Proceedings of the International MultiConference of Engineers and Computer Scientists 2012 Vol II,
IMECS 2012, March 14 - 16, 2012, Hong Kong

A Model of Adding Relation between the Top and a Member of a Linking Pin Organization Structure with K Subordinates

Kiyoshi Sawada, Hidefumi Kawakatsu, and Takashi Mitsuishi

***Abstract*—This study considers the addition of relation to an organization structure such that the communication of information between every member in the organization becomes the most efficient. This paper proposes a model of adding relation to a linking pin organization structure where every pair of siblings in a complete K-ary tree of height H is adjacent. When a new edge between the root and a node with a depth N is added, an optimal depth N^* is obtained by minimizing the total distance which is the sum of lengths of shortest paths between every pair of all nodes.**

***Index Terms*—organization structure, linking pin, complete K-ary tree, total distance.**

I. Introduction

A pyramid organization is a hierarchical structure based on the principle of unity of command [1] that every member except the top in the organization should have a single immediate superior. On the other hand an organization characterized by System 4 [2] has a structure in which relations between members of the same section are added to the pyramid organization structure. Members of middle layers of System 4 which are both members of the upper units and chiefs of the lower units are called linking pins, and this type of organization is called a linking pin organization. In the linking pin organization there exist relations between each superior and his direct subordinates and those between members which have the same immediate superior.

The linking pin organization structure can be expressed as a structure where every pair of siblings which are nodes which have the same parent in a rooted tree is adjacent, if we let nodes and edges in the structure correspond to members and relations between members in the organization respectively. Then the linking pin organization structure is characterized by the number of subordinates of each member, that is, the number of children of each node and the number of levels in the organization, that is, the height of the rooted tree, and so on [3], [4]. Moreover, the path between a pair of nodes in the structure is equivalent to the route of communication of information between a pair of members in the organization, and adding edges to the structure is equivalent to forming additional relations other than those between each superior and his direct subordinates and between members which have the same direct subordinate.

Manuscript received January 8, 2012.

K. Sawada is with the Department of Policy Studies, University of Marketing and Distribution Sciences, Kobe, 651-2188 Japan (e-mail: sawada@umds.ac.jp).

H. Kawakatsu is with the Department of Economics, Management and Information Science, Onomichi University, Onomichi, 722-8506 Japan.

T. Mitsuishi is with the Department of Commerce, University of Marketing and Distribution Sciences, Kobe, 651-2188 Japan.

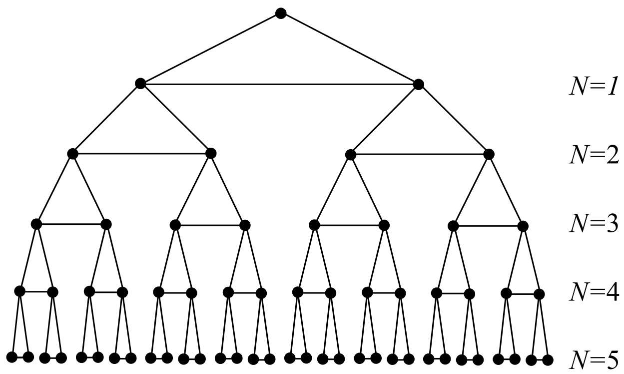

Fig. 1. An Example of a Complete K-ary Linking Pin Structure of $K = 2$ and $H = 5$

The purpose of our study is to obtain an optimal set of additional relations to the linking pin organization such that the communication of information between every member in the organization becomes the most efficient. This means that we obtain a set of additional edges to the structure minimizing the sum of lengths of shortest paths between every pair of all nodes.

We have obtained an optimal depth for each of the following two models of adding relations in the same level to a complete K-ary linking pin structure of height $H(H = 2, 3, \dots)$ where every pair of siblings in a complete K-ary tree of height H is adjacent: (i) a model of adding an edge between two nodes with the same depth and (ii) a model of adding edges between every pair of nodes with the same depth [6]. A complete K-ary tree is a rooted tree in which all leaves have the same depth and all internal nodes have $K(K = 2, 3, \dots)$ children [7]. Figure 1 shows an example of a complete K-ary linking pin structure of $K = 2$ and $H = 5$. In Figure 1 the value of N expresses the depth of each node.

This paper proposes a model of adding an edge between the root and a node with a depth $N(N = 2, 3, \dots, H)$ in a complete K-ary linking pin structure of height $H(H = 2, 3, \dots)$. This model corresponds to the formation of an additional relation between the top and one member of an organization.

