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Monte Carlo Simulation of a Two-Factor Stochastic Volatility Model

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Abstract—**Empirical phenomenon in financial markets such as volatility smiles and term structure of implied volatilities made stochastic volatility models more attractive. In this paper we consider a multi-factor stochastic volatility model with two mean reverting factors and the analytical approximation formula given by Fouque et al. [5] for a vanilla European call. Using the European call option pricng problem as our test problem we compared crude MC estimator with the randomized quasi-Monte Carlo method. Our findings show that using the randomized low-discrepancy sequences such as Faure sequence, one can reduce the variance of the estimator and achieve faster convergence compared to crude Monte Carlo simulation.**

Keywords: ***Two-factor stochastic volatility model, Monte Carlo simulation, Randomized Quasi-Monte Carlo, Scrambled Faure sequence***

1 Introduction

Volatility is one of the most important factors in the pricing of financial derivatives. Although the standard Black-Scholes model with the constant volatility assumption serves as a good basis for modeling the asset price dynamics, there has been a wide-range of research done to improve upon this model. Black-Scholes model has been extended to incorporate either constant volatility plus jumps or stochastic volatility with/without jumps to provide a better fit of implied volatility surfaces. In particular, a mean-reverting stochastic process is suitable for modeling volatility. Mean-reversion refers to a linear pull back term in the mean growth rate of the volatility process. This class of models is known as stochastic volatility models. A recent alternative to these approaches is to consider volatility driven by two factors.

The mean reversion of volatility is captured using two stochastic processes one with a slow mean reversion and the other with a fast mean reversion property. Under this framework Fouque et al. [7] used a combination of singular and regular perturbations to derive a closed form formula to approximate European option prices. In an earlier work by Fouque et al. [4] fast mean reverting volatility was considered. In another study by Christoffersen et al. [2] two factor stochastic volatility model is used and the use of two factors is justified with principle component analysis of the Black-Scholes implied volatility surface. Principle components analysis shows that the first two eigenvalues account for more than 95% of the total variation in the volatility process. Hence, this shows actually few number of factors would be sufficient to explain most of the variation in the volatility process. However, mean reversing parameters are not categorized as fast and slow mean reverting as it is done in the studies by Fouque et al. [5], [6], and [7]. Market data shows the need for introducing also a slowly varying factor into the model for stochastic volatility. In particular, for options with longer maturities the addition of slow varying factor is useful to improve the fit. The calibration to the market data is also presented in Fouque et al. [5] with supporting evidence in favor of using fast and slow mean reversion parameters.

Empirical evidence suggests that multi-factor stochastic volatility models provide significant improvements in explaining the term structure of volatility. Under the multi-factor stochastic volatility models Monte Carlo simulation methods can be used to evaluate option prices efficiently. Fouque and Han [3] proposed the use of variance reduction techniques such as importance sampling and control variate methods to evaluate risk neutral pricing of financial derivatives in a multi-factor volatility model. In this paper, we consider the European call option problem using the same multi-factor volatility model and test the efficiency of randomized quasi-Monte Carlo method, particulary the linear and digit scrambled Faure sequence.

This paper is organized as follows: In section 2, the two factor stochastic volatility model is introduced with option price approximations for the European call option. In Section 3 we briefly discuss the quasi-Monte Carlo and randomization of quasi-Monte Carlo sequences such as the Faure sequence. In this section we also present our numerical results. Finally, the conclusion in Section 5.

2 Multi-Factor Stochastic Volatility Model

In the study by Fouque et al. [7], a combination of singular and regular perturbations are used to derive the

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asymptotic approximation formula for the vanilla European call option price. A class of multi-factor volatility models has been introduced which are driven by two diffusions, one fluctuating on a fast time scale and another one fluctuating on a slower time scale. It has been shown that it is possible to combine a singular perturbation expansion with respect to the fast scale with a regular perturbation expansion with respect to the slow scale. This again leads to a leading term which is the Black-Scholes price with a constant effective volatility.

