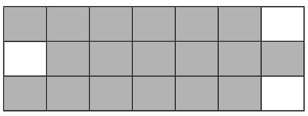
$$\begin{aligned} 2|A| + |B| \quad = \quad & |E| + 2|F| + 3|G| + |H| + 2|I| + \\ & 3|J| + 4|K| + 5|L| + 6|M| \end{aligned} \qquad (1)$$

Let us prove by contradiction that Vertical can make a larger number of moves than Horizontal. Assume therefore $moves(V) \leq moves(H)$ using the data in Table III

$$\begin{aligned} & 5|A| + 3|B| + |C| + 2|D| + \lceil |D|/2 \rceil + \\ & \qquad \lceil |E|/2 \rceil + 2|H| + \lceil |I|/2 \rceil \quad \leq \\ & 2|D| + 2|E| + |F| + 2|H| + 4|I| + \end{aligned}$$

ISBN: 978-988-19251-9-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

TABLE III
THE 13 CLASSES FOR 3×7 SUB-RECTANGLES

	vt	ht	vm	hm	Example
A	2	0	$5\|A\|$	0	
B	2	1	$3\|B\|$	0	
C	2	2	$\|C\|$	0	
D	1	1	$*$	$2\|D\|$	
E	1	2	$*$	$2\|E\|$	
F	1	3	0	$\|F\|$	
G	1	4	0	0	
H	0	1	$*$	$2\|H\|$	
I	0	2	$*$	$4\|I\|$	
J	0	3	0	$3\|J\|$	
K	0	4	0	$2\|K\|$	
L	0	5	0	$\|L\|$	
M	0	6	0	0	

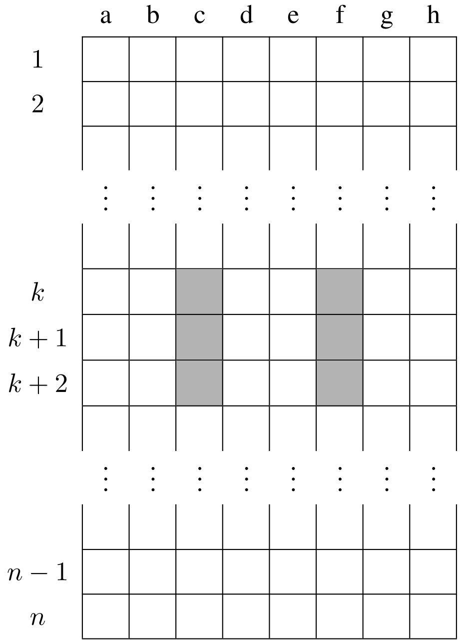


Fig. 2. Vertical strategy on the $n \times 8$ board of Synchronized Triomineering.

$$3|J| + 2|K| + |L| + 4$$

and applying Equation 1

$$\begin{aligned} |A| + |B| + |C| + \lceil |D|/2 \rceil + \lceil |E|/2 \rceil + \\ 3|F| + 6|G| + 2|H| + \lceil |I|/2 \rceil + \\ 3|J| + 6|K| + 9|L| + 12|M| \quad \leq \quad 4 \end{aligned}$$

which is false because

$$\begin{aligned} |A| + |B| + |C| + |D| + |E| + |F| + |G| + \\ |H| + |I| + |J| + |K| + |L| + |M| \quad = \quad \lfloor n/3 \rfloor \end{aligned}$$

and by hypothesis $n \geq 27$. Therefore, $moves(V) \leq moves(H)$ does not hold and consequently $moves(H) < moves(V)$. We observe that if $n \equiv 1 \pmod 3$, then the theorem holds for $n \geq 16$ and if $n \equiv 0 \pmod 3$, then the theorem holds for $n \geq 3$. ■

Theorem 2: Let G be a $n \times 8$ board of Synchronized Triomineering with $n \geq 15$. Then, Vertical has a winning strategy.

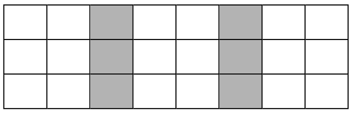
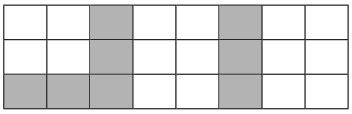
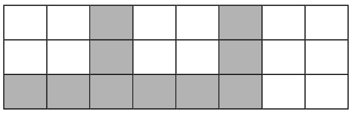
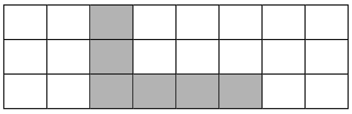
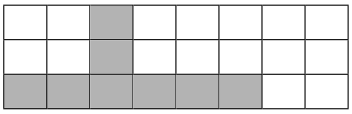
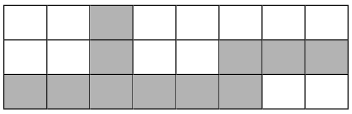
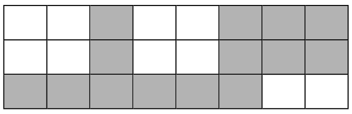
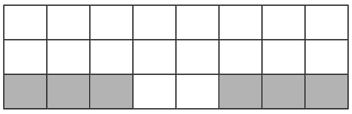
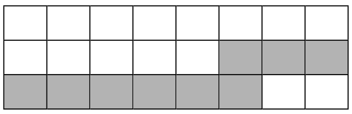
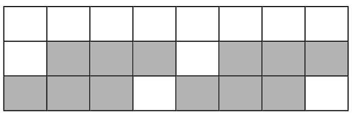
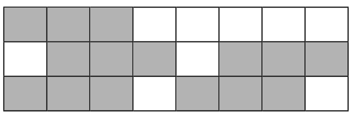
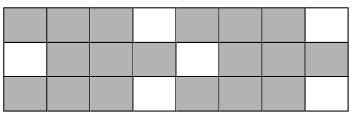
Proof: In the beginning, Vertical will always move into the third and the sixth column of the board, i.e., (k, c), $(k+1, c)$, $(k+2, c)$, (k, f), $(k+1, f)$, and $(k+2, f)$, where $k \equiv 1 \pmod 3$, as shown in Fig. 2. When Vertical cannot move anymore into the third and the sixth column, let us imagine that we divide the main rectangle into 3×8 sub-rectangles starting from the top of the board (by using horizontal cuts). Of course, if $n \not\equiv 0 \pmod 3$, then the last sub-rectangle will be of size either 1×8 or 2×8, and Horizontal will be able to make respectively either two more move or four more moves.

We can classify all these sub-rectangles into 12 different classes according to:

- The number of vertical triominoes already placed in the sub-rectangle (vt),
- The number of horizontal triominoes already placed in the sub-rectangle (ht),

ISBN: 978-988-19251-9-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

TABLE IV
THE 12 CLASSES FOR 3×8 SUB-RECTANGLES

	vt	ht	vm	hm	Example
A	2	0	$6\|A\|$	0	
B	2	1	$4\|B\|$	0	
C	2	2	$2\|C\|$	0	
D	1	1	$*$	$2\|D\|$	
E	1	2	$*$	$2\|E\|$	
F	1	3	0	$\|F\|$	
G	1	4	0	0	
H	0	2	$*$	$4\|H\|$	
I	0	3	0	$3\|I\|$	
J	0	4	0	$2\|J\|$	
K	0	5	0	$\|K\|$	
L	0	6	0	0	

- The number of moves that Vertical is able to make in the worst case, in all the sub-rectangles of that class (vm),
- The number of moves that Horizontal is able to make in the best case, in all the sub-rectangles of that class (hm),

as shown in Table IV.

We denote with $|A|$ the number of sub-rectangles in the A class, with $|B|$ the number of sub-rectangles in the B class, and so on. The value of vm in all the sub-rectangles belonging to the class D, E, and H considered as a group is

$$3|D| + |E| + \lceil 3|I|/4 \rceil$$

The last statement is true under the assumption that Vertical moves first into the sub-rectangles of class H as long as they exist, second into the sub-rectangles of class D and E as long as they exist, and finally into the sub-rectangles of the other classes.

When Vertical cannot move anymore into the third and the sixth column, both Vertical and Horizontal have placed the same number of triominoes, therefore

$$\begin{aligned} 2|A| + |B| \quad = \quad & |E| + 2|F| + 3|G| + 2|H| + 3|I| + \\ & 4|J| + 5|K| + 6|L| \end{aligned} \tag{2}$$

Let us prove by contradiction that Vertical can make a larger number of moves than Horizontal. Assume therefore $moves(V) \leq moves(H)$ using the data in Table IV

$$\begin{aligned} 6|A| + 4|B| + 2|C| + 3|D| + |E| + \lceil 3|H|/4 \rceil \quad \leq \\ 2|D| + 2|E| + |F| + 4|H| + 3|I| + 2|J| + |K| + 4 \end{aligned}$$

and applying Equation 2

$$\begin{aligned} 2|A| + 2|B| + 2|C| + |D| + |E| + 3|F| + 6|G| + \\ \lceil 3|H|/4 \rceil + 3|I| + 6|J| + 9|K| + 12|L| \quad \leq \quad 4 \end{aligned}$$

which is false because

$$\begin{aligned} |A| + |B| + |C| + |D| + |E| + |F| + |G| + \\ |H| + |I| + |J| + |K| + |L| \quad = \quad \lfloor n/3 \rfloor \end{aligned}$$

and by hypothesis $n \geq 15$. Therefore, $moves(V) \leq moves(H)$ does not hold and consequently $moves(H) < moves(V)$. We observe that if $n \equiv 1 \pmod 3$, then the theorem holds for $n \geq 10$ and if $n \equiv 0 \pmod 3$, then the theorem holds for $n \geq 3$. ■

By symmetry the following two theorems hold.

Theorem 3: Let G be a $7 \times n$ board of Synchronized Triomineering with $n \geq 27$. Then, Horizontal has a winning strategy.

Theorem 4: Let G be a $8 \times n$ board of Synchronized Triomineering with $n \geq 15$. Then, Horizontal has a winning strategy.

REFERENCES

[1] S. A. Blanco and A. S. Fraenkel, "Triomineering, Tridomineering, and L-Tridomineering," *Technical Report MCS04-10, Department of Computer Science and Applied Mathematics, The Weizmann Institute of Science.* Available: http://wisdomarchive.wisdom.weizmann.ac.il:81/archive/00000367/

[2] A. Cincotti, S. Komori and H. Iida, "The game of Synchronized Triomineering and Synchronized Tridomineering," *International Journal of Computational and Mathematical Sciences*, vol. 2, no. 3, pp. 143–148, 2008.

ISBN: 978-988-19251-9-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)