If $l_{i,j}(= l_{j,i})$ denotes the distance, which is the number of edges in the shortest path from a node v_i to a node v_j $(i, j = 1, 2, \dots, (K^{H+1}-1)/(K-1))$ in the complete K-ary linking pin structure of height H, then $\sum_{i<j} l_{i,j}$ is the total distance. Furthermore, if $l'_{i,j}$ denotes the distance from v_i to v_j after adding an edge in this model, $l_{i,j}-l'_{i,j}$ is called the shortening distance between v_i and v_j, and $\sum_{i<j}(l_{i,j} - l'_{i,j})$ is called

ISBN: 978-988-19251-9-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

the *total shortening distance*. Minimizing the total distance is equivalent to maximizing the total shortening distance. When an edge between the root and a node with a depth N is added to the complete K-ary linking pin structure of height H, an optimal depth N^* is obtained by maximizing the total shortening distance.

In Section 2 we formulate the total shortening distance of the above model. In Section 3 we show an optimal depth N^* which maximizes the total shortening distance.

II. Formulation of Total Shortening Distance

This section formulates the total shortening distance when an edge between the root and a node with a depth $N(N = 2, 3, \ldots,\ H)$ is added to a complete K-ary linking pin structure of height $H(H = 2, 3, \ldots)$.

Let v_N denote the node of depth N which gets adjacent to the root. The set of descendants of v_N is denoted by V_1. The set of ancestors of parent of v_N is denoted by V_2. (Note that every node is a descendant and an ancestor of itself [7].) Let V_3 denote the set obtained by removing V_1 and V_2 from the set of all nodes of the complete K-ary linking pin structure.

The sum of shortening distances between every pair of nodes in V_1 and nodes in V_2 is given by

$$A_H(N) = M(H-N) \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor} (N - 2i + 1)\,, \tag{1}$$

where $M(h)$ denotes the number of nodes of a complete K-ary tree of height $h(h = 0, 1, 2, \ldots)$, and $\lfloor x \rfloor$ denotes the maximum integer which is equal to or less than x. The sum of shortening distances between every pair of nodes in V_2 is given by

$$B(N) = \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor - 1} \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor - i} (N - 2i - 2j + 1)\,, \tag{2}$$

where we define $\sum_{i=1}^{0} \cdot = 0$. The sum of shortening distances between every pair of nodes in V_1 and nodes in V_3 is given by

$$\begin{aligned} C_H(N) &= M(H-N) \\ &\quad \times \sum_{i=1}^{\lfloor \frac{N-1}{2} \rfloor} (K-1) M(H-i)(N-2i)\,. \end{aligned} \tag{3}$$

The sum of shortening distances between every pair of nodes in V_2 and nodes in V_3 is given by

$$\begin{aligned} D_H(N) &= \sum_{i=1}^{\lfloor \frac{N-1}{2} \rfloor - 1} (K-1) M(H-i) \\ &\quad \times \sum_{j=1}^{\lfloor \frac{N-1}{2} \rfloor - i} (N - 2i - 2j) \\ &\quad + \sum_{i=1}^{\lfloor \frac{N-1}{2} \rfloor} (K-1) M(H-N+i-1) \\ &\quad \times \sum_{j=1}^{\lfloor \frac{N-1}{2} \rfloor - i + 1} (N - 2i - 2j + 2)\,, \end{aligned} \tag{4}$$

where we define $\sum_{i=1}^{-1} \cdot = 0$. The sum of shortening distances between every pair of nodes in V_3 is given by

$$\begin{aligned} E_H(N) &= \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor - 1} (K-1) M(H-N+i-1) \\ &\quad \times \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor - i} (K-1) M(H-j) \\ &\quad \times (N - 2i - 2j + 1)\,. \end{aligned} \tag{5}$$

From the above equations, the total shortening distance $S_H(N)$ is given by

$$\begin{aligned} &S_H(N) \\ &\quad = A_H(N) + B(N) + C_H(N) + D_H(N) + E_H(N)\,. \end{aligned} \tag{6}$$

III. An Optimal Depth

This section obtains an optimal depth N^* which maximizes the total shortening distance $S_H(N)$.