Following the multi-factor stochastic volatility model in [7] and in [3], we consider the family of stochastic volatility model, where S_t is the underlying price, Y_t evolves as an Ornstein-Uhlenbeck (OU) process, and Z_t follows another diffusion process. Given the risk neutral probability measure $\mathbb{P}^*$, the stochastic volatility model is described with the following equations:

$$dS_t = rS_t dt + \sigma_t S_t dW_t^0 \tag{1}$$

$$\sigma_t = f(Y_t, Z_t) \tag{2}$$

$$dY_t = \left(\alpha(m_f - Y_t) - \nu_f\sqrt{2\alpha}\Lambda_f(Y_t, Z_t)\right)dt + \nu_f\sqrt{2\alpha}\left(\rho_1 dW_t^0 + \sqrt{1-\rho_1^2}dW_t^1\right). \tag{3}$$

$$dZ_t = \left(\delta(m_s - Z_t) - \nu_s\sqrt{2\delta}\Lambda_s(Y_t, Z_t)\right)dt + \nu_s\sqrt{2\delta}\left(\rho_2 dW_t^0 + \rho_{12}dW_t^1 + \sqrt{1-\rho_2^2-\rho_{12}^2}dW_t^2\right). \tag{4}$$

Here (W_t^0, W_t^1, W_t^2) are independent standard Brownian motions with instant correlation coefficients ρ_1, ρ_2, and ρ_{12} such that $\rho_1^2 < 1$, and $\rho_2^2 + \rho_{12}^2 < 1$, whereas S_t is the underlying stock price process with the risk free rate equal to r. The volatility process σ_t is driven by two factors; Y_t and Z_t. As stated in [3] the risk neutral probability measure $\mathbb{P}^*$ is determined by the market prices of volatility risk captured in Λ_f and Λ_s, which are assumed to be bounded and independent of the stock price S. The driving processes for Y_t and Z_t are mean reverting around their long run mean m_f and m_s, respectively when Λ_f and Λ_s are equal to zero. The mean reversion rate for Y_t and Z_t are $\alpha > 0$ and $\delta > 0$ respectively, where Y_t is fast mean reverting on a short time scale $1/\alpha$ and Z_t is slowly varying on a long time scale $1/\delta$ (i.e. $\alpha^{-1} < 1 < \delta^{-1}$). The time scales of Y_t and Z_t are given as $1/\alpha$ and $1/\delta$ with the "vol-vol" parameter equal to $\nu_f\sqrt{2\alpha}$ and $\nu_s\sqrt{2\delta}$, with long run distribution of OU process given as $N(m_f, \nu_f^2)$ and $N(m_s, \nu_s^2)$ as prototypes of more general ergodic diffusions. Following Fouque and Han [3], in the numerical computations we use the the asymptotic theory in the mean reversion regime where $\alpha \to \infty$, $\delta \to 0$.

Let $H(S_T)$ denote the payoff from an European call option, which is a function of the final stock price S_T. Hence under the risk neutral measure $\mathbb{P}^*$ the no arbitrage price is given by the conditional expectation of the discounted payoff given the current levels of (S_t, Y_t, Z_t):

$$P(t,x,y,z) = \mathbb{E}^*\left[e^{-r(T-t)}H(S_T)|S_t = x, Y_t = y, Z_t = z\right]. \tag{5}$$

The crude Monte Carlo estimate of the option price is given by

$$P(t,x,y,z) \approx \frac{1}{N}\sum_{k=1}^{N} e^{-r(T-t)}H(S_T^{(k)}), \tag{6}$$

where $S_T^{(k)}$ is the price at maturity found by the kth simulation using the Euler discretization of the process, and N is the total number of simulations.

