Interactive Fuzzy Decision Making for Hierarchical Multiobjective Stochastic Linear Programming Problems

Hitoshi Yano

***Abstract*—In this paper, we focus on hierarchical multiobjective stochastic linear programming problems (HMOP) where multiple decision makers in a hierarchical organization have their own multiple objective linear functions together with common linear constraints. In order to deal with HMOP, a probability maximization model and a fractile optimization model are applied. By considering the conflict between permissible objective levels and permissible probability levels in such two models, it is assumed that each of the decision makers has fuzzy goals for permissible objective levels and permissible probability levels, and such fuzzy goals can be quantified by eliciting the membership functions. Through the fuzzy decision, such membership functions are integrated. In the integrated membership space, Pareto optimality concept is introduced. The interactive algorithm to obtain a satisfactory solution from among a Pareto optimal solution set is proposed on the basis of linear programming technique, in which the hierarchical decision structure is reflected by the decision power and the proper balance between permissible objective levels and the corresponding probability function is attained.**

***Index Terms*—hierarchical multiobjective stochastic linear programming, decision power, a probability maximization model, a fractile optimization model, interactive decision making.**

I. INTRODUCTION

The decision makers in practical hierarchical decision making situations often encounter two kinds of decision making processes, one is well known as a multi-level programming process and the other is the interactive decision making process [3]. The Stackelberg games are well-known as multilevel programming problems with multiple decision makers, in which the decision maker in each level makes his/her decision independently in order to optimize his/her own objective function [1], [8]. On the other hand, the interactive decision making process can be found in large scale hierarchical organizations such as multi-hierarchical companies, in which the decision maker in each level makes his/her decision through the interaction between the decision makers and the lower level decision makers submit their own decision and then such decision is modified by the upper level decision makers with considerations of the overall benefits [3]. In order to deal with such an interactive decision making process, Lai [2], Shih et al.[7] and Lee et al.[3] introduced concepts of memberships of optimalities and degrees of decision powers and proposed fuzzy approaches to obtain a satisfactory solution. As a natural extension of their approaches, Yano [9] proposed a fuzzy approach for hierarchical multiobjective linear programming problems. On the other hand, in the actual decision making situations, the decision makers often encounter difficulties to deal with vague information or uncertain data. Sakawa et al.[5],[6] formulated multiobjective stochastic linear programming problems through a probability maximization model and a fractile optimization model, and proposed interactive algorithm to obtain a satisfactory solution from among a Pareto optimal solution set. Using a probability maximization model or a fractile optimization model, it is required for the decision maker to specify parameters called permissible objective levels or permissible probability levels in advance. However, it seems to be very difficult to specify such values in advance. In order to cope with such difficulties, Yano et al.[10] proposed fuzzy approaches to multiobjective stochastic linear programming problems, where the decision maker has fuzzy goals for permissible objective levels and permissible probability levels, and such fuzzy goals are quantified by eliciting the membership functions. Unfortunately, in the proposed method, it is assumed that the decision maker adopts the fuzzy decision [4] to obtain the satisfactory solution.

In this paper, we focus on hierarchical multiobjective stochastic linear programming problems, and propose an interactive algorithm to obtain a satisfactory solution from among a Pareto optimal solution set. In the proposed method, by considering the conflict between permissible objective levels and and permissible probability levels, the corresponding membership functions are integrated through the fuzzy decision. In the integrated membership space, Pareto optimal concept is introduced. In section II, hierarchical multiobjective programming problems through a probability maximization model is formulated. In section III, hierarchical multiobjective programming problems through a fractile optimization model is formulated. It is shown that the two kinds of formulations to obtain Pareto optimal solutions are same. In section IV, an interactive algorithm based on linear programming technique is proposed to obtain a satisfactory solution.

II. HIERARCHICAL MULTIOBJECTIVE STOCHASTIC LINEAR PROGRAMMING PROBLEMS THROUGH A PROBABILITY MAXIMIZATION MODEL

We consider the following hierarchical multiobjective stochastic linear programming problem (HMOP), where each of the decision makers ($\mathrm{DM}_r, r = 1, \cdots, q$) has his/her own multiple objective linear functions together with common linear constraints, and random variable coefficients are involved in each objective function.

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[HMOP1]

first level decision maker : DM$_1$

$$\min_{\boldsymbol{x}\in X} \bar{\boldsymbol{z}}_1(\boldsymbol{x}) = (\bar{z}_{11}(\boldsymbol{x}), \cdots, \bar{z}_{1k_1}(\boldsymbol{x}))$$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

q-th level decision maker : DM$_q$

$$\min_{\boldsymbol{x}\in X} \bar{\boldsymbol{z}}_q(\boldsymbol{x}) = (\bar{z}_{q1}(\boldsymbol{x}), \cdots, \bar{z}_{qk_q}(\boldsymbol{x}))$$

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T$ is n-dimensional decision column vector whose elements $x_i, i = 1, \cdots, n$ are non-negative, X is a linear constraint set with respect to $\boldsymbol{x}$. Each objective function of DM$_r, r = 1, \cdots, q$ is defined by $\bar{z}_{r\ell}(\boldsymbol{x}) = \bar{\boldsymbol{c}}_{r\ell}\boldsymbol{x} + \bar{\alpha}_{r\ell}$, $\bar{\boldsymbol{c}}_{r\ell} = \boldsymbol{c}^1_{r\ell} + \bar{t}_{r\ell}\boldsymbol{c}^2_{r\ell}$, $\bar{\alpha}_{r\ell} = \alpha^1_{r\ell} + \bar{t}_{r\ell}\alpha^2_{r\ell}$, where $\bar{\boldsymbol{c}}_{r\ell}, \ell = 1, \cdots, k_r$ are n dimensional random variable row vectors, $\bar{\alpha}_{r\ell}, \ell = 1, \cdots, k_r$ are random variables, and $\bar{t}_{r\ell}$ is a random variable whose cumulative distribution function $T_{r\ell}(\cdot)$ is assumed to be strictly monotone increasing and continuous.

Similar to the formulations of multilevel linear programming problems proposed by Lee et al.[3], it is assumed that the upper level decision makers make their decisions with consideration of the overall benefits for the hierarchical organization, although they can take priority for their objective functions over the lower level decision makers.

In order to deal with HMOP1, we adopt stochastic linear programming techniques. Using a probability maximization model [6], we substitute the minimization of the objective function $\bar{z}_{r\ell}(\boldsymbol{x})$ for the maximization of the probability that $\bar{z}_{r\ell}(\boldsymbol{x})$ is less than or equal to a certain permissible objective level $f_{r\ell}$. Such a probability $p_{r\ell}(\boldsymbol{x}, f_{r\ell})$ can be defined as follows.

$$p_{r\ell}(\boldsymbol{x}, f_{r\ell}) \stackrel{\text{def}}{=} \Pr(\omega \mid z_{r\ell}(\boldsymbol{x}, \omega) \le f_{r\ell}), \quad (1)$$

where $\Pr(\cdot)$ denotes a probability measure, ω is an event, and $z_{r\ell}(\boldsymbol{x}, \omega)$ is a realization of the random objective function $\bar{z}_{r\ell}(\boldsymbol{x})$ under the occurrence of each elementary event ω. Each of the decision makers (DM$_r, r = 1, \cdots, q$) subjectively specifies certain permissible objective levels $\boldsymbol{f}_r = (f_{r1}, \cdots, f_{rk_r}), r = 1, \cdots, q, \boldsymbol{f} = (\boldsymbol{f}_1, \cdots, \boldsymbol{f}_q)$. Then, HMOP1 can be transformed into the following problem involving permissible objective levels as parameters.

[HMOP2($\boldsymbol{f}$)]

first level decision maker : DM$_1$

$$\max_{\boldsymbol{x}\in X} (p_{11}(\boldsymbol{x}, f_{11}), \cdots, p_{1k_1}(\boldsymbol{x}, f_{1k_1}))$$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

q-th level decision maker: DM$_q$

$$\max_{\boldsymbol{x}\in X} (p_{q1}(\boldsymbol{x}, f_{q1}), \cdots, p_{qk_q}(\boldsymbol{x}, f_{qk_q}))$$

Under the assumption that $\boldsymbol{c}^2_{r\ell}\boldsymbol{x} + \alpha^2_{r\ell} > 0, r = 1, \cdots, q, \ell = 1, \cdots, k_r$, the objective function $p_{r\ell}(\boldsymbol{x}, f_{r\ell})$ in HMOP1($\boldsymbol{f}$) is expressed as follows.

$$\begin{aligned} p_{r\ell}(\boldsymbol{x}, f_{r\ell}) &= \Pr(\omega \mid z_{r\ell}(\boldsymbol{x}, \omega) \le f_{r\ell}) \\ &= T_{r\ell}\left(\frac{f_{r\ell} - (\boldsymbol{c}^1_{r\ell}\boldsymbol{x} + \alpha^1_{r\ell})}{\boldsymbol{c}^2_{r\ell}\boldsymbol{x} + \alpha^2_{r\ell}}\right) \end{aligned}$$

In HMOP2($\boldsymbol{f}$), each decision maker (DM$_r$) seems to prefer not only the less values of permissible objective levels $\boldsymbol{f}_r$, but also the larger values of the corresponding distribution functions $p_{r\ell}(\boldsymbol{x}, f_{r\ell}), \ell = 1, \cdots, k_r$. Since these values conflict with each other, the less value of permissible objective level results in the less value of the corresponding probability function. Therefore, it is very important for each decision maker (DM$_r$) to determine appropriate values of permissible objective levels $\boldsymbol{f}_r$. Unfortunately, it seems to be difficult for the decision maker to find appropriate values of permissible objective levels. In order to circumvent such difficulties, Yano et al. [10] proposed a fuzzy approach for multiobjective stochastic linear programming problems. From a similar point of view, instead of HMOP2($\boldsymbol{f}$), we consider the following multiobjective programming problem in which permissible objective levels are not constant values but the decision variables.

[HMOP3]

first level decision maker : DM$_1$

$$\max_{\boldsymbol{x}\in X, \boldsymbol{f}_1\in \mathrm{R}^{k_1}} (p_{11}(\boldsymbol{x}, f_{11}), \cdots, p_{1k_1}(\boldsymbol{x}, f_{1k_1}), -f_{11}, \cdots, -f_{1k_1})$$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

q-th level decision maker: DM$_q$

$$\max_{\boldsymbol{x}\in X, \boldsymbol{f}_q\in \mathrm{R}^{k_q}} (p_{q1}(\boldsymbol{x}, f_{q1}), \cdots, p_{qk_q}(\boldsymbol{x}, f_{qk_q}), -f_{q1}, \cdots, -f_{qk_q})$$

Considering the imprecise nature of the decision maker's judgment, it is natural to assume that the decision maker have a fuzzy goal for each objective function in HMOP3. In this section, it is assumed that such a fuzzy goal can be quantified by eliciting a corresponding membership function. Let us denote a membership function of $p_{r\ell}(\boldsymbol{x}, f_{r\ell})$ as $\mu_{\tilde{p}_{r\ell}}(p_{r\ell}(\boldsymbol{x}, f_{r\ell}))$, and a membership function of $f_{r\ell}$ as $\mu_{\tilde{f}_{r\ell}}(f_{r\ell})$ respectively. Then, HMOP3 can be transformed to the following multiobjective programming problem.

[HMOP4]

first level decision maker : DM$_1$

$$\begin{aligned} \max_{\boldsymbol{x}\in X, \boldsymbol{f}_1\in \mathrm{R}^{k_1}} \quad & (\mu_{\tilde{p}_{11}}(p_{11}(\boldsymbol{x}, f_{11})), \cdots, \mu_{\tilde{p}_{1k_1}} \\ & (p_{1k_1}(\boldsymbol{x}, f_{1k_1})), \mu_{\tilde{f}_{11}}(f_{11}), \cdots, \mu_{\tilde{f}_{1k_1}}(f_{1k_1})) \end{aligned}$$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

q-th level decision maker: DM$_q$

$$\begin{aligned} \max_{\boldsymbol{x}\in X, \boldsymbol{f}_q\in \mathrm{R}^{k_q}} \quad & (\mu_{\tilde{p}_{q1}}(p_{q1}(\boldsymbol{x}, f_{q1})), \cdots, \mu_{\tilde{p}_{qk_q}} \\ & (p_{qk_q}(\boldsymbol{x}, f_{qk_q})), \mu_{\tilde{f}_{q1}}(f_{q1}), \cdots, \mu_{\tilde{f}_{qk_q}}(f_{qk_q})) \end{aligned}$$

Throughout this section, we make the assumptions that $\mu_{\tilde{f}_{r\ell}}(f_{r\ell})$ is strictly monotone decreasing and continuous with respect to $f_{r\ell}$, and $\mu_{\tilde{p}_{r\ell}}(p_{r\ell}(\boldsymbol{x}, f_{r\ell}))$ is strictly monotone increasing and continuous with respect to $p_{r\ell}(\boldsymbol{x}, f_{r\ell})$ for any $r = 1, \cdots, q, \ell = 1, \cdots, k_r$.

It should be noted here that $\mu_{\tilde{p}_{r\ell}}(p_{r\ell}(\boldsymbol{x}, f_{r\ell}))$ and $\mu_{\tilde{f}_{r\ell}}(f_{r\ell})$ are conflict each other for any $\boldsymbol{x} \in X$. Here, let us assume that the decision maker adopts the fuzzy decision [4] in order to integrate both the membership functions $\mu_{\tilde{p}_{r\ell}}(p_{r\ell}(\boldsymbol{x}, f_{r\ell}))$ and $\mu_{\tilde{f}_{r\ell}}(f_{r\ell})$. Then, the integrated membership function can be defined as follows.

$$\mu_{D_{p_{r\ell}}}(\boldsymbol{x}, f_{r\ell}) \stackrel{\text{def}}{=} \min\{\mu_{\tilde{f}_{r\ell}}(f_{r\ell}), \mu_{\tilde{p}_{r\ell}}(p_{r\ell}(\boldsymbol{x}, f_{r\ell}))\} \quad (2)$$

Using the integrated membership functions $\mu_{D_{p_{r\ell}}}(\boldsymbol{x}, f_{r\ell})$, HMOP4 can be transformed into the following form.

[HMOP5]

first level decision maker : DM$_1$

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$$\max_{\boldsymbol{x}\in X, f_{1\ell}\in \mathrm{R}^1, \ell=1,\cdots,k_1} \left(\mu_{D_{p_{11}}}(\boldsymbol{x}, f_{11}), \cdots, \mu_{D_{p_{1k_1}}}(\boldsymbol{x}, f_{1k_1}))\right)$$

....

q-th level decision maker: DM$_q$

$$\max_{\boldsymbol{x}\in X, f_{q\ell}\in \mathrm{R}^1, \ell=1,\cdots,k_q} \left(\mu_{D_{p_{q1}}}(\boldsymbol{x}, f_{q1}), \cdots, \mu_{D_{p_{qk_q}}}(\boldsymbol{x}, f_{qk_q}))\right)$$

In order to deal with HMOP5, we introduce a D_p-Pareto optimal solution concept.

Definition 1.

$\boldsymbol{x}^* \in X, f^*_{r\ell} \in \mathrm{R}^1, r = 1,\cdots,q, \ell = 1,\cdots,k_r$ is said to be a D_p-Pareto optimal solution to HMOP5, if and only if there does not exist another $\boldsymbol{x} \in X, f_{r\ell} \in \mathrm{R}^1, r = 1,\cdots,q, \ell = 1,\cdots,k_r$ such that $\mu_{D_{p_{r\ell}}}(\boldsymbol{x}, f_{r\ell}) \geq \mu_{D_{p_{r\ell}}}(\boldsymbol{x}^*, f^*_{r\ell})$, $r = 1,\cdots,q, \ell = 1,\cdots,k_r$, with strict inequality holding for at least one r and ℓ.

For generating a candidate of a satisfactory solution which is also D_p-Pareto optimal, each decision maker (DM$_r$) is asked to specify the reference membership values $\hat{\boldsymbol{\mu}}_{\boldsymbol{r}} = (\hat{\mu}_{r1},\cdots,\hat{\mu}_{rk_r})$ [4]. Once the reference membership values are specified, the corresponding D_p-Pareto optimal solution is obtained by solving the following minmax problem.

[MINMAX1($\hat{\boldsymbol{\mu}}$)]

$$\min_{\boldsymbol{x}\in X, f_{r\ell}\in \mathrm{R}^1, r=1,\cdots,q, \ell=1,\cdots,k_r, \lambda\in\Lambda} \lambda \tag{3}$$

subject to

$$\hat{\mu}_{r\ell} - \mu_{\tilde{p}_{r\ell}}(p_{r\ell}(\boldsymbol{x}, f_{r\ell})) \leq \lambda, \tag{4}$$

$$\hat{\mu}_{r\ell} - \mu_{\tilde{f}_{r\ell}}(f_{r\ell}) \leq \lambda, \tag{5}$$

$$r = 1,\cdots,q, \ell = 1,\cdots,k_r$$

where

$$\Lambda = [\max_{r=1,\cdots,q,\ell=1,\cdots,k_r} \hat{\mu}_{r\ell} - 1, \min_{r=1,\cdots,q,\ell=1,\cdots,k_r} \hat{\mu}_{r\ell}]. \tag{6}$$

It should be noted here that, in general, the optimal solution of MINMAX1($\hat{\boldsymbol{\mu}}$) does not reflect the hierarchical structure between q decision makers where the upper level decision maker can take priority for his/her distribution functions over the lower level decision makers. In order to cope with such a hierarchical preference structure between q decision makers, we introduce the concept of the decision power [2] $\boldsymbol{w} = (w_1,\cdots,w_q)$, where the r-th level decision maker (DM$_r$) can specify the decision power w_{r+1} in his/her subjective manner and the last decision maker (DM$_q$) has no decision power. In order to reflect the hierarchical preference structure between multiple decision makers, the decision powers $\boldsymbol{w} = (w_1,\cdots,w_q)$ have to satisfy the following inequality conditions.

$$w_1 = 1 \geq w_2 \geq \cdots\cdots \geq w_{q-1} \geq w_q > 0 \tag{7}$$

Then, the corresponding modified MINMAX1($\hat{\boldsymbol{\mu}}$) is reformulated as follows.

[MINMAX2($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$)]

$$\min_{\boldsymbol{x}\in X, f_{r\ell}\in \mathrm{R}^1, r=1,\cdots,q, \ell=1,\cdots,k_r, \lambda\in\Lambda} \lambda \tag{8}$$

subject to

$$\hat{\mu}_{r\ell} - \mu_{\tilde{p}_{r\ell}}(p_{r\ell}(\boldsymbol{x}, f_{r\ell})) \leq \lambda w_r, \tag{9}$$

$$\hat{\mu}_{r\ell} - \mu_{\tilde{f}_{r\ell}}(f_{r\ell}) \leq \lambda w_r, \tag{10}$$

$$r = 1,\cdots,q, \ell = 1,\cdots,k_r$$

Since $\mu_{\tilde{p}_{r\ell}}(p_{r\ell}(\boldsymbol{x}, f_{r\ell}))$ is strictly monotone increasing and continuous and $\boldsymbol{c}^2_{r\ell}\boldsymbol{x} + \alpha^2_{r\ell} > 0$, the constraint (9) can be transformed as follows.

$$\begin{aligned} & \hat{\mu}_{r\ell} - \mu_{\tilde{p}_{r\ell}}(p_{r\ell}(\boldsymbol{x}, f_{r\ell})) \leq \lambda w_r, \\ \Leftrightarrow\ & p_{r\ell}(\boldsymbol{x}, f_{r\ell}) \geq \mu^{-1}_{\tilde{p}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda w_r), \\ \Leftrightarrow\ & T_{r\ell}\left(\frac{f_{r\ell} - (\boldsymbol{c}^1_{r\ell}\boldsymbol{x} + \alpha^1_{r\ell})}{\boldsymbol{c}^2_{r\ell}\boldsymbol{x} + \alpha^2_{r\ell}}\right) \geq \mu^{-1}_{\tilde{p}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda w_r), \\ \Leftrightarrow\ & f_{r\ell} - (\boldsymbol{c}^1_{r\ell}\boldsymbol{x} + \alpha^1_{r\ell}) \\ & \geq T^{-1}_{r\ell}(\mu^{-1}_{\tilde{p}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda w_r)) \cdot (\boldsymbol{c}^2_{r\ell}\boldsymbol{x} + \alpha^2_{r\ell}). \end{aligned} \tag{11}$$

where $\mu^{-1}_{\tilde{p}_{r\ell}}(\cdot)$ and $T^{-1}_{r\ell}(\cdot)$ are inverse functions with respect to $\mu_{\tilde{p}_{r\ell}}(\cdot)$ and $T_{r\ell}(\cdot)$ respectively. Moreover, it holds that $f_{r\ell} \leq \mu^{-1}_{\tilde{f}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda w_r)$, because $\mu_{\tilde{f}_{r\ell}}(f_{r\ell})$ is strictly monotone decreasing and continuous. As a result, the constraints (9) and (10) can be reduced to the following inequality where a permissible objective level $f_{r\ell}$ is removed.

$$\begin{aligned} & \mu^{-1}_{\tilde{f}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda w_r) - (\boldsymbol{c}^1_{r\ell}\boldsymbol{x} + \alpha^1_{r\ell}) \\ & \geq T^{-1}_{r\ell}(\mu^{-1}_{\tilde{p}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda w_r)) \cdot (\boldsymbol{c}^2_{r\ell}\boldsymbol{x} + \alpha^2_{r\ell}) \end{aligned} \tag{12}$$

Then, MINMAX2($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$) is equivalently transformed to the following problem.

[MINMAX3($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$)]

$$\min_{\boldsymbol{x}\in X, \lambda\in\Lambda} \lambda \tag{13}$$

subject to

$$\begin{aligned} & \mu^{-1}_{\tilde{f}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda w_r) - (\boldsymbol{c}^1_{r\ell}\boldsymbol{x} + \alpha^1_{r\ell}) \\ & \geq T^{-1}_{r\ell}(\mu^{-1}_{\tilde{p}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda w_r)) \cdot (\boldsymbol{c}^2_{r\ell}\boldsymbol{x} + \alpha^2_{r\ell}) \\ & r = 1,\cdots,q, \ell = 1,\cdots,k_r \end{aligned} \tag{14}$$

It should be noted here that the constraints (14) can be reduced to a set of linear inequalities for some fixed value $\lambda \in \Lambda$. This means that an optimal solution $(\boldsymbol{x}^*, \lambda^*)$ of MINMAX3($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$) is obtained by combined use of the bisection method with respect to λ and the first-phase of the two-phase simplex method of linear programming.

The relationship between the optimal solution $(\boldsymbol{x}^*, \lambda^*)$ of MINMAX3($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$) and D_p-Pareto optimal solutions can be characterized by the following theorem.

Theorem 1.

If $(\boldsymbol{x}^*, \lambda^*)$ is a unique optimal solution of MINMAX3($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$), then $\boldsymbol{x}^* \in X$, $f^*_{r\ell} = \mu^{-1}_{\tilde{f}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r)$, $r = 1,\cdots,q, \ell = 1,\cdots,k_r$ is a D_p-Pareto optimal solution to HMOP5.

(Proof)

Since an optimal solution $(\boldsymbol{x}^*, \lambda^*)$ satisfies the constraints (14), it holds that

$$\begin{aligned} & T_{r\ell}\left(\frac{\mu^{-1}_{\tilde{f}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r) - (\boldsymbol{c}^1_{r\ell}\boldsymbol{x}^* + \alpha^1_{r\ell})}{\boldsymbol{c}^2_{r\ell}\boldsymbol{x}^* + \alpha^2_{r\ell}}\right) \\ & = p_{r\ell}(\boldsymbol{x}^*, \mu^{-1}_{\tilde{f}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r)) \\ & \geq \mu^{-1}_{\tilde{p}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r), r = 1,\cdots,q, \ell = 1,\cdots,k_r. \end{aligned}$$

Assume that $\boldsymbol{x}^* \in X$, $f^*_{r\ell} = (\mu^{-1}_{\tilde{f}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r)$, $r = 1,\cdots,q, \ell = 1,\cdots,k_r$ is not a D_p-Pareto optimal solution to HMOP5, then there exists $\boldsymbol{x} \in X$ $f_{r\ell}$, $r = 1,\cdots,q, \ell = 1,\cdots,k_r$ such that $\mu_{D_{p_{r\ell}}}(\boldsymbol{x}, f_{r\ell}) = \min\{\mu_{\tilde{f}_{r\ell}}(f_{r\ell}), \mu_{\tilde{p}_{r\ell}}(p_{r\ell}(\boldsymbol{x}, f_{r\ell}))\} \geq \mu_{D_{p_{r\ell}}}(\boldsymbol{x}^*, \mu^{-1}_{\tilde{f}_{r\ell}}(\hat{\mu}_{r\ell} -$

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$\lambda^* w_r)) = \hat{\mu}_{r\ell} - \lambda^* w_r, r = 1, \cdots, q, \ell = 1, \cdots, k_r$, with strict inequality holding for at least one r and ℓ. Then it holds that

$$\mu_{\tilde{f}_{r\ell}}(f_{r\ell}) \geq \hat{\mu}_{r\ell} - \lambda^* w_r, \quad (15)$$

$$\mu_{\tilde{p}_{r\ell}}(p_{r\ell}(\boldsymbol{x}, f_{r\ell})) \geq \hat{\mu}_{r\ell} - \lambda^* w_r, \quad (16)$$

$r = 1, \cdots, q, \ell = 1, \cdots, k_r$. From the definition (1), the inequalities (15) and (16) can be transformed into the inequalities, $f_{r\ell} \leq \mu_{\tilde{f}_{r\ell}}^{-1}(\hat{\mu}_{r\ell} - \lambda^* w_r), f_{r\ell} \geq T_{r\ell}^{-1}(\mu_{\tilde{p}_{r\ell}}^{-1}(\hat{\mu}_{r\ell} - \lambda^* w_r)) \cdot (\boldsymbol{c}_{r\ell}^2 \boldsymbol{x} + \alpha_{r\ell}^2) + (\boldsymbol{c}_{r\ell}^1 \boldsymbol{x} + \alpha_{r\ell}^1)$. This means that there exists some $\boldsymbol{x} \in X$ such that $\mu_{\tilde{f}_{r\ell}}^{-1}(\hat{\mu}_{r\ell} - \lambda^* w_r) \geq T_{r\ell}^{-1}(\mu_{\tilde{p}_{r\ell}}^{-1}(\hat{\mu}_{r\ell} - \lambda^* w_r)) \cdot (\boldsymbol{c}_{r\ell}^2 \boldsymbol{x} + \alpha_{r\ell}^2) + (\boldsymbol{c}_{r\ell}^1 \boldsymbol{x} + \alpha_{r\ell}^1)$, $r = 1, \cdots, q, \ell = 1, \cdots, k_r$, which contradicts the fact that $\boldsymbol{x}^* \in X, \lambda^* \in \Lambda$ is a unique optimal solution to MINMAX3$(\hat{\boldsymbol{\mu}}, \boldsymbol{w})$.

III. Hierarchical Multiobjective Stochastic Linear Programming Problems Through a Fractile Optimization Model

If we adopt a fractile optimization model [5] for HMOP1, we can convert HMOP1 to the following multiobjective programming problem.

[HMOP6($\hat{\boldsymbol{p}}$)]

first level decision maker : DM$_1$

$$\min_{\boldsymbol{x} \in X, f_{1\ell} \in R^1, \ell=1,\cdots,k_r} (f_{11}, \cdots, f_{1k_1})$$

. . . .

q-th level decision maker: DM$_q$

$$\min_{\boldsymbol{x} \in X, f_{q\ell} \in R^1, \ell=1,\cdots,k_q} (f_{q1}, \cdots, f_{qk_q})$$

subject to

$$p_{r\ell}(\boldsymbol{x}, f_{r\ell}) \geq \hat{p}_{r\ell}, r = 1, \cdots, q, \ell = 1, \cdots, k_r \quad (17)$$

where $\hat{\boldsymbol{p}}_{\boldsymbol{r}} = (\hat{p}_{r1}, \cdots, \hat{p}_{rk_r}), r = 1, \cdots, q, \hat{\boldsymbol{p}} = (\hat{\boldsymbol{p}}_{\boldsymbol{1}}, \cdots, \hat{\boldsymbol{p}}_{\boldsymbol{q}})$ are vectors of permissible probability levels which are specified by the decision maker (DM$_r, r = 1, \cdots, q$) in his/her subjective manner.

In HMOP6($\hat{\boldsymbol{p}}$), the constraint (17) can be transformed into the following form.

$$\hat{p}_{r\ell} \leq p_{r\ell}(\boldsymbol{x}, f_{r\ell}) = T_{r\ell}\left(\frac{f_{r\ell} - (\boldsymbol{c}_{r\ell}^1 \boldsymbol{x} + \alpha_{r\ell}^1)}{\boldsymbol{c}_{r\ell}^2 \boldsymbol{x} + \alpha_{r\ell}^2}\right)$$

$$\Leftrightarrow f_{r\ell} \geq T_{r\ell}^{-1}(\hat{p}_{r\ell}) \cdot (\boldsymbol{c}_{r\ell}^2 \boldsymbol{x} + \alpha_{r\ell}^2) + (\boldsymbol{c}_{r\ell}^1 \boldsymbol{x} + \alpha_{r\ell}^1)$$

Let us define the right-hand side of the above inequality as follows.

$$f_{r\ell}(\boldsymbol{x}, \hat{p}_{r\ell}) \stackrel{\text{def}}{=} T_{r\ell}^{-1}(\hat{p}_{r\ell}) \cdot (\boldsymbol{c}_{r\ell}^2 \boldsymbol{x} + \alpha_{r\ell}^2) + (\boldsymbol{c}_{r\ell}^1 \boldsymbol{x} + \alpha_{r\ell}^1) \quad (18)$$

Then, HMOP6($\hat{\boldsymbol{p}}$) can be equivalently reduced to the following simple form.

[HMOP7($\hat{\boldsymbol{p}}$)]

first level decision maker : DM$_1$

$$\min_{\boldsymbol{x} \in X} (f_{11}(\boldsymbol{x}, \hat{p}_{11}), \cdots, f_{1k_1}(\boldsymbol{x}, \hat{p}_{1k_1}))$$

. . . .

q-th level decision maker: DM$_q$

$$\min_{\boldsymbol{x} \in X} (f_{q1}(\boldsymbol{x}, \hat{p}_{q1}), \cdots, f_{qk_q}(\boldsymbol{x}, \hat{p}_{qk_q}))$$

In order to deal with HMOP7($\hat{\boldsymbol{p}}$), the decision maker must specify permissible probability levels $\hat{\boldsymbol{p}}$ in advance. However, in general, the decision maker seems to prefer not only the less value of the objective function $f_{r\ell}(\boldsymbol{x}, \hat{p}_{r\ell})$ but also the larger value of the permissible probability level $\hat{p}_{r\ell}$. From such a point of view, we consider the following multiobjective programming problem which can be regarded as a natural extension of HMOP7($\hat{\boldsymbol{p}}$).

[HMOP8]

first level decision maker : DM$_1$

$$\min_{\boldsymbol{x} \in X} (f_{11}(\boldsymbol{x}, \hat{p}_{11}), \cdots, f_{1k_1}(\boldsymbol{x}, \hat{p}_{1k_1}), -\hat{p}_{11}, \cdots, -\hat{p}_{1k_1})$$

. . . .

q-th level decision maker: DM$_q$

$$\min_{\boldsymbol{x} \in X} (f_{q1}(\boldsymbol{x}, \hat{p}_{q1}), \cdots, f_{qk_q}(\boldsymbol{x}, \hat{p}_{qk_q}), -\hat{p}_{q1}, \cdots, -\hat{p}_{qk_q})$$

Considering the imprecise nature of the decision maker's judgment, we assume that the decision maker has a fuzzy goal for each objective function in HMOP8. Such a fuzzy goal can be quantified by eliciting the corresponding membership function. Let us denote a membership function of an objective function $f_{r\ell}(\boldsymbol{x}, \hat{p}_{r\ell})$ as $\mu_{\tilde{f}_{r\ell}}(f_{r\ell}(\boldsymbol{x}, \hat{p}_{r\ell}))$, and a membership function of a permissible probability level $\hat{p}_{r\ell}$ as $\mu_{\tilde{\hat{p}}_{r\ell}}(\hat{p}_{r\ell})$ respectively. Then, HMOP8 can be transformed as the following problem.

[HMOP9]

first level decision maker : DM$_1$

$$\max_{\boldsymbol{x} \in X, \hat{\boldsymbol{p}}_1 \in (0,1)^{k_1}} \Big(\mu_{\tilde{f}_{11}}(f_{11}(\boldsymbol{x}, \hat{p}_{11})), \cdots, \mu_{\tilde{f}_{1k_1}}(f_{1k_1}(\boldsymbol{x}, \hat{p}_{1k_1})), \mu_{\tilde{\hat{p}}_{11}}(\hat{p}_{11}), \cdots, \mu_{\tilde{\hat{p}}_{1k_1}}(\hat{p}_{1k_1})\Big)$$

. . . .

q-th level decision maker: DM$_q$

$$\max_{\boldsymbol{x} \in X, \hat{\boldsymbol{p}}_q \in (0,1)^{k_q}} \Big(\mu_{\tilde{f}_{q1}}(f_{q1}(\boldsymbol{x}, \hat{p}_{q1})), \cdots, \mu_{\tilde{f}_{qk_q}}(f_{qk_q}(\boldsymbol{x}, \hat{p}_{qk_q})), \mu_{\tilde{\hat{p}}_{q1}}(\hat{p}_{q1}), \cdots, \mu_{\tilde{\hat{p}}_{qk_q}}(\hat{p}_{qk_q})\Big)$$

Throughout this section, we make the assumptions that $\mu_{\tilde{\hat{p}}_{r\ell}}(\hat{p}_{r\ell})$ is strictly monotone increasing and continuous with respect to $\hat{p}_{r\ell}$, and $\mu_{\tilde{f}_{r\ell}}(f_{r\ell}(\boldsymbol{x}, \hat{p}_{r\ell}))$ is strictly monotone decreasing and continuous with respect to $f_{r\ell}(\boldsymbol{x}, \hat{p}_{r\ell})$ for any $r = 1, \cdots, q, \ell = 1, \cdots, k_r$.

It should be noted here that, from (18), $\mu_{\tilde{f}_{r\ell}}(f_{r\ell}(\boldsymbol{x}, \hat{p}_{r\ell}))$ and $\mu_{\tilde{\hat{p}}_{r\ell}}(\hat{p}_{r\ell})$ are conflict each other for any $\boldsymbol{x} \in X$. Here, let us assume that the decision maker adopts the fuzzy decision [4] in order to integrate both the membership functions $\mu_{\tilde{f}_{r\ell}}(f_{r\ell}(\boldsymbol{x}, \hat{p}_{r\ell}))$ and $\mu_{\tilde{\hat{p}}_{r\ell}}(\hat{p}_{r\ell})$. Then, the integrated membership function can be defined as follows.

$$\mu_{D_{f_{r\ell}}}(\boldsymbol{x}, \hat{p}_{r\ell}) \stackrel{\text{def}}{=} \min\{\mu_{\tilde{\hat{p}}_{r\ell}}(\hat{p}_{r\ell}), \mu_{\tilde{f}_{r\ell}}(f_{r\ell}(\boldsymbol{x}, \hat{p}_{r\ell}))\} \quad (19)$$

Using the membership functions $\mu_{D_{f_{r\ell}}}(\boldsymbol{x}, \hat{p}_{r\ell})$, HMOP9 can be transformed into the following form.

[HMOP10]

first level decision maker : DM$_1$

$$\max_{\boldsymbol{x} \in X, \hat{p}_{1\ell} \in (0,1), \ell=1,\cdots,k_1} \Big(\mu_{D_{f_{11}}}(\boldsymbol{x}, \hat{p}_{11}), \cdots, \mu_{D_{f_{1k_1}}}(\boldsymbol{x}, \hat{p}_{1k_1})\Big)$$

. . . .

q-th level decision maker: DM$_q$

$$\max_{\boldsymbol{x} \in X, \hat{p}_{q\ell} \in (0,1), \ell=1,\cdots,k_q} \Big(\mu_{D_{f_{q1}}}(\boldsymbol{x}, \hat{p}_{q1}), \cdots, \mu_{D_{f_{qk_q}}}(\boldsymbol{x}, \hat{p}_{qk_q})\Big)$$

In order to deal with HMOP10, we introduce a D_f-Pareto optimal solution concept.

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Definition 2.
$\boldsymbol{x}^* \in X, \hat{p}^*_{r\ell} \in (0,1), r = 1,\cdots,q, \ell = 1,\cdots,k_r$ is said to be a D_f-Pareto optimal solution to HMOP10, if and only if there does not exist another $\boldsymbol{x} \in X, \hat{p}_{r\ell} \in (0,1), r = 1,\cdots,q, \ell = 1,\cdots,k_r$ such that $\mu_{D_{f_{r\ell}}}(\boldsymbol{x}, \hat{p}_{r\ell}) \geq \mu_{D_{f_{r\ell}}}(\boldsymbol{x}^*, \hat{p}^*_{r\ell})$ $r = 1,\cdots,q, \ell = 1,\cdots,k_r$, with strict inequality holding for at least one r and ℓ.

For generating a candidate of the satisfactory solution which is also D_f-Pareto optimal, the decision maker is asked to specify the reference membership values [4]. Similar to MINMAX2($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$) in the previous section, once the reference membership values $\hat{\boldsymbol{\mu}}_{\boldsymbol{r}} = (\hat{\mu}_{r1},\cdots,\hat{\mu}_{rk_r})$ and the decision power w_{r+1} are specified by each of the decision makers ($\text{DM}_r, r = 1,\cdots,q$), the corresponding D_f-Pareto optimal solution is obtained by solving the following minmax problem.

[MINMAX4($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$)]

$$\min_{\boldsymbol{x}\in X, \hat{p}_{r\ell}\in(0,1), r=1,\cdots,q, \ell=1,\cdots,k_r, \lambda\in\Lambda} \lambda \tag{20}$$

subject to

$$\hat{\mu}_{r\ell} - \mu_{\tilde{f}_{r\ell}}(f_{r\ell}(\boldsymbol{x}, \hat{p}_{r\ell})) \leq \lambda w_r, \tag{21}$$
$$\hat{\mu}_{r\ell} - \mu_{\tilde{\hat{p}}_{r\ell}}(\hat{p}_{r\ell}) \leq \lambda w_r, \tag{22}$$
$$r = 1,\cdots,q, \ell = 1,\cdots,k_r$$

Because of $\boldsymbol{c}^2_{r\ell}\boldsymbol{x} + \alpha^2_{r\ell} > 0$, the constraints (21) can be transformed as follows.

$$\hat{p}_{r\ell} \leq T_{r\ell}\left(\frac{\mu^{-1}_{\tilde{f}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda w_r) - (\boldsymbol{c}^1_{r\ell}\boldsymbol{x} + \alpha^1_{r\ell})}{\boldsymbol{c}^2_{r\ell}\boldsymbol{x} + \alpha^2_{r\ell}}\right) \tag{23}$$

where $\mu^{-1}_{\tilde{f}_{r\ell}}(\cdot)$ is an inverse function of $\mu_{\tilde{f}_{r\ell}}(\cdot)$. From the constraints (22), it holds that $\hat{p}_{r\ell} \geq \mu^{-1}_{\tilde{\hat{p}}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda)$ where $\mu^{-1}_{\tilde{\hat{p}}_{r\ell}}(\cdot)$ is an inverse function of $\mu_{\tilde{\hat{p}}_{r\ell}}(\cdot)$. Therefore, the constraint (23) can be reduced to the following inequality where a permissible probability level $\hat{p}_{r\ell}$ is disappeared.

$$\begin{aligned} & \mu^{-1}_{\tilde{f}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda w_r) - (\boldsymbol{c}^1_{r\ell}\boldsymbol{x} + \alpha^1_{r\ell}) \\ \geq \; & T^{-1}_{r\ell}(\mu^{-1}_{\tilde{\hat{p}}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda w_r)) \cdot (\boldsymbol{c}^2_{r\ell}\boldsymbol{x} + \alpha^2_{r\ell}) \end{aligned} \tag{24}$$

Then, MINMAX4($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$) can be equivalently reduced to the following problem.

[MINMAX5($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$)]

$$\min_{\boldsymbol{x}\in X, \lambda\in\Lambda} \lambda \tag{25}$$

subject to

$$\begin{aligned} & \mu^{-1}_{\tilde{f}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda w_r) - (\boldsymbol{c}^1_{r\ell}\boldsymbol{x} + \alpha^1_{r\ell}) \\ \geq \; & T^{-1}_{r\ell}(\mu^{-1}_{\tilde{\hat{p}}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda w_r)) \cdot (\boldsymbol{c}^2_{r\ell}\boldsymbol{x} + \alpha^2_{r\ell}), \\ & r = 1,\cdots,q, \ell = 1,\cdots,k_r \end{aligned} \tag{26}$$

It should be noted here that MINMAX5($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$) is same as MINMAX3($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$). Therefore, an optimal solution $(\boldsymbol{x}^*, \lambda^*)$ of MINMAX5($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$) can be obtained by combined use of the bisection method with respect to λ and the first-phase of the two-phase simplex method of linear programming.

The relationship between the optimal solution $(\boldsymbol{x}^*, \lambda^*)$ of MINMAX5($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$) and D_f-Pareto optimal solutions can be characterized by the following theorem.

Theorem 2.
If $\boldsymbol{x}^* \in X, \lambda^* \in \Lambda$ is a unique optimal solution of MINMAX5($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$), then $\boldsymbol{x}^* \in X, \hat{p}^*_{r\ell} = \mu^{-1}_{\tilde{\hat{p}}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r)), r = 1,\cdots,q, \ell = 1,\cdots,k_r$ is a D_f-Pareto optimal solution.

(Proof)
From (26), it holds that $\hat{\mu}_{r\ell} - \lambda^* w_r \leq \mu_{\tilde{f}_{r\ell}}(f_{r\ell}(\boldsymbol{x}^*, \mu^{-1}_{\tilde{\hat{p}}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r)))$, $r = 1,\cdots,q, \ell = 1,\cdots,k_r$. Assume that $\boldsymbol{x}^* \in X$, $\mu^{-1}_{\tilde{\hat{p}}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r)$, $r = 1,\cdots,q, \ell = 1,\cdots,k_r$ is not a D_f-Pareto optimal solution. Then, there exist $\boldsymbol{x} \in X, \hat{p}_{r\ell}$, $r = 1,\cdots,q, \ell = 1,\cdots,k_r$ such that $\mu_{D_{f_{r\ell}}}(\boldsymbol{x}, \hat{p}_{r\ell}) = \min\{\mu_{\tilde{\hat{p}}_{r\ell}}(\hat{p}_{r\ell}), \mu_{\tilde{f}_{r\ell}}(f_{r\ell}(\boldsymbol{x}, \hat{p}_{r\ell})),\}$ $\geq \mu_{D_{f_{r\ell}}}(\boldsymbol{x}^*, \mu^{-1}_{\tilde{\hat{p}}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r)) = \hat{\mu}_{r\ell} - \lambda^* w_r$, $r = 1,\cdots,q, \ell = 1,\cdots,k_r$, strict inequality holding for at least one r and ℓ. Then it holds that

$$\mu_{\tilde{\hat{p}}_{r\ell}}(\hat{p}_{r\ell}) \geq \hat{\mu}_{r\ell} - \lambda^* w_r, \tag{27}$$
$$\mu_{\tilde{f}_{r\ell}}(f_{r\ell}(\boldsymbol{x}, \hat{p}_{r\ell})) \geq \hat{\mu}_{r\ell} - \lambda^* w_r, \tag{28}$$

$r = 1,\cdots,q, \ell = 1,\cdots,k_r$. From the definition (18), the inequalities (27) and (28) can be transformed into the inequalities, $\hat{p}_{r\ell} \geq \mu^{-1}_{\tilde{\hat{p}}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r), \hat{p}_{r\ell} \leq T_{r\ell}\left(\frac{\mu^{-1}_{\tilde{f}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r) - (\boldsymbol{c}^1_{r\ell}\boldsymbol{x}^* + \alpha^1_{r\ell})}{\boldsymbol{c}^2_{r\ell}\boldsymbol{x}^* + \alpha^2_{r\ell}}\right)$. This means that there exists some $\boldsymbol{x} \in X$ such that $\mu^{-1}_{\tilde{f}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r) - (\boldsymbol{c}^1_{r\ell}\boldsymbol{x} + \alpha^1_{r\ell}) \geq T^{-1}_{r\ell}(\mu^{-1}_{\tilde{\hat{p}}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r)) \cdot (\boldsymbol{c}^2_{r\ell}\boldsymbol{x} + \alpha^2_{r\ell}), r = 1,\cdots,q, \ell = 1,\cdots,k_r$, which contradicts the fact that $\boldsymbol{x}^* \in X, \lambda^* \in \Lambda$ is a unique optimal solution to MINMAX5($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$).

IV. An Interactive Algorithm

In this section, we propose an interactive algorithm to obtain a satisfactory solution of the decision maker from among D_f-Pareto optimal solution set. Unfortunately, it is not guaranteed that the optimal solution $(\boldsymbol{x}^*, \lambda^*)$ of MINMAX5($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$) is D_f-Pareto optimal, if $(\boldsymbol{x}^*, \lambda^*)$ is not unique. In order to guarantee D_f-Pareto optimality, we first assume that $\sum_{r=1}^{q} k_r$ constraints (26) of MINMAX5($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$) are active at the optimal solution $(\boldsymbol{x}^*, \lambda^*)$. If one of the constraints of (26) is inactive, *i.e.*,

$$\begin{aligned} & \mu^{-1}_{\tilde{f}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r) - (\boldsymbol{c}^1_{r\ell}\boldsymbol{x}^* + \alpha^1_{r\ell}) \\ > \; & T^{-1}_{r\ell}(\mu^{-1}_{\tilde{\hat{p}}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r)) \cdot (\boldsymbol{c}^2_{r\ell}\boldsymbol{x}^* + \alpha^2_{r\ell}), \end{aligned} \tag{29}$$

we can convert the inactive constraint (29) into the active one by applying the bisection method, where

$$G_{r\ell}(\hat{\mu}_{r\ell}) \stackrel{\text{def}}{=} \mu^{-1}_{\tilde{f}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r) - f_{r\ell}(\boldsymbol{x}^*, \mu^{-1}_{\tilde{\hat{p}}_{r\ell}}(\hat{\mu}_{r\ell} - \lambda^* w_r)).$$

[The bisection method for the inactive constraint]

Step 1. Set $q^L_{r\ell} \leftarrow \lambda^* w_r$, $q^R_{r\ell} \leftarrow \lambda^* w_r + 1$.

Step 2. Set $q_{r\ell} \leftarrow (q^L_{r\ell} + q^R_{r\ell})/2$.

Step 3. If $G_{r\ell}(q_{r\ell}) > 0$ then $q^L_{r\ell} \leftarrow q_{r\ell}$ and go to Step 2, else if $G_{r\ell}(q_{r\ell}) < 0$ then $q^R_{r\ell} \leftarrow q_{r\ell}$ and go to Step 2, else if $G_{r\ell}(q_{r\ell}) = 0$, then update the reference membership value as $\hat{\mu}_{r\ell} \leftarrow q_{r\ell}$ and stop.

For the optimal solution $(\boldsymbol{x}^*, \lambda^*)$ of MINMAX5($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$), where the active conditions of the constraints (26) are satisfied, we solve the D_f-Pareto optimality test problem formulated as follows.

[Test problem for D_f-Pareto optimality]

$$\max_{\boldsymbol{x}\in X, \epsilon_{r\ell}\geq 0, r=1,\cdots,q, \ell=1,\cdots,k_r} w = \sum_{r=1}^{q}\sum_{\ell=1}^{k_r} \epsilon_{r\ell} \tag{30}$$

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subject to

$$\begin{aligned} & T_{r\ell}^{-1}(\mu_{\tilde{p}_{r\ell}}^{-1}(\hat{\mu}_{r\ell}-\lambda^* w_r))\cdot(\boldsymbol{c}_{r\ell}^2\boldsymbol{x}+\alpha_{r\ell}^2) \\ & +(\boldsymbol{c}_{r\ell}^1\boldsymbol{x}+\alpha_{r\ell}^1)+\epsilon_{r\ell} \\ = \ & T_{r\ell}^{-1}(\mu_{\tilde{p}_{r\ell}}^{-1}(\hat{\mu}_{r\ell}-\lambda^* w_r))\cdot(\boldsymbol{c}_{r\ell}^2\boldsymbol{x}^*+\alpha_{r\ell}^2) \\ & +(\boldsymbol{c}_{r\ell}^1\boldsymbol{x}^*+\alpha_{r\ell}^1), r=1,\cdots,q, \ell=1,\cdots,k_r \end{aligned} \quad (31)$$

For the optimal solution of the above test problem, the following theorem holds.

Theorem 3.

Let $\check{\boldsymbol{x}} \in X, \check{\epsilon}_{r\ell} \geq 0, r=1,\cdots,q, \ell=1,\cdots,k_r$ be an optimal solution of the test problem (30)-(31). If $w=0$, $\boldsymbol{x}^* \in X, \mu_{\tilde{p}_{r\ell}}^{-1}(\hat{\mu}_{r\ell}-\lambda^* w_r), r=1,\cdots,q, \ell=1,\cdots,k_r$ is a D_f-Pareto optimal solution.

(Proof)

From the active conditions of the constraints (26), it holds that $\hat{\mu}_{r\ell}-\lambda^* w_r = \mu_{\tilde{f}_{r\ell}}(f_{r\ell}(\boldsymbol{x}^*, \mu_{\tilde{p}_{r\ell}}^{-1}(\hat{\mu}_{r\ell}-\lambda^* w_r))), r=1,\cdots,q, \ell=1,\cdots,k_r$. If $\boldsymbol{x}^* \in X, \mu_{\tilde{p}_{r\ell}}^{-1}(\hat{\mu}_{r\ell}-\lambda^* w_r), r=1,\cdots,q, \ell=1,\cdots,k_r$ is not a D_f-Pareto optimal solution, there exists some $\boldsymbol{x} \in X, \hat{p}_{r\ell}, r=1,\cdots,q, \ell=1,\cdots,k_r$ such that $\mu_{D_{f_{r\ell}}}(\boldsymbol{x}, \hat{p}_{r\ell}) = \min\{\mu_{\tilde{p}_{r\ell}}(\hat{p}_{r\ell}), \mu_{\tilde{f}_{r\ell}}(f_{r\ell}(\boldsymbol{x}, \hat{p}_{r\ell}))\} \geq \mu_{D_{f_{r\ell}}}(\boldsymbol{x}^*, \mu_{\tilde{p}_{r\ell}}^{-1}(\hat{\mu}_{r\ell}-\lambda^*)) = \hat{\mu}_{r\ell}-\lambda^*, r=1,\cdots,q, \ell=1,\cdots,k_r$, with strict inequality holding for at least one r and ℓ. This means that the following inequalities hold.

$$\mu_{\tilde{p}_{r\ell}}(\hat{p}_{r\ell}) \geq \hat{\mu}_{r\ell}-\lambda^* w_r, \quad (32)$$

$$\mu_{\tilde{f}_{r\ell}}(f_{r\ell}(\boldsymbol{x}, \hat{p}_{r\ell})) \geq \hat{\mu}_{r\ell}-\lambda^* w_r, \quad (33)$$

$r=1,\cdots,q, \ell=1,\cdots,k_r$. This means that there exists some $\boldsymbol{x} \in X$, $\hat{p}_{r\ell}, r=1,\cdots,q, \ell=1,\cdots,k_r$ such that $\mu_{\tilde{f}_{r\ell}}^{-1}(\hat{\mu}_{r\ell}-\lambda^*) \geq (\boldsymbol{c}_{r\ell}^1\boldsymbol{x}+\alpha_{r\ell}^1) + T_{r\ell}^{-1}(\mu_{\tilde{p}_{r\ell}}^{-1}(\hat{\mu}_{r\ell}-\lambda^*))\cdot(\boldsymbol{c}_{r\ell}^2\boldsymbol{x}+\alpha_{r\ell}^2)$. Because of the active conditions of the constraints (26), it holds that $T_{r\ell}^{-1}(\mu_{\tilde{p}_{r\ell}}^{-1}(\hat{\mu}_{r\ell}-\lambda^*))\cdot(\boldsymbol{c}_{r\ell}^2\boldsymbol{x}^*+\alpha_{r\ell}^2)+(\boldsymbol{c}_{r\ell}^1\boldsymbol{x}^*+\alpha_{r\ell}^1) \geq T_{r\ell}^{-1}(\mu_{\tilde{p}_{r\ell}}^{-1}(\hat{\mu}_{r\ell}-\lambda^*))\cdot(\boldsymbol{c}_{r\ell}^2\boldsymbol{x}+\alpha_{r\ell}^2)+(\boldsymbol{c}_{r\ell}^1\boldsymbol{x}+\alpha_{r\ell}^1)$, $r=1,\cdots,q, \ell=1,\cdots,k_r$, with strict inequality holding for at least one r and ℓ. This contradicts the fact that $w=0$.

Now, following the above discussions, we can present the interactive algorithm in order to derive a satisfactory solution from among a D_f-Pareto optimal solution set.

[An interactive algorithm]

Step 1: Each of the decision maker ($\mathrm{DM}_r, r=1,\cdots,q$) sets his/her membership functions $\mu_{\tilde{p}_{r\ell}}(\hat{p}_{r\ell}), \ell=1,\cdots,k_r$ in his/her subjective manner.

Step 2: Corresponding to the membership functions $\mu_{\tilde{p}_{r\ell}}(\hat{p}_{r\ell}), \ell=1,\cdots,k_r$, each of the decision maker ($\mathrm{DM}_r, r=1,\cdots,q$) sets his/her membership functions $\mu_{\tilde{f}_{r\ell}}(f_{r\ell}(\boldsymbol{x}, \hat{p}_{r\ell})), \ell=1,\cdots,k_r$.

Step 3: Set the initial reference membership values as $\hat{\mu}_{r\ell}=1, r=1,\cdots,q, \ell=1,\cdots,k_r$, and the initial decision power as $w_r=1, r=1,\cdots,q$.

Step 4: Solve MINMAX5($\hat{\boldsymbol{\mu}}, \boldsymbol{w}$) by combined use of the bisection method and the first-phase of the two-phase simplex method of linear programming. If the active condition of the constraints (26) is not satisfied at the optimal solution $(\boldsymbol{x}^*, \lambda^*)$, then the bisection method with respect to the reference membership value is applied, and D_f-Pareto optimality test problem is solved.

Step 5: If each of the decision makers ($\mathrm{DM}_r, r=1,\cdots,q$) is satisfied with the current values of the D_f-Pareto optimal solution $\mu_{D_{f_{r\ell}}}(\boldsymbol{x}^*, \hat{p}_{r\ell}^*), \ell=1,\cdots,k_r$, where $\hat{p}_{r\ell}^* = \mu_{\tilde{p}_{r\ell}}^{-1}(\hat{\mu}_{r\ell}-\lambda^*)$,then stop. Otherwise, let the s-th level decision maker (DM_s) be the uppermost of the decision makers who are not satisfied with the current values. Considering the current values of his/her membership functions, DM_s updates his/her decision power w_{s+1} and/or his/her reference membership values $\hat{\mu}_{s\ell}, \ell=1,\cdots,k_s$ according to the following two rules, and return to Step 4.

Rule 1 w_{s+1} must be set as $w_{s+1} \leq w_s$. After updating w_{s+1}, if $w_{s+1} < w_t, s+2 \leq t \leq q$, w_t is replaced by w_{s+1} ($w_t \leftarrow w_{s+1}$).

Rule 2 Before updating DM_s's reference membership values $\hat{\mu}_{s\ell}, \ell=1,\cdots,k_s$, the other decision makers' reference membership values are fixed as the current values ($\hat{\mu}_{r\ell} \leftarrow \mu_{D_{f_{r\ell}}}(\boldsymbol{x}^*, \hat{p}_{r\ell}^*), r=1,\cdots,q, r\neq s, \ell=1,\cdots,k_r$) .

V. Conclusion

In this paper, we have proposed an interactive decision making method for hierarchical multiobjective stochastic linear programming problems to obtain a satisfactory solution from among a Pareto optimal solution set. In the proposed method, by considering the conflict between permissible objective levels and and permissible probability levels, the corresponding membership functions are integrated through the fuzzy decision. In the integrated membership space, the candidate of a satisfactory solution is obtained from among Pareto optimal solution set by updating the reference membership values and/or the decision powers. In our proposed method, it is expected to obtain the satisfactory solution, in which the proper balance between permissible objective values and permissible probability levels are attained.

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