Let us classify $S_H(N)$ into two cases of $N = 2L$ where $L = 1, 2, \ldots,\ \lfloor \frac{H}{2} \rfloor$ and $N = 2L + 1$ where $L = 1, 2, \ldots,\ \lfloor \frac{H-1}{2} \rfloor$. Since the number of nodes of a complete K-ary tree of height h is

$$M(h) = \frac{K^{h+1} - 1}{K - 1}, \tag{7}$$

$S_H(2L)$ and $S_H(2L+1)$ becomes

$$\begin{aligned} &S_H(2L) \\ &= \frac{K^{H-2L+1} - 1}{K-1} \sum_{i=1}^{L} (2L - 2i + 1) \\ &\quad + \sum_{i=1}^{L-1} \sum_{j=1}^{L-i} (2L - 2i - 2j + 1) \\ &\quad + \frac{K^{H-2L+1} - 1}{K-1} \sum_{i=1}^{L-1} \left(K^{H-i+1} - 1\right)(2L - 2i) \\ &\quad + \sum_{i=1}^{L-2} \left(K^{H-i+1} - 1\right) \sum_{j=1}^{L-i-1} (2L - 2i - 2j) \\ &\quad + \sum_{i=1}^{L-1} \left(K^{H-2L+i} - 1\right) \sum_{j=1}^{L-i} (2L - 2i - 2j + 2) \\ &\quad + \sum_{i=1}^{L-1} \left(K^{H-2L+i} - 1\right) \sum_{j=1}^{L-i} \left(K^{H-j+1} - 1\right) \\ &\quad \times (2L - 2i - 2j + 1) \\ &= \frac{1}{(K-1)^3} \Big\{ K^{2H-3L+3} - K^{2H-2L+3} - K^{2H-2L+2} \\ &\quad + K^{2H-L+2} - K^{H-2L+2} + K^{H-2L+1} - K^{H-L+3} \\ &\quad - K^{H-L+1} - (L-1) K^{H+3} + K^{H+2} + L K^{H+1} \\ &\quad - L(K-1)^2 \Big\}, \end{aligned} \tag{8}$$

ISBN: 978-988-19251-9-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$
\begin{aligned}
& S_H(2L+1) \\
&= \frac{K^{H-2L}-1}{K-1}\sum_{i=1}^{L}(2L-2i+2) \\
&\quad + \sum_{i=1}^{L-1}\sum_{j=1}^{L-i}(2L-2i-2j+2) \\
&\quad + \frac{K^{H-2L}-1}{K-1}\sum_{i=1}^{L}\left(K^{H-i+1}-1\right)(2L-2i+1) \\
&\quad + \sum_{i=1}^{L-1}\left(K^{H-i+1}-1\right)\sum_{j=1}^{L-i}(2L-2i-2j+1) \\
&\quad + \sum_{i=1}^{L}\left(K^{H-2L+i-1}-1\right)\sum_{j=1}^{L-i+1}(2L-2i-2j+3) \\
&\quad + \sum_{i=1}^{L-1}\left(K^{H-2L+i-1}-1\right)\sum_{j=1}^{L-i}\left(K^{H-j+1}-1\right) \\
&\quad \times (2L-2i-2j+2) \\
&= \frac{1}{(K-1)^3}\Bigg\{\frac{K-1}{K+1}K^{2H-3L+2}+K^{2H-3L+1} \\
&\quad - K^{2H-2L+2}-K^{2H-2L+1}+\frac{2}{K+1}K^{2H-L+2} \\
&\quad - K^{H-2L+2}+K^{H-2L+1}-K^{H-L+3}+K^{H-L+2} \\
&\quad - 2K^{H-L+1}-(L-1)K^{H+3}+(L+1)K^{H+1} \\
&\quad - L(K-1)^3-L(K-1)^2\Bigg\}\,. \qquad (9)
\end{aligned}
$$

Lemma 1:
(i) If $L=1$, then $S_H(2L) < S_H(2L+1)$.
(ii) If $L \geq 2$, then $S_H(2L) > S_H(2L+1)$.

Proof:
(i) If $L=1$, then

$$
\begin{aligned}
& S_H(2L)-S_H(2L+1) \\
&\quad = \frac{K^{2H-2}}{K-1}\left(-1+\frac{1}{K^{H-2}}+\frac{K-1}{K^{2H-2}}\right) \\
&\quad < 0\,. \qquad (10)
\end{aligned}
$$

(ii) If $L \geq 2$, then

$$
\begin{aligned}
& S_H(2L)-S_H(2L+1) \\
&\quad = \frac{1}{(K-1)^2}\Bigg\{\frac{K^{2H-L+2}}{K+1} \\
&\qquad \times \left(1-\frac{K^2+2K+1}{K^{L+1}}+\frac{K^2+K+1}{K^{2L+1}}\right) \\
&\qquad + K^{H+1}\left(1-\frac{1}{K^L}\right)+L(K-1)^2\Bigg\} \\
&\quad > 0\,, \qquad (11)
\end{aligned}
$$

where $L=2,3,\ldots,\left\lfloor\frac{H-1}{2}\right\rfloor$. The proof is complete.