2.1 Option Price Approximations

An approximation under the two-factor stochastic volatility model is given by Fouque et al. [4] using the asymptotic expansion theory. Asymptotic expansion theory can be applied by introducing the parameter $\epsilon = 1/\alpha$. Hence, both ϵ and δ are small quantities such that $0 < \epsilon, \delta << 1$. We use a singular expansion about ϵ and a regular expansion about δ. By applying the Feynman-Kac formula to (5), the formula for $P^{\epsilon,\delta}$ can be obtained as given in [4] with the price dependent on ϵ and δ. The pointwise price approximation is

$$P^{\epsilon,\delta}(t,x,y,z) \approx \widetilde{P}(t,x,z),$$

where

$$\widetilde{P} = P_{BS(\bar{\sigma})} + (T-t)\times \left(V_0\frac{\partial}{\partial\sigma} + V_1 x\frac{\partial^2}{\partial x\partial\sigma} + V_2 x^2\frac{\partial^2}{\partial x^2} + V_3 x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2}{\partial x^2}\right)\right)P_{BS(\bar{\sigma})}, \tag{7}$$

with an accuracy of order $(\epsilon|\log\epsilon| + \delta)$ for call options. The price $P_{BS}(\bar{\sigma})$ is the homogenized price which solves the Black-Scholes equation

$$\mathcal{L}_{BS(\bar{\sigma})} = 0 \tag{8}$$

$$P_{BS(\bar{\sigma})}(T,x) = H(x), \tag{9}$$

where $\bar{\sigma}$ is called the effective volatility and is defined by

$$\bar{\sigma}^2 =< f^2(.,z) > . \tag{10}$$

The brackets are used to denote the average with respect to the invariant distribution $N(m_f, \nu_f^2)$ of the fast factor (Y_t). The parameters (V_0, V_1, V_2, V_3) are

$$V_0 = -\frac{\nu_s\sqrt{\delta}}{\sqrt{2}}\langle\Lambda_s\rangle\bar{\sigma}', \tag{11}$$

$$V_1 = \frac{\rho_2\nu_s\sqrt{\delta}}{\sqrt{2}}\langle f\rangle\bar{\sigma}', \tag{12}$$

$$V_2 = \frac{\nu_f\sqrt{\epsilon}}{\sqrt{2}}\langle\Lambda_f\frac{\partial\phi}{\partial y}\rangle, \tag{13}$$

$$V_3 = -\frac{\rho_1\nu_f\sqrt{\epsilon}}{\sqrt{2}}\langle f\frac{\partial\phi}{\partial y}\rangle, \tag{14}$$

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where $\bar{\sigma}'$ denotes the derivative of $\bar{\sigma}$. The function $\phi(y,z)$ is the solution of the Poisson equation

$$\mathcal{L}_0\phi(y,z) = f^2(y,z) - \bar{\sigma}^2(z).$$

Equation (7) gives us two approximations. The first is the actual solution $\widetilde{P}$ and if we ignore all terms of order $\sqrt{\epsilon}$, $\sqrt{\delta}$ or higher we get the first order approximation $P_{BS(\bar{\sigma})}$. It should also be noted that if a higher order approximation is desired it is possible to redo the asymptotic expansion keeping more terms to get a higher order approximation, but the approximation will be much more complicated than the already complicated $\widetilde{P}$.

Table 1: Parameter values for our numerical experiments

S_0	Y_0	Z_0	K	T	r	m_f	m_s	n_f	n_s
55	-1	-1	50	1	1	-0.8	-0.8	0.5	0.8

ρ_1	ρ_2	ρ_{12}	Λ_f	Λ_s	$f(y,z)$	Δt	N
-0.2	-0.2	0	0	0	$\exp(y+z)$	0.01	5000

3 Quasi-Monte Carlo (QMC) Method

The general problem for which QMC methods have been proposed as an alternative to the MC method is multidimensional numerical integration. One might consider the QMC method as the deterministic version of the Monte Carlo simulation. Convergence of the MC estimator is based on the central limit theorem, whereas in the QMC method a deterministic upper bound for the error is given by an upper bound.

