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When to Refinance Mortgage Loans in a Stochastic Interest Rate Environment

Siwei Gan, Jin Zheng, Xiaoxia Feng, and Dejun Xie

***Abstract*—Refinancing refers to the replacement of an existing debt obligation with another debt obligation to take the advantage of a lower interest rate. This paper reflects our ongoing work to find a desirable refinancing time for mortgage borrowers to minimize the total payments in a dynamic interest rate environment. To simulate the alternative financial service that the market may offer, it is assumed that the future interest rate follows a stochastic model with mean-reverting property, which is essentially the only required market condition to implement our method. To make it more applicable to the real financial practice, two balance settlement schemes are considered and compared. Numerical simulations with varying samplings lead to several interesting characteristics pertaining to the optimal mortgage refinancing period. Our method is robust and user friendly, thus is useful for financial institutions and individual property investors.**

***Index Terms*—mortgage refinancing, loan valuation, financial optimization, Monte-Carlo simulation, stochastic interest rate model**

I. Introduction

Debtors refinance to improve the leverage efficiency of their loan portfolios by taking advantage of lower borrowing rates. Many previous studies are addressing the problem from the perspective of optimal refinancing differentials, where the optimal differential is reached when the net present value of the interest payment saved equals the sum of refinancing costs (see [1] and relevant references contained therein). In comparison to those studies, one important distinction of our work is to simulate alternative mortgage rate instead of assuming a known forward interest term structure. Based on the idea of mean reverting, the Vasicek Model is one of the earliest non-arbitrage interest rate models. It has been used not only for characterizing the prices of basic discount bond (see [6]), but also for valuing complicated financial products, including residential mortgages (see [4], [5], [7], for instance). Another reason for using Vasicek model to demonstrate our algorithm is the existence of convenient parameter estimation procedures for the model, including maximum likelihood method or Bayesian based method. References of such estimations can be found in [3], for instance.

Two types of loan payment are commonly adopted in mortgage industry: one is to match the principal repayment method and the other is to match the repayment of principal plus interest. In this work, both methods are used, actually contrasted, to generate the streams of loan instalments during the whole contracted duration of mortgage contract. To start the analysis, it is worthwhile to specify two important restrictions on the practice of refinance. First, the debtor is allowed to refinance only once after the mortgage is signed but before the expiry of the contract. To minimize the total financial payment, the debtor should grasp the best opportunity where the interest rate is lower enough. Second, if the debtor decides to refinance, the total amount of money he shall pay at the refinancing time is the outstanding loan balance, which means the subsequent residuary interest collapses upon refinancing. This assumption is reasonable since otherwise the debtor will have no motivation to refinance and the liquidity of financial market will be adversely affected. In reality, the original lender may charge certain amount of refinance fee in compensation of its interest loss from the early closing of the contract. The impact of refinance fee is not included in the current work.

The rest of this paper is organized as follows. We choose a model to simulate alternative mortgage rate, then derive the cash flow schemes for the two types of loan settlement in Section 2. In Section 3, we formulate and present our algorithm for finding the period which is desirable for debtors to refinance. In Section 4, we provide numerical experiments for model calibration with varying samplings. We summarize in Section 6 with concluding remarks and possible applications in related fields.

II. Model derivation and interest rate simulation

The Vasicek short term interest rate process is a mathematical model describing the evolution of interest rate (see [2]). The model specifies that the instantaneous interest rate follows the stochastic differential equation:

$$dr_t = k(\theta - r_t)dt + \sigma dW_t \tag{1}$$

where k is the reversion rate, θ is long$-$term mean interest rate and σ is the standard deviation, all of which are positive constants. We let r_t denote the instantaneous spot rate at time t, and W_t is the standard Brownian Motion.