Lemma 2: If $L \geq 2$, then $L^*=2$ maximizes $S_H(2L)$.

Proof: If $L \geq 2$, then $L^*=2$ maximizes $S_H(2L)$ since

$$
\begin{aligned}
& S_H(2L)-S_H(2L+2) \\
&\quad = \frac{1}{(K-1)^2}\Bigg\{K^{2H-L+1} \\
&\qquad \times \left(1-\frac{K^2+2K+1}{K^{L+1}}+\frac{K^2+K+1}{K^{2L+1}}\right) \\
&\qquad + K^{H+2}\left(1+\frac{1}{K}-\frac{K^2+1}{K^{L+2}}-\frac{K^2-1}{K^{2L+3}}\right) \\
&\qquad + K-1\Bigg\} \\
&\quad > 0\,, \qquad (12)
\end{aligned}
$$

where $L=2,3,\ldots,\left\lfloor\frac{H}{2}\right\rfloor-1$. The proof is complete.

Lemma 3:
(i) If $H=4$, then $S_H(3) > S_H(4)$.
(ii) If $H \geq 5$, then $S_H(3) < S_H(4)$.

Proof:
(i) If $H=4$, then

$$
\begin{aligned}
S_H(3)-S_H(4) &= K^2-2 \\
&> 0\,. \qquad (13)
\end{aligned}
$$

(ii) If $H \geq 5$, then

$$
\begin{aligned}
& S_H(3)-S_H(4) \\
&\quad = \frac{K^{2H-3}}{K-1}\left(-1+\frac{K^4+K^2-K-1}{K^H}-\frac{K-2}{K^{2H-3}}\right) \\
&\quad < 0\,. \qquad (14)
\end{aligned}
$$

The proof is complete.

Theorem 4:
(i) If $H=2$, then $N^*=2$ maximizes $S_H(N)$.
(ii) If $3 \leq H \leq 4$, then $N^*=3$ maximizes $S_H(N)$.
(iii) If $H \geq 5$, then $N^*=4$ maximizes $S_H(N)$.

Proof:
(i) If $H=2$, then $N^*=2$ trivially.
(ii) If $H=3$, then $N^*=3$ since $S_H(2) < S_H(3)$ from (i) of Lemma 1. If $H=4$, then $N^*=3$ since $S_H(2) < S_H(3)$ from (i) of Lemma 1 and $S_H(3) > S_H(4)$ from (i) of Lemma 3.
(iii) If $H \geq 5$, then $N^*=4$ since
(a) $S_H(2) < S_H(3)$ from (i) of Lemma 1,
(b) $S_H(3) < S_H(4)$ from (ii) of Lemma 3,
(c) if $L \geq 2$, then $S_H(2L) > S_H(2L+1)$ from (ii) of Lemma 1, and
(d) if $L \geq 2$, then $L^*=2$ maximizes $S_H(2L)$ from Lemma 2.
The proof is complete.

IV. Conclusions

This study considered the addition of relation to a linking pin organization structure such that the communication of information between every member in the organization becomes the most efficient. For a model of adding an edge between the root and a node with a depth N to a complete K-ary linking pin structure of height H where every pair of siblings in a complete K-ary tree of height H is adjacent, we obtained an optimal depth N^* which maximizes the total shortening distance. Theorem 4 shows that the most efficient manner of adding relation between the top and a subordinate

ISBN: 978-988-19251-9-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

is to add the relation to a subordinate of the second, the third or the fourth level depending on the number of levels in the organization structure.

REFERENCES

[1] H. Koontz, C. O'Donnell, and H. Weihrich, *Management*, 7th ed. New York: McGraw-Hill, 1980.

[2] R. Likert and J. G. Likert, *New Ways of Managing Conflict*. New York: McGraw-Hill, 1976.

[3] S. P. Robbins, *Essentials of Organizational Behavior*, 7th ed. Upper Saddle River, N.J.: Prentice Hall, 2003.

[4] Y. Takahara and M. Mesarovic, *Organization Structure: Cybernetic Systems Foundation*. New York: Kluwer Academic / Plenum Publishers, 2003.

[5] K. Sawada and R. Wilson, "Models of Adding Relations to an Organization Structure of a Complete K-ary Tree," *European Journal of Operational Research*, vol. 174, pp. 1491-1500, 2006.

[6] K. Sawada, "Adding Relations in the Same Level of a Linking Pin Type Organization Structure," *IAENG International Journal of Applied Mathematics*, vol. 38, pp. 20-25, 2008.

[7] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, 2nd ed. Cambridge, Mass.: MIT Press, 2001.

ISBN: 978-988-19251-9-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)