The idea of QMC method is to use a more regularly distributed point set, so that a better sampling of the function can be achieved. An important difference with the MC method is that the point set P_N is deterministic. A detailed discussion of these methods can be found in Niederreiter [19]. The use of more "regular" or "uniform" points in the QMC method means these sequences satisfy certain uniformity conditions. A commonly used measure for uniformity of a QMC sequence is the *star discrepancy* $D^*(P_N)$ of a point set P_N, which looks at the difference between the volume of a rectangular box aligned with axes of $[0,1)^s$ and having a corner at the origin, and the fraction of points from P_N contained in the box, and then the maximum difference over all such boxes. Typically, a point set P_N is called a *low discrepancy sequence* if $D^*(P_N) = O(N^{-1}\log^s N)$. For a function of bounded variation in the sense of Hardy and Krause, the integration error $|I - \hat{I}|$ is in $O(N^{-1}\log^s N)$ when we use a low-discrepancy sequence in the integration (see Niederreiter [19] for details).

The convergence rate for QMC method by using low-discrepancy sequences suggests that if the dimension of the integration is high the advantages of QMC might be lost or more precisely QMC methods will require a very large sample size when the dimension s is high. However, we should note that our error analysis in QMC method is based on an upper bound for the error and for some problems with smooth integrands the observed convergence rate might be much faster than the convergence rate of the upper bound. There are several studies in the literature showing this phenomenon some examples can be found in [1], [10], and [15].

3.1 Faure Sequence

In this subsection we describe the generation of a commonly used quasi-Monte Carlo sequence, namely Faure sequence. Faure sequence has certain advantages for the valuation of high dimensional integrals, which leads to the efficient algorithms for the computation of complex financial derivatives. Applications of the Faure sequence in complex financial derivatives can be found in [12].

To compute the nth element of the Faure sequence, we start by representing any integer n in terms of the base b,

$$n = \sum_{j=0}^{m} a_j^1 b^j. \tag{15}$$

The first element of the Faure sequence is given by reflection about the decimal point as before,

$$\phi_b^1(n) = \sum_{j=0}^{m} a_j^1 b^{-j-1}. \tag{16}$$

The remaining elements of the sequence can be found recursively. As also explained in Joy et al. [12], given $a_j^{k-1}(n)$ the next term can be written as

$$a_j^k(n) = \sum_{j=1}^{m} C_j^i a_i^{k-1}(n) \pmod b, \tag{17}$$

where $C_j^i = \frac{i!}{j!(i-j)!}$. Hence, to generate the next dimension in the sequence we multiply with the generator matrix, which is the upper triangular matrix with entries as the binomial coefficients,

$$\begin{pmatrix} C_0^0 & C_0^1 & C_0^2 & C_0^3 & \vdots \\ 0 & C_1^1 & C_1^2 & C_1^3 & \\ & & & & \\ 0 & 0 & C_2^2 & C_2^3 & \vdots \\ 0 & 0 & 0 & C_3^3 & \end{pmatrix}.$$

Hence, the rest of the points can be generated by,

$$\phi_b^k(n) = \sum_{j=0}^{m} a_j^k(n) b^{-j-1}, \quad 2 \le k \le d. \tag{18}$$

We can represent the nth element of the d dimensional Faure sequence by,

$$\phi_n = (\phi_n^1, ..., \phi_n^d).$$

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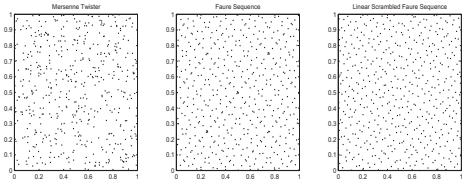

Figure 1: Two dimensional projection of the generated points from Mersenne Twister, Faure Sequence and Scrambled Faure Sequence ($d1 = 1, d2 = 2, base = 2$)

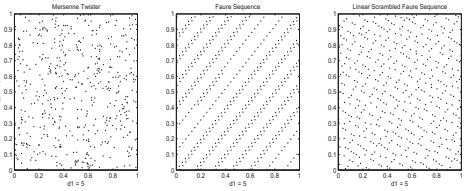

Figure 2: Two dimensional projection of the generated points from Mersenne Twister, Faure Sequence and Scrambled Faure Sequence ($d1 = 5, d2 = 40, base = 41$)

Example 1 *The following table displays the first ten elements of the Faure sequence.*

Construction of the Faure sequence

n	$a_0(n)$	$a_1(n)$	$a_2(n)$	ϕ_n^1	ϕ_n^2	ϕ_n^3
1	1	0	0	1/3	1/3	1/3
2	2	0	0	2/3	2/3	2/3
3	0	1	0	1/9	4/9	7/9
4	1	1	0	4/9	7/9	1/9
5	2	1	0	7/9	1/9	4/9
6	0	2	0	2/9	8/9	5/9
7	1	2	0	5/9	2/9	8/9
8	2	2	0	8/9	5/9	2/9
9	0	0	1	1/27	16/27	13/27
10	1	0	1	10/27	25/27	22/27

3.2 Randomized Quasi-Monte Carlo

Quasi-Monte Carlo is based on the use of deterministic low discrepancy sequences with nice uniformity properties. However, for two main reasons often we are interested in using the randomized version of these sequences. First reason is the convenience of using confidence intervals while preserving much of the accuracy of pure quasi-Monte Carlo. Thus, randomized quasi-Monte Carlo method seeks to combine the best features of Monte Carlo and quasi-Monte Carlo methods. Secondly, there are settings in which randomizing a low discrepancy sequence actually improves accuracy. An important result showing that the root mean square error of integration using a class of randomized nets is $O(1/N^{1.5-\epsilon})$, whereas the error without randomization is $O(1/N^{1-\epsilon})$ for smooth integrands. Even though Owen's [18] result is restrictive in terms of pricing financial derivatives it is an important example to show that randomization can take advantage of the smoothness of the integrand which quasi-Monte Carlo alone cannot achieve. Discussion on improving accuracy through randomization of low discrepancy sequences can be found in Hickernell [11], Matousek [14], and L'Ecuyer and Lemieux [13].

It is often useful to randomize QMC point sets for the purpose of error analysis. Two desirable properties that a given randomization should have are: (i) each point in the randomized point set should have a uniform distribution on the s-dimensional hypercube; (ii) the uniformity of the original point set should be preserved.

Owen's scrambling method is computationally demanding. Matoušek [14] introduced an alternative scrambling approach which is not only efficient but also satisfies some of the theoretical properties of Owen's scrambled nets and sequences. In Owen's scrambling is given in [18] and in Ökten and Eastman [16]. Instead of Owen's scrambling method we prefer to use the computationally less costly Matousek scrambling method (see Glasserman [9] for details on scrambling quasi-Monte Carlo sequences).

As well known, quasi-Monte Carlo methods have better convergence rate, at least asymptotically, of $O(\log^d N/N)$, whereas Monte Carlo methods have convergence rate of $O(N^{-1/2})$, where N is the sample size or the number of simulations. In many problems we do not have analytical formulas, this increased the popularity of quasi-Monte Carlo methods and special softwares has been designed for this purpose. Low discrepancy sequences are deterministic, hence we get a single estimate of the result. This is a drawback for QMC method, since having many estimates of the unknown quantity we can construct confidence intervals. Furthermore, the deterministic error bound due to Koksma-Hlawka inequality is not computationally feasible for most of the problems. Therefore, computing the standard deviation and constructing a confidence band for our estimates is quite desirable in a quasi-Monte Carlo simulation. To address this drawback of quasi-Monte Carlo methods, researchers introduced randomized versions of quasi-Monte Carlo methods, where we still have the good uniformity properties of low discrepancy sequences but also a statistical error analysis is available. A good discussion can be found in L'Ecuyer and Lemieux [13] and in Ökten and Eastman [16].

Estimate

$$I = \int_{[0,1)^d} f(x)dx \tag{19}$$

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using sums of the form

$$Q(q_u) = \frac{1}{N}\sum_{n=1}^{N} f(q_u^{(n)}) \tag{20}$$

where q_u is a family of d-dimensional low-discrepancy sequences indexed by the random parameter u.

Matoušek [14] proposes a linear digit scrambling method, by multiplying the generator matrix of a low discrepancy sequence by a random matrix for which entries consisting of $0, 1, 2, ..., b-1$ and adding a random vector $\vec{U}$ in mod b. This method is easier to implement compared to the full scrambling, and for the jth permutation it is applied to the digit a_j by a simple choice of π_j given by

$$\pi_j(a_j) = h_j a_j + g_j \pmod b \tag{21}$$

where $h_j \in \{0, 1, 2, ..., b-1\}$ and $g_j \in \{0, 1, 2, ..., b-1\}$ are random integers.

Another choice is given by

$$\pi_j(a_j) = \sum_{i=1}^{j} h_{ij} a_j + g_j \pmod b \tag{22}$$

which is used to generate the Generalized Faure sequence. Here, again the arithmetic is done in mode b, where g_j and h_{ij} are selected at random and uniformly from $\{0, 1, 2, ..., b-1\}$. In matrix notation, we can write the same expression as

$$\pi(a) = H^T a + g$$

where H^T is nonsingular lower-triangular matrix with random entries. Matoušek [14] shows that scrambling methods both in Equations 21 and 22 preserves the net property.

In Figure 1 we plot the first two dimensions of a two dimensional uniform sequence. The first plot coming from the uniform random number generated by Mersenne Twister pseudo-random number generator. The second and third plots belong to the Faure sequence and its linear scrambled version. On Figure 2 we plot a two dimensional projection of a 40 dimensional sequence for the first 500 forty dimensional vectors. Figure 2 shows that when we plot dimension 5 versus dimension 40 coming from these 40 dimensional vector of points, Faure sequence starts to show linear patterns. These linear patterns are undesirable in an integration problem since the "good quality" of the Faure sequence is lost as the dimension gets higher. On the same figure, the third plot shows that the linear scrambled Faure sequence still preserves most of the uniformity of the Faure sequence at higher dimensions. Therefore, scrambling a low-discrepancy sequence such as the Faure sequence gives us the chance to deal with high dimensional problems without losing the nice features of the original low-discrepancy sequence.

4 Numerical Results

The multi-stochastic volatility model discussed is applied to the vanilla European option pricing problem with the given parameter values in Table 1. This problem is used for both testing the effectiveness of importance sampling with tilting parameters derived from the Black-Scholes model and asymptotic approximation methods. Furthermore, randomized quasi-Monte Carlo sequences are tested in this high dimensional problem. As given in Table 1 $\Delta t = 0.01$ and $T = 1$, thus we have 100 time increments $(T/\Delta t)$ in each simulated path of the process, requiring the use of 300 dimensional quasi-Monte Carlo sequence at each sample path. Thus, the dimension of the problem is very high for expecting good performance from the quasi-Monte Carlo simulation. For the crude Monte Carlo, results are obtained using the Mersenne twister pseudo-random number generator. For generating the normal deviates Box Muller transformation is used as discussed in [17].

In our numerical experiments we used the digit scrambling given Equation 21 and linear scrambling given in Equation 22.

Table 2: European Call Option RMSE: MC versus RQMC

Sample Size	MC	Linear Scr.	Matousek Scr.
1000	0.0560	0.0151	0.0171
5000	0.0282	0.0043	0.0037
10000	0.0190	0.0020	0.0019

The randomized quasi-Monte Carlo method is then applied to our vanilla European call option pricing problem under the two factor stochastic volatility model. Room mean squared error (RMSE) results show that the best performer is Matousek scrambled Faure sequence at sample sizes 5000 and 10000. At a sample size of 1000 linear scrambled Faure sequence performed best. By using the scrambled Faure sequence we observed factors of improvement around 6-10 compared to the crude Monte Carlo variance. Overall, both scrambling methods were superior to the crude Monte Carlo estimator with small differences between two scrambling methods.

5 Conclusion

In this study one of the popular randomized quasi-Monte Carlo methods, namely scrambled Faure sequence is discussed and applied to the pricing of a European call option problem in a multi-factor stochastic volatility setting. In numerical experiments, we compared crude Monte Carlo and randomized quasi-Monte Carlo estimators. The two factor stochastic volatility used in this study is described in [4]. Numerical findings show that both linear and digit scrambling methods for the Faure

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sequence outperforms the crude Monte Carlo estimator by offering significant reductions in the variance.

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