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Numerical Solution of Heat Transfer in a Viscoelastic Boundary Layer Flow over a Stretching Sheet

V. Dhanalaxmi and B. Shanker

***Abstract*—This paper presents a study on visco elastic boundary layer flow and heat transfer over a stretching sheet in the presence of viscous dissipation and non-uniform heat source. A quasilinearization technique is used to solve velocity and temperature profiles. Two cases are considered, namely, (i) Prescribed surface temperature (PST) and (ii) Prescribed wall heat flux (PHF). The effect of various parameters on velocity and temperature profiles is depicted in graphs and discussed.**

***Index Terms*—viscoelastic fluid flow, heat transfer, stretching sheet, non-uniform heat source, Walters' liquid B model.**

I. Introduction

BOUNDARY layer behavior over a moving continuous solid surface is an important type of flow occurring in several engineering processes. Since the pioneering work of Sakiadis [1]-[2]. Various aspects of the problem have been investigated by many authors. Erickson [3] extended this problem to the case for which suction or blowing existed at the moving surface. Crane [4] extended the problem of Sakiadis to the stretching sheet whose velocity is proportional to the distance from the slit.

Since the physical properties of the ambient fluid effectively influence the boundary layer characteristics, the study of non-Newtonian fluid flow over a moving sheet has gained considerable importance. Therefore several authors [5]-[11] studied viscoelastic boundary layer flow along a stretching sheet for Non-Newtonian fluids.

In the present study, an incompressible viscoelastic (Walters' liquid B model) fluid over a stretching sheet with viscous dissipation and non-uniform heat source is considered. A numerical method, quasilinearization technique is used to find velocity and temperature profiles. Results are in good agreement with available literature.

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II. Mathematical Formulation

Consider the flow of an incompressible viscoelastic (Walters' liquid B model) fluid over a wall coinciding with the plane y = 0, the flow being confined to y > 0.Two equal and opposite forces are applied along the x-axis, so that the wall is stretched keeping the origin fixed. The steady two dimensional boundary layer equations for this fluid were derived by Beard and Walters [12]. In usual notation these equations are given as:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}-k_0\left\{\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}+v\frac{\partial^3 u}{\partial y^3}\right\} \tag{2}$$

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\frac{\partial^2 T}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2+q''' \tag{3}$$

$$where \quad \upsilon=\frac{\mu}{\rho},\ k_0>0$$

where u and v are the velocity components respectively along the x and y directions, υ is the kinematic viscosity, k_0 is the co-efficient of elasticity, k is the thermal conductivity, ρ is the density, T is the temperature, c_p is the specific heat at constant pressure and q''' is the space and temperature dependent internal heat generation/absorption (non-uniform heat source/sink) [13] which can be expressed in simplest form as

$$q'''=\left(ku_w(x)/x\upsilon\right)\left[A^*\left(T_w-T_\infty\right)f'(\eta)+B^*\left(T-T_\infty\right)\right] \tag{4}$$

Here A* and B* are parameters of space and temperature dependent internal heat generation/absorption. It is to be noted that A* > 0 and B* > 0 correspond to internal heat generation while A* < 0 and B* < 0 correspond to internal heat absorption.

The boundary conditions for the velocity field are:

$$u=u_w=bx,\ v-0 \qquad \text{at } y=0,\ b>0$$

$$u\to 0,\ \partial u/\partial y\to 0\ \ as\ y\to\infty \tag{5}$$

where $\partial u/\partial y\to 0\ \ as\ \ y\to\infty$ is the augmented condition given by K. R. Rajagopal [14]. Here the flow is caused solely by the stretching of the sheet, since the free stream

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velocity is zero.

In the present study, the thermal transport phenomenon has been investigated for two general boundary conditions, namely, (i) Prescribed surface temperature (PST case) and (ii) Prescribe surface heat flux(PHF case).

The thermal boundary conditions for the equation of energy (3) are:

At y = 0

$$T = T_w(x) + A(x/l)^2 \quad \text{(PST case)} \quad (6a)$$

$$q_w(x) = -k\frac{\partial T}{\partial y} = D(x/l)^2 \quad \text{(PHF case)} \quad (6b)$$

$$T \to T_\infty \quad as \quad y \to \infty$$

where T_w the temperature of the wall is, T_∞ is the temperature outside the dynamic region. A is a constant depends on thermal properties of the liquid, $l = \sqrt{\upsilon/b}$ is chosen as characteristic length, q_w is the wall heat flux, D is a constant and k is the thermal conductivity.

Defining new variables:

$$u = bxf_\eta(\eta),\ v = -\sqrt{b\upsilon}\, f(\eta),\ \eta = \sqrt{b/\upsilon}\, y, \quad (7)$$
$$\theta(\eta) = (T - T_\infty / \Delta T\ )$$

where

$$\Delta T = A(x/l)^2 \quad \text{for PST case} \quad (8a)$$

and

$$\Delta T = (D/k)(x/l)^2\sqrt{\upsilon/b} \quad \text{for PHF case} \quad (8b)$$

Equations (2) and (3) are transformed as

$$f_\eta^2 - ff_{\eta\eta} = f_{\eta\eta\eta} - k_1\{2f_\eta f_{\eta\eta\eta} - ff_{\eta\eta\eta\eta} - f_{\eta\eta}^2\} \quad (9)$$

$$\theta_{\eta\eta} + \Pr f\theta_\eta - (\Pr f_\eta - B^*)\theta + (A^* - \Pr\theta)f_\eta = 0 \quad (10)$$

where $\Pr = (\mu C_p / k)$ is the Prandtl number, and $k_1 = (k_0 b / \upsilon)$ is the viscoelastic parameter.

The boundary conditions (5), (6) reduce to:

$$f(0) = 0,\ f_\eta(0) = 1,\ \theta(0) = 1 \quad \text{for PST case} \quad (11a)$$

$$f(0) = 0,\ f_\eta(0) = 1,\ \theta_\eta(0) = -1 \quad \text{for PHF case} \quad (11b)$$

$$f_\eta(\eta) \to 0,\ f_{\eta\eta}(\eta) \to 0,\ \theta(\eta) \to 0\ as\ \eta \to \infty \quad (12)$$

III. NUMERICAL SOLUTION OF THE PROBLEM

The flow equation (9) coupled with energy equation (10) constitute a set of highly nonlinear differential equations for which obtaining closed form solution is difficult. Hence quasilinearization technique, given by Bellman & Kalaba [15] is used to solve this system. This method converts the nonlinear two-point boundary value problem into an iterative scheme of solution, which involves the step-by-step integration of linearised differential equations, with two point boundary conditions. This method is quadratically convergent, starting from the initial guess value and solution obtained is valid for a large range of parameters. Even when the required number of initial conditions is not given, this method converges at a fast speed. In order to implement the quasilinearization technique, the system of equations (9) and (10) are converted to a system of first order differential equations as follows:

Substitute

$$(f, f', f'', f''', \theta, \theta') = (x_1, x_2, x_3, x_4, x_5, x_6)$$

Then equations (9) and (10) reduce to:

$$\frac{dx_1}{d\eta} = x_2$$
$$\frac{dx_2}{d\eta} = x_3$$
$$\frac{dx_3}{d\eta} = x_4$$
$$\frac{dx_4}{d\eta} = \frac{1}{k_1 x_1}\{x_2^2 - x_1x_3 - x_4 + 2k_1x_2x_4 - k_1x_3^2\} \quad (13)$$
$$\frac{d(x_5)}{d\eta} = x_6$$
$$\frac{dx_6}{d\eta} = (2\Pr x_2 - B^*)x_5 - \Pr x_1x_6 - (E\Pr x_3^2 + A^*x_2)$$

Let x_i^r (i =1,2,…6) be an approximate current solution and x_i^{r+1} (i = 1,2,…6) be an improved solution of (13). By taking tailor's series expansion around the current solution and neglecting the second and higher order derivatives, the coupled first order system (13) is linearized as:

$$\frac{dx_1^{r+1}}{d\eta} = x_2^{r+1}$$
$$\frac{dx_2^{r+1}}{d\eta} = x^{r+1}$$
$$\frac{dx_3^{r+1}}{d\eta} = x_4^{r+1}$$

$$\frac{dx_4^{r+1}}{d\eta} = Px_1^{r+1} + Qx_2^{r+1} + Rx_3^{r+1} + Sx_4^{r+1} + \left(\frac{-x_4^r}{k_1 x_1^r}\right)$$

$$\frac{dx_5^{r+1}}{d\eta} = x_6^{r+1} \tag{14}$$

$$\frac{dx_6^{r+1}}{d\eta} = \left(-\Pr x_6^r\right)x_1^{r+1} + \left(2\Pr x_5^r - A^*\right)x_2^{r+1} - \left(2E\Pr x_3^r\right)x_3^{r+1} + \left(2\Pr x_2^r - B^*\right)x_5^{r+1} - \Pr x_1^r x_6^{r+1} + \left(E\Pr\left(x_3^r\right)^2 - 2\Pr x_2^r x_5^r + \Pr x_1^r x_6^r\right)$$

where

$$P = \left(\frac{-1}{k_1\left(x_1^r\right)^2}\left(\left(x_2^r\right)^2 - x_4^r + 2k_1 x_2^r x_4^r - k_1\left(x_3^r\right)^2\right)\right)$$

$$Q = \left(\frac{1}{k_1 x_1^r}\left(2x_2^r + 2k_1 x_4^r\right)\right)$$

$$R = \left(\frac{1}{k_1 x_1^r}\left(-x_1^r - 2k_1 x_3^r\right)\right)$$

$$S = \left(\frac{1}{k_1 x_1^r}\left(-1 + 2k_1 x_2^r\right)\right)$$

The above system of equations is linear in x_i^{r+1} and general solution can be obtained by using the principle of superposition.

The boundary conditions given by (11) and (12) reduce to

$$x_1^{r+1}(0)=0,\ x_2^{r+1}(0)=1,\ x_5^{r+1}(0)=1 \quad \text{for PST case} \tag{15a}$$

$$x_1^{r+1}(0)=0,\ x_2^{r+1}(0)=1,\ x_6^{r+1}(0)=-1 \quad \text{for PHF case} \tag{15b}$$

$$x_2^{r+1}(\eta)\to 0, x_3^{r+1}(\eta)\to 0, x_5^{r+1}(\eta)\to 0 \quad as\ \eta\to\infty$$

The initial values are chosen as follows:

For the homogeneous solution:

$$x_i^{h_1}(\eta) = [0\ 0\ 1\ 0\ 0\ 0]$$
$$x_i^{h_2}(\eta) = [0\ 0\ 0\ 1\ 0\ 0] \tag{16}$$
$$x_i^{h_3}(\eta) = [0\ 0\ 0\ 0\ 0\ 1]$$

For particular solution:

$$x_i^p(\eta) = [0\ 1\ 0\ 0\ 1\ 0] \tag{17a}$$

(OR)

$$x_i^p(\eta) = [0\ 1\ 0\ 0\ 0\ -1] \tag{17b}$$

The general solution of system of equations is given by

$$x_i^{r+1}(\eta) = C_1 x_i^{h_1}(\eta) + C_2 x_i^{h_2}(\eta) + C_3 x_i^{h_3}(\eta) + x_i^p(\eta) \tag{18}$$

where C_1, C_2, C_3 are the unknown constants and are determined by considering the boundary conditions as $\eta \to \infty$. This solution $\left(x_i^{r+1}, i = 1,2,\dots 6\right)$ is then compared with solution at the previous step $x_i^r, i = 1,2,\dots 6$ and next iteration is performed if the convergence has not been achieved or greater accuracy is desired.

IV. RESULT AND DISCUSSION

Heat transfer in the steady laminar flow of an incompressible viscoelastic fluid over a stretching sheet with prescribed surface temperature and prescribed heat flux, including viscous dissipation and the non-uniform heat source has been examined. Numerical computations of the results are depicted in Fig [1-4].

Fig. 1 depicts temperature profiles θ (η) versus η in PST and PHF cases respectively. It shows that temperature θ (η) increases with the increase in the value of viscoelastic parameter k_1 in both the cases. This is due to the fact that an increase of viscoelastic normal stress gives rise to thickening of thermal boundary layer.

Fig 2(a) and 2(b) reveal that temperature θ (η) decreases with increase in Prandtl number (Pr), which implies viscous boundary layer is thicker than the thermal boundary layer.

Fig 3(a) and 3(b) depict that temperature θ (η) increases with Eckert number. This is due to the fact that heat energy is stored in the liquid due to the frictional heating. The effect of increasing Eckert number is to enhance the temperature at any point.

Figs 4 depicts that temperature θ (η) increases when A* is positive (heat source), since thermal boundary layer generates energy. Temperature θ (η) decreases when A* is negative (absorption). The effect of the parameter B^* on Temperature θ (η) is same as A^*.

From our numerical results for both PST/PHF cases, the following conclusions may be drawn.

Temperature of the fluid

i. increases with increase in viscoelastic parameter k_1,
ii. increases with increase in viscous dissipation (Ec),
iii. increases with increase in non-uniform heat source/sink parameters A* and B*.
iv. decreases with increase in Prandtl number (Pr).

Finally, the values of the wall temperature gradient (-θ' (0)) and the wall temperature g (0) as a function of all the parameters of the thermal boundary layer analyzed and tabulated in Table1. This shows that the effect of viscoelastic parameter (k_1) is to increase the wall temperature gradient in PST case and wall temperature in PST case. Effect of Prandtl number (Pr), Eckert number (E) is to decrease the magnitude of both wall temperature gradient (-θ' (0)) and the wall temperature g (0), whereas opposite behavior is seen with the both the parameters A* and B*.

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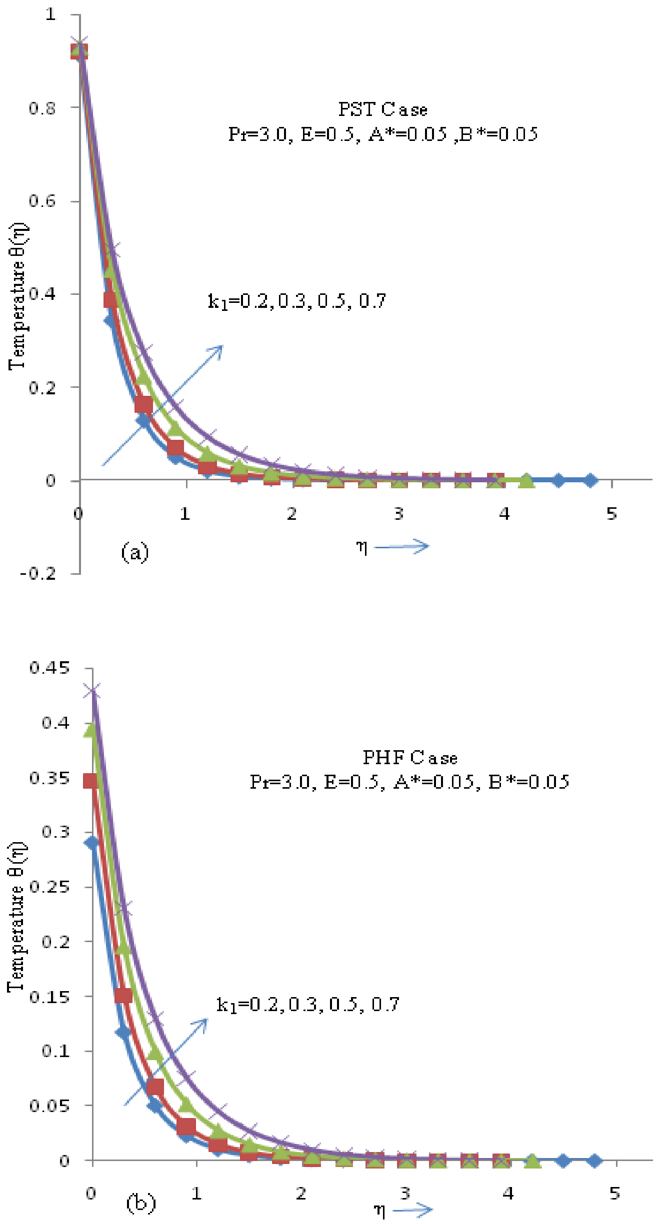

Fig. 1. Effect of visco-elasticity (k_1) on temperature distribution

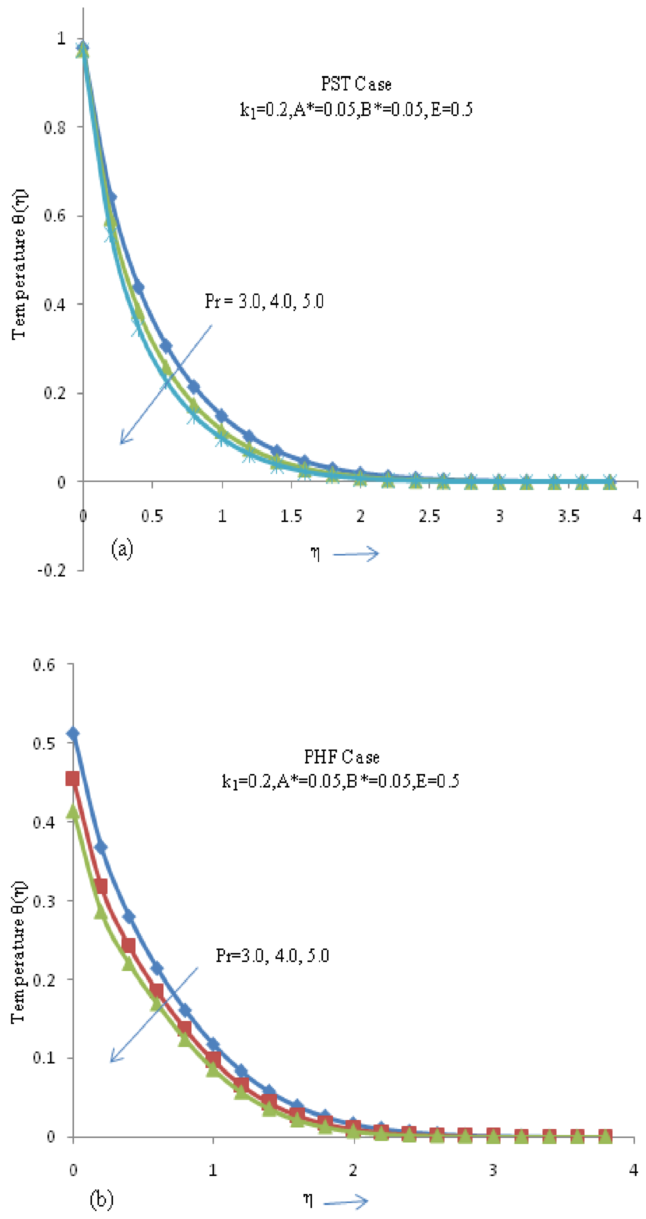

Fig. 2. Effect of Prandtl number (Pr) on temperature distribution.

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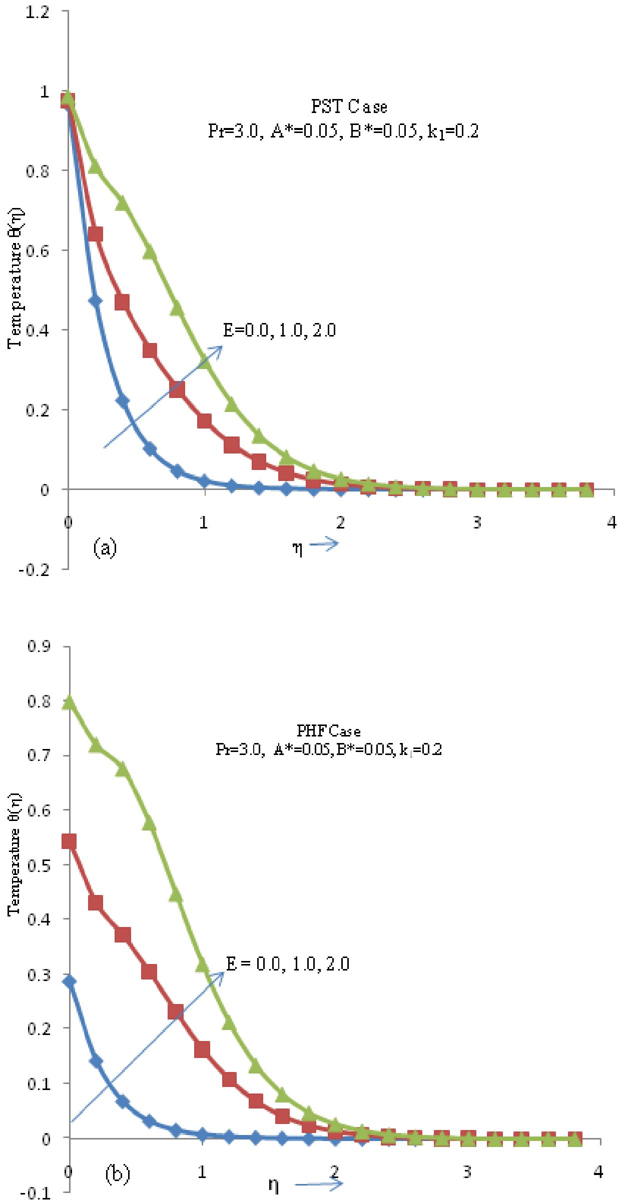

Fig. 3. Effect of Eckert number (E) on temperature distribution.

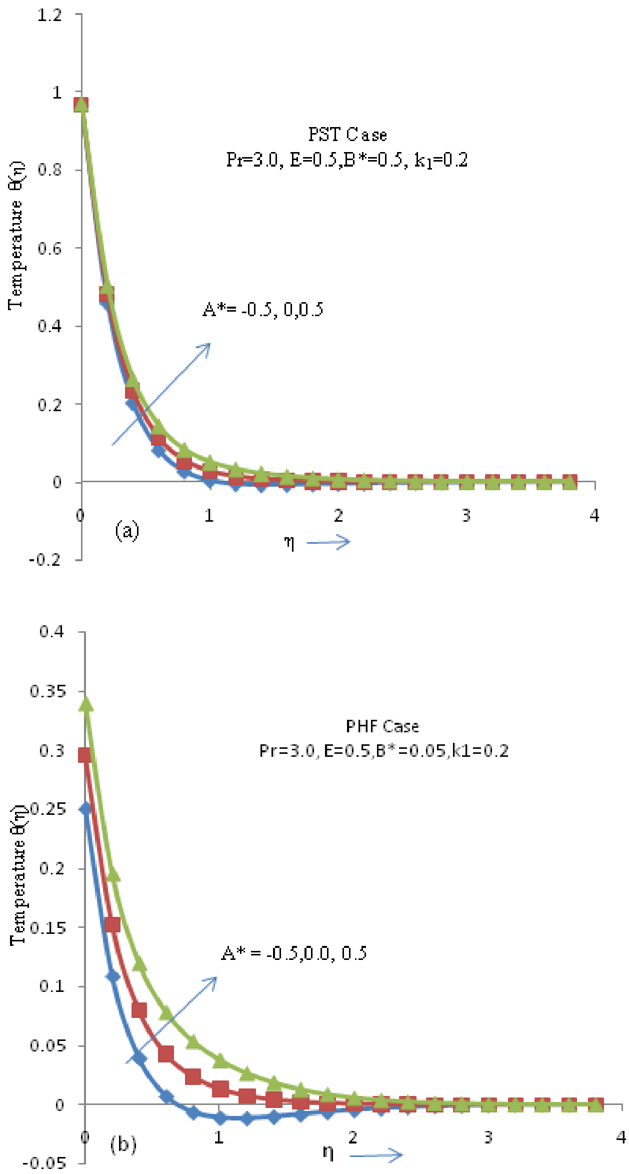

Fig. 4. Effect of non-uniform heat source/sink parameter (A*) on temperature distribution

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TABLE I
HEAT TRANSFER CHARACTERISTICS AT THE WALL

E	K_1	Pr	A^*	B^*	PST case _θ'(0)	PHF case g(0)
0.0	0.2	3.0	0.05	0.05	2.86091	0.30401
0.5					2.75725	0.32600
1.0					2.57130	0.35560
0.5	0.2	3.0	0.05	0.05	2.75752	0.29600
	0.3				2.56567	0.34610
	0.4				2.038502	0.36926
0.5	0.2	3.0	0.05	0.05	2.75752	0.29600
		4.0			3.15299	0.24762
		5.0			3.48365	0.21479
0.5	0.2	3.0	-0.05	0.05	2.79910	0.28153
			0.0		2.77831	0.35877
			0.05		2.75752	0.42600
0.5	0.2	3.0	0.05	-0.05	2.77143	0.29417
				0.0	2.76449	0.29508
				0.05	2.75752	0.29600

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