The original stochastic differential equation can be solved by noticing that

$$d(e^{kt}r_t) = e^{kt}dr_t + ke^{kt}r_t dt \tag{2}$$

Substituting Vasicek Model into equation(2), we have

$$\begin{aligned} d(e^{kt}r_t) &= e^{kt}(\theta k - r_t k)dt + e^{kt}\sigma dW_t + ke^{kt}r_t dt \\ &= \theta k e^{kt}dt + e^{kt}\sigma dW_t \end{aligned} \tag{3}$$

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Siwei Gan (Siwei.Gan10 at student.xjtlu.edu.cn) and Dejun Xie (Dejun.Xie at xjtlu.edu.cn) are with the Department of Mathematical Sciences, Xi'an Jiaotong Liverpool University, China.

Jin Zheng (Jin.Zheng at liv.ac.uk) is with the School of Mathematical Sciences, University of Liverpool, UK; and the Department of Mathematical Sciences, Xi'an Jiaotong Liverpool University, China.

Xiaoxia Feng (x.x.f.09 at stu.xjtu.edu.cn) is with the Department of Mathematics, Xi'an Jiaotong University, China.

Correspondence author email: Jin.Zheng at liv.ac.uk or x.x.f.09 at stu.xjtu.edu.cn.

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Integrating both sides with respect to t:

$$\begin{aligned} e^{kt}r_t &= r_0 + k\theta \int_0^t e^{ks}ds + \sigma \int_0^t e^{ks}dW_s \\ &= r_0 + \theta(e^{kt}-1) + \sigma \int_0^t e^{ks}dW_s \end{aligned} \quad (4)$$

which yields the explicit solution for equation (1)

$$r_t = e^{-kt}r_0 + \theta(1-e^{-kt}) + \sigma \int_0^t e^{-k(t-s)}dW_s \quad (5)$$

Under the Euler approximation, equation (1) can be rewritten as:

$$\Delta r = k(\theta - r_t)\Delta t + \sigma \Delta W_t \quad (6)$$

Both equation (1) and equation (5) can be used equivalently to describe the alternative mortgage rate that a loan borrower may choose from the open market. But equation (6) is often more useful for simulation purposes. We would like to remark that although Vasicek model is considered in the current paper, our method is equally applicable to many other classes of stochastic models.

A. Matching The Principal Repayment Method

Suppose the debtor borrows P_0 with monthly interest rate r_0 during the time period $[0\ T]$ and repays m_t at the beginning of each month, where t denotes the t^{th} month. According to matching the principal repayment method, m_t equals to a certain portion of principal plus a decreasing value of interest.

$$m_t = \frac{P_0}{n} + (1 - \frac{t-1}{n})P_0 r_0 \quad (7)$$

where n is the total number of repayment times.

The term $\frac{P_0}{n}$ could be explained as a fixed portion of principal, and $(1 - \frac{t-1}{n})P_0 r_0$ is an amount of decreasing interest due to the reduction of principal every month.

At time k, the debtor prefers to refinance the debt with another lender when a lower interest rate r_k is offered. On the k^{th} month, he owes the previous bank P_k and has paid A_k.

$$\begin{aligned} P_k &= (1 - \frac{k-1}{n})P_0 \\ A_k &= \sum_{i=1}^{k-1} m_i = P_0(k-1)(r_0 + \frac{1}{n} - \frac{k-2}{2n}r_0) \end{aligned} \quad (8)$$

The amount of money $P(t)$ is the new principal the debtor borrows from another bank with the interest rate $r(t)$. This transaction will last from time k to time T. The total payment over time $[0\ T]$ could be described as follows:

$$\begin{aligned} P(T) &= A_k + \sum_{i=k}^{n} m_i \\ &= P_0(k-1)(r_0 + \frac{1}{n} - \frac{k-2}{2n}r_0) \\ &\quad + P_k[1 + \frac{(n^*+1)r_k}{2}] \end{aligned} \quad (9)$$

where $n^* = n - k + 1$

B. Matching the Repayment of Principal and Interest Method

The second method to repay loan is to match the repayment of principal and interest. Assume the debtor borrows P_0 with interest rate r_0 over time $[0\ T]$ and the amount of monthly payment is kept the same. In the beginning of the contract, the interest accounts for most of payment due to a large amount of loan while principal is small. Let $P(t)$ denote the amount of money owed at time t and m is the monthly payment.

$$\begin{cases} dP(t) = -mdt + r_0 P(t)dt \\ P(0) = P_0 \end{cases} \quad (10)$$

The monthly payment m, should be:

$$m = \frac{P_0 r_0 (1+r_0)^n}{(1+r_0)^n - 1} \quad (11)$$

where n is the number of total repayment times.

At time k, the debtor owes the $P(k)$ to the previous bank. Again, due to the lower interest rate r_k , the debtor would borrow $P(k)$ from another bank to repay the remaining debts $P(k)$. The total payment over time $[0\ T]$ could be described as follows:

$$P(T) = m_1(k) + m_2(n-k) \quad (12)$$

where

$$\begin{cases} m_1(k) = \dfrac{P_0 r_0 (1+r_0)^k}{(1+r_0)^k - 1} \\ m_2(n-k) = \dfrac{P_k r_k (1+r_k)^{n-k}}{(1+r_k)^{n-k} - 1} \end{cases} \quad (13)$$

To carry out numerical simulations for both repayment schemes, we assume that the principal P_0 is 100,000, the initial lending rate r_0 is 5%, and the total payment period, counted in number of months, is $T = 240$.

III. NUMERICAL EXPERIMENTATION

In this section, we use simulated data to carry out the experiment. The aim of our model is to obtain the best period to refinance. The 'best period' in our experiment means the month during which to refinance yields a lowest total payment. We simulate both methods to obtain the frequency distributions.

A. Matching The Principal Payment Method

The following Figure 1 provides the information of the frequency distribution of the best period throughout the contracted duration. The frequency space is 6 months. It can be seen that the frequency arrives the peak at the second half of the first year. The frequency of following months declines over time. From the results reported in Table 1, we find that until the $5th$ year, the total times to refinance is up to 9252 (the frequency rate is 92.52 %), which implies it is better to refinance early.

We include the interest rate factor into our implementation and discussion. Assume the best opportunity to refinance arises when the total payment is comparatively low. We define a new variable 'count' to record the times that the best month to refinance (m_t) coincides with the month when the smallest interest rate (m_r) occurs. Hence, 'count' plays

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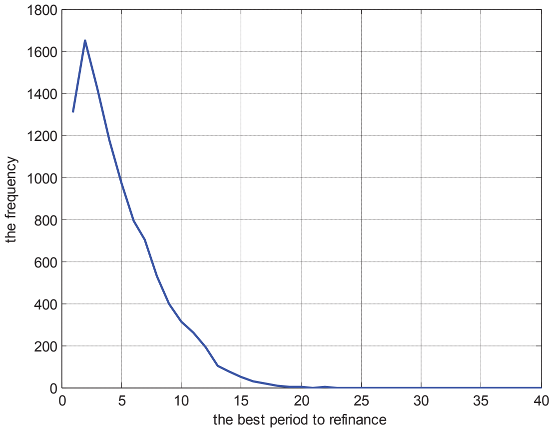

Fig. 1. The frequency distribution over 240 months' duration by 10000 times of simulations with matching the principal payment method.

TABLE I
FREQUENCY AND CUMULATIVE FREQUENCY OF THE BEST TIME TO REFINANCE

Months	Frequency	Cumulative Frequency
1-6	1037	1037
7-12	1649	2956
13-18	1424	4380
19-24	1174	5554
25-30	971	6525
31-36	791	7316
37-42	701	8017
43-48	527	8544
49-54	395	8939
55-60	313	9252
61-66	258	9510
67-72	193	9703
73-78	105	9808
79-84	78	9886
85-90	50	9936
91-96	30	9966
97-102	21	9987
103-108	7	9994
109-114	3	9997
115-120	2	9999
121-126	0	9999
127-132	1	10000
133-240	0	10000

a role on quantifying and measuring the coincidence. In each simulation, if the difference between them is less than 3 months, we regard them to be coincidence and the value of 'count' increases by 1.

The above procedure is circulated 10 times and we choose (m_r) in different time intervals. We use a variable 'count' to measure the coincidence and the results are shown in Table 2. The second column $1-36$ represents the time interval from the $1st$ month to the $36th$ month of the contract. Similarly, $1-60$, $1-90$ and $1-240$ mean the corresponding month intervals. For instance, the times that the optimal refinance period locates in the interval from the first month to the $90th$ month is 6295 in the first simulation.

The bottom row in Table 2 displays the average value of 'count'. It is observed that the average percentage value

TABLE II
FREQUENCY AND CUMULATIVE FREQUENCY OF THE BEST TIME TO REFINANCE

Times	1-36	1-60	1-90	1-240
1	5612	6765	6295	2721
2	5584	6823	6446	2820
3	5721	6838	6321	2761
4	5666	6973	6393	2749
5	5660	6890	6442	2760
6	5731	6853	6348	2708
7	5607	6710	6260	2671
8	5626	6770	6326	2743
9	5679	6829	6351	2807
10	5714	6767	6335	2713
Average	5660	6821.8	6447	2745.3

of 'count' during $1st-240th$ is only 27.45%, which is the lowest compared to others. This result is not surprising since it has been shown in above that the possibility of refinance is up to 92.52% in the first five years. Furthermore, the interest rate movement is simulated by the Vasicek model, indicating that the lowest interest rate could appear anytime. Both reasons lead to such a low coincidence. As the time interval is shortened to, say, the first three years or the first five years, the percentage of coincidence substantially increases.

From Table 1, we have seen that the best refinancing month is considerably more possibly located in the earlier time. But how early is still a problem deserving prudent consideration. The duration of the first 90 months apparently shows the highest possibility 99.36%. However, the interval is so long that it may not be an operative suggestion to debtors. In fact, the frequency rate steadily increases after the $60th$ month. On the other hand, when we inspect the first 36 months' duration, it is noted that although the range becomes small, the possibility that the best period to refinance locates in this range is still as high as 73.16%. As for the duration of the first 60 months, the frequency rate is 92.52%, and the corresponding average percentage of coincidence is the highest among all these three cases. This comparison provides a useful hint on the distributional pattern of the best refinance period, which, taken in conjunction with the observations of the real market interest rate, will facilitate the borrower's financial decisions.

B. *Matching The Payment of Principal and Interest*

Figure 2 is the frequency distribution generated by simulating 10000 times of matching payment of principal and interest method. It has the similar but not identical properties compared to Figure 1. In this payment scheme, the principal balance decreases rather slowly at early stage while in the first payment scheme (matching the payment of principle) that the principle decreases by an equal amount each month. Thus, the less indifference of change of principle leads to the more divergent distribution.

Again, the interest rate factor should be involved in our discussion. As presented above, we use 'count' to record the times that the best month to refinance (m_t) coincides with the month when the smallest interest rate (m_r) occurs.

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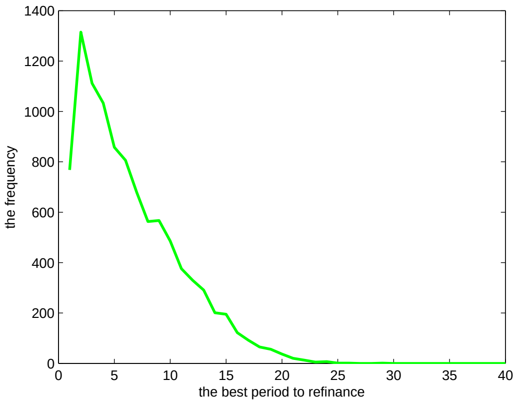

Fig. 2. The frequency distribution over 240 months' duration by 10000 times of simulations with matching the payment of principal and interest method.

TABLE III
FREQUENCY AND CUMULATIVE FREQUENCY OF THE BEST TIME TO REFINANCE

Months	Frequency	Cumulative Frequency
1-6	727	727
7-12	1038	1765
13-18	1259	3024
19-24	1162	4186
25-30	961	5047
31-36	766	5913
37-42	670	6583
43-48	588	7171
49-54	553	7704
55-60	468	8172
61-66	381	8553
67-72	334	8887
73-78	274	9161
79-84	233	9494
85-90	161	9625
91-96	136	9791
97-102	106	9797
103-108	58	9855
109-114	45	9900
115-120	38	9938
121-126	34	9972
127-132	15	9987
133-138	11	9999
139-142	1	10000
143-240	0	10000

Figure 2 and Table 3 reveal the frequency distribution during 20 years. The results we have obtained are similar to the previous method. The frequency first increases, reaching the peak during the $13th$ month to the $18th$ month. Afterwards it decreases gradually, down to 0 after 11 years. Until the $7th$ year, the cumulative frequency is 9494 in total, which provides a strong evidence for early refinance. As for coincidence, again, the duration of 90 months has the highest value in these three periods.

TABLE IV
FREQUENCY AND CUMULATIVE FREQUENCY OF THE BEST TIME TO REFINANCE

Times	1-36	1-60	1-90	1-240
1	5579	6862	6465	2777
2	5622	6833	6460	2824
3	5572	6821	6549	2877
4	5531	6769	6456	2757
5	5587	6764	6528	2800
6	5647	6856	6485	2802
7	5574	6798	6450	2761
8	5624	6847	6478	2903
9	5633	6821	6435	2739
10	5540	6914	6510	2809
Average	5590.9	6828.5	6481.6	2804.9

C. Comments on the Results

In our paper, we wish to determine which period is a better choice for debtors to refinance. The study has found some important properties for refinancing. First, the possibility of refinancing in the early stage may surpass 90%, which implies that debtors should refinance early. Second, the frequency curve arrives its peak at the last half of the first year. After that, the frequency of refinancing will drop and the coincidence increases at first and decreases after its peak value. Finally, a duration neither relatively too long nor too short is regarded as a perfect solution, i.e., a duration of 90 months (7.5 years) is relatively too long to the whole duration of 20 years. In consideration of these four properties, the debtors should refinance in the period of the $1st$ to the $60th$ month when the interest rate is locally low, for contract conditions and market rate movement specified in this paper. That means in certain month when the interest rate will be expected to fall down to certain lower enough level, it is probably the best time to refinance.

IV. APPLICATIONS

A. Model Calibration

The above results show that the debtors should refinance as earlier as possible when the lending rate is relatively low. In this section, we examine the correction of the conclusion and calibrate our model.

We assume that the debtor adopts the matching the principal payment method to pay back his debt. Let $t = k - 1$, then equation (9) yields

$$\begin{aligned}\frac{P(T)}{P_0} &= [r_0 - \frac{r_{t+1}}{2} + \frac{r_0}{2n} - \frac{(n+1)r_{t+1}}{2n}]t \\ &\quad + \frac{r_{t+1} - r_0}{2n}t^2 + \frac{n+1}{2}r_{t+1} + 1 \\ &= \frac{r_{t+1} - r_0}{2n}t^2 + (1 + \frac{1}{2n})(r_0 - r_{t+1})t \\ &\quad + \frac{n+1}{2}r_{t+1} + 1 \qquad (14)\end{aligned}$$

We proceed the analysis by identifying the following two scenarios.

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1) $r_0 = \theta$: When the initial borrowing rate equals to the long term mean rate, the stochastic process for the market interest rate becomes

$$\begin{aligned} r_t &= e^{-kt} r_0 + \theta(1 - e^{-kt}) + \sigma \int_0^t e^{-k(t-s)} dW_s \\ &= r_0 + \sigma \int_0^t e^{-k(t-s)} dW_s \qquad (15) \end{aligned}$$

It is intuitive and worthwhile to note that the debtor is likely to refinance only when the instantaneous spot rate is less than the initial borrowing rate, i.e., only when the stochastic integral term $\sigma \int_0^t e^{-k(t-s)} dW_s$ results in a negative value. But even with this in mind, the statistically measured minimizer t to the stochastic function $\frac{P(T)}{P_0}$ is not immediate since the equation (14), as a quadratic form in t with stochastic coefficients, is composed of terms with different signs in differentials in t. For instance, one might want t go to zero on the set of t where $r_0 > r_t$ if only the first order term of t is concerned, but this move may not grant enough time for r_t to achieve sufficiently lower level, which is desirable if the second order or zero order term of t is concerned. An equilibrium of the opposing factors in (14), as shown by our simulated results in Figure 4 and 6, says that the best refinance time is most likely located in the early stage of the contract for the usual conditions set in this paper. This is true despite that the expectation of $\frac{P(T)}{P_0}$ is independent of time t. The result is consistent with the numerical results contained in the previous section and offers a statistical explanation to the optimal strategy that a borrower should take to minimize his total financial cost.

2) $r_0 > \theta$: When the initial borrowing rate is higher than the long term mean rate, note that the stochastic process for the market interest rate can be written as

$$r_t = (r_0 - \theta)e^{-kt} + \theta + \sigma \int_0^t e^{-k(t-s)} dW_s \qquad (16)$$

Figure 3 reveals that when the value of σ is small (i.e. 0.001 or less), the simulated interest rates are fluctuating around the 'drift' with very small deviations. In this scenario, the general trend of interest rate drops exponentially to the mean level. With the parameters we choose for the model, and with the current simulation specifications, such as the time step for the Euler approximation and the maximum number of simulated trajectories, contained in this paper, we find that the stochastic integral term $\sigma \int_0^t e^{-k(t-s)} dW_s$ is negligible in statistical sense for understanding the refinancing strategy.

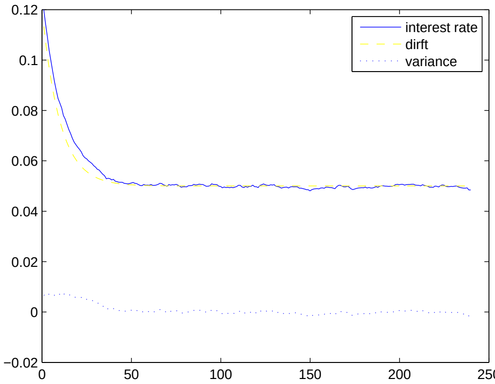

Fig. 3. 'drift' represents the term $(r_0 - \theta)e^{-kt} + \theta$, 'variance' represents the term $\sigma \int_0^t e^{-k(t-s)} dW_s$ and 'interest rates' are the simulated spot instantaneous rates, where r_0=0.12, k=0.1 and σ=0.001.

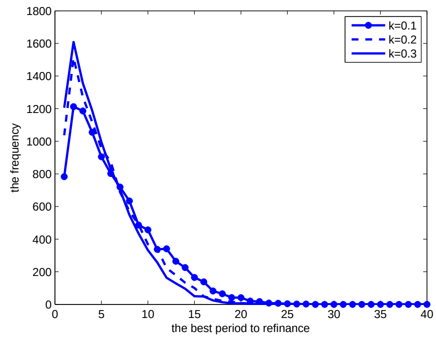

Fig. 4. The frequency distribution at different values of k, when θ =0.05, σ =0.003 and r_0=0.05.

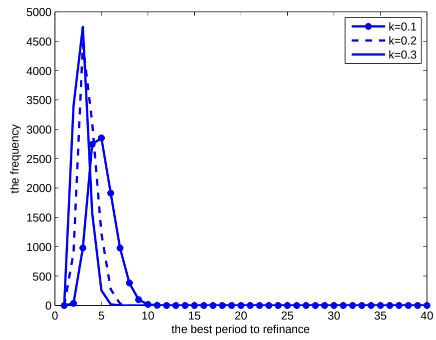

Fig. 5. The frequency distribution at different values of k, when θ =0.05, σ =0.003 and r_0=0.12.

B. Implementation With Different Values for Parameters

We simulate the process for 10000 times for different values of parameters and adopt the matching the principal payment method in all simulations. We also involve two conditions into simulation. One is normal that the initial interest rate equals to the long-term mean interest rate. The other is an extreme condition that the initial interest rate is greater than the long-term mean interest rate.

1) Parameter k: It has been pointed out that the parameter k is the reversion rate and σ is the long-term mean interest rate. As mentioned before, the Vasicek Model is notable for its reversion property, which means after a positive change in the actual returns, mean-reversion causes a negative subsequent change and vice versa (see [3]). Figure 4 and 5 show the fact that, as the value of k rises, the likelihood of refinancing in the last half of the first year sees a growth when the initial lending rate equals to the mean lending rate. In the extreme condition that the initial interest rate is greater than the initial lending rate, the increase of the reversion speed leads to early refinancing.

2) Parameter σ: To observe the effect of market rate volatility on the refinance frequency distribution, we change the value of σ while keeping other parameters fixed. Figure 6 provides the numerical outputs when r_0 equals to the long-

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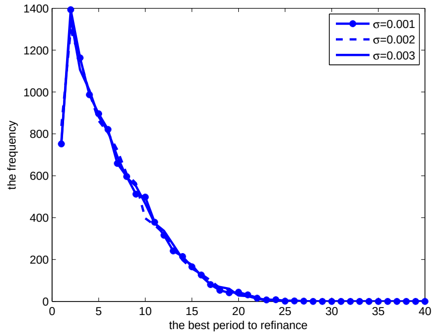

Fig. 6. The frequency distribution at different values of σ, when θ=0.05, k=0.1 and r_0=0.05.

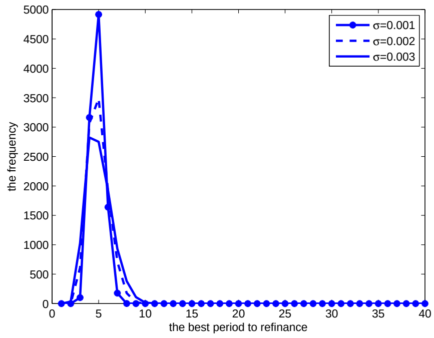

Fig. 7. The frequency distribution at different values of σ, when θ=0.05, k=0.1 and r_0=0.12.

term mean. In this example, changes in the value of σ do not lead to significant changes in the frequency distribution. When r_0 is relatively higher than the long-term mean interest rate θ, the consequence is more apparent. Figure 7 shows the numerical plots for this scenario. An apparent convergent pattern can be drawn from Figure 7, where the best refinance period converges to around the $25th$ month as σ decreases.

V. Concluding Remarks

This paper focuses on the numerical simulation approach for finding the best refinancing strategy for mortgage borrowers in a stochastic interest environment. Interesting properties of the optimal refinancing time, including its relative closeness to the origination of the contract and the statistically lowest point of the interest curve, are discovered. In this work, Vasicek Model is applied to simulate the monthly interest rate and both matching the principal payment method and matching the payment of principal and interest method are considered to generate the total payment. Results from these empirical experiments tend to suggest relatively early refinancing for both scenarios under the conditions of the mortgage contracts set in the paper, particulary when the initial borrowing rate is large compared to the long term mean rate. These findings shed lights on the very important financial queries for many property investors.

In addition, since mortgage contract is also a type of option, the usefulness of our approach is not limited to the problem at hand. Traditional analytical techniques for characterizing option contracts, if possible, usually require mathematically strong and sometimes parameter sensitive properties attached to the formulation of the problem, such as the convexity existed in the early exercise boundary of the classic American put option (see [2], [8], for instance). In comparison to such analytical methods, our approach is robust and easy to implement. The algorithms contained in this work can be readily applied to a broad class of problems arising from financial optimization and option pricing.

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