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Embedding Two Edge-Disjoint Hamiltonian Cycles and Two Equal Node-Disjoint Cycles into Twisted Cubes

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***Abstract*—The presence of edge-disjoint Hamiltonian cycles provides an advantage when implementing algorithms that require a ring structure by allowing message traffic to be spread evenly across the network. Edge-disjoint Hamiltonian cycles also provide the edge-fault tolerant Hamiltonicity of an interconnection network. Two node-disjoint cycles in a network are called equal if the number of nodes in the two cycles are the same and every node appears in one cycle exactly once. The presence of two equal node-disjoint cycles provides algorithms that require a ring structure to be preformed in the network simultaneously. The hypercube is one of the most popular interconnection networks since it has simple structure and is easy to implement. The n-dimensional twisted cube, an important variation of the hypercube, possesses some properties superior to the hypercube. In this paper, we present linear time algorithms to construct two edge-disjoint Hamiltonian cycles and two equal node-disjoint cycles in an n-dimensional twisted cube.**

***Index Terms*—edge-disjoint Hamiltonian cycles, equal node-disjoint cycles, twisted cubes, parallel computing, inductive construction**

I. INTRODUCTION

PARALLEL computing is important for speeding up computation. The topology design of an interconnection network is the first thing to be considered. Many topologies have been proposed in the literature [4], [6], [8], [9], [10], [13], [18], and the desirable properties of an interconnection network include symmetry, relatively small degree, small diameter, embedding capabilities, scalability, robustness, and efficient routing. Among those proposed interconnection networks, the hypercube is a popular interconnection network with many attractive properties such as regularity, symmetry, small diameter, strong connectivity, recursive construction, partition ability, and relatively low link complexity [24]. The topology of an interconnection network is usually modeled by a graph, where nodes represent the processing elements and edges represent the communication links. In this paper, we will use graphs and networks interchangeably.

The n-dimensional twisted cube TQ_n, an important variation of the hypercube, was first proposed by Hilbers et al. [13] and possesses some properties superior to the hypercube. The twisted cube is derived from the hypercube by twisting some edges. Due to these twisted edges, the diameter, wide diameter, and fault diameter of TQ_n are about half of those of the comparable hypercube [5]. An n-dimensional twisted cube is $(n-3)$-Hamiltonian connected [16] and $(n-2)$-pancyclic [22], whereas the hypercube is not. Moreover, its performance is better than that of the hypercube even if it is asymmetric [1]. Recently, some interesting properties, such as conditional link faults, of the twisted cube TQ_n were investigated. Yang et al. [27] showed that, with $n_e+n_v \leqslant n-2$, a faulty TQ_n still contains a cycle of length l for every $4 \leqslant l \leqslant |V(TQ_n)| - n_v$, where n_e and n_v are the numbers of faulty edges and faulty nodes in TQ_n, respectively, and $|V(TQ_n)|$ denotes the number of nodes in TQ_n. In [12], Fu showed that TQ_n can tolerate up to $2n-5$ edge faults, while retaining a fault-free Hamiltonian cycle. Fan et al. [11] showed that the twisted cube TQ_n, with $n \geqslant 3$, is edge-pancyclic and provided an $O(l \log l + n^2 + nl)$-time algorithm to find a cycle of length l containing a given edge of the twisted cube. In [11], the author also asked if TQ_n is edge-pancyclic with $(n-3)$ faults for $n \geqslant 3$. Yang [28] answered the question and showed that TQ_n is not edge-pancyclic with only one faulty edge for any $n \geqslant 3$, and that TQ_n is node-pancyclic with $(\lfloor \frac{n}{2} \rfloor - 1)$ faulty edges for every $n \geqslant 3$. Lai et al. [20] embedded a family of 2-dimensional meshes into a twisted cube.

A *Hamiltonian cycle* in a graph is a simple cycle that passes through every node of the graph exactly once. The ring structure is important for distributed computing, and its benefits can be found in [19]. Two Hamiltonian cycles in a graph are said to be *edge-disjoint* if there exists no common edge in them. The edge-disjoint Hamiltonian cycles can provide an advantage for algorithms that make use of a ring structure [25]. Consider the problem of all-to-all broadcasting in which each node sends an identical message to all other nodes in the network. There is a simple solution for the problem using an n-node ring that requires $n-1$ steps, i.e., at each step, every node receives a new message from its ring predecessor and passes the previous message to its ring successor. If the network admits edge-disjoint rings, then messages can be divided and the parts broadcast along different rings without any edge (link) contention. If the network can be decomposed into edge-disjoint Hamiltonian cycles, then the message traffic will be evenly distributed across all communication links. Edge-disjoint Hamiltonian cycles also form the basis of an efficient all-to-all broadcasting algorithm for networks that employ wormhole or cut-through routing [21]. Further, edge-disjoint Hamiltonian cycles also provide the edge-fault tolerant Hamiltonicity of an interconnected network; that is, when a Hamiltonian cycle of an interconnected network contains one faulty edge, then the other edge-disjoint Hamiltonian cycle can be used to replace it for transmission. The existence of a Hamiltonian cycle in twisted cubes has been verified [16]. However, there has been little work reported so far on edge-disjoint

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properties in the twisted cubes. In this paper, we show that, for any odd integer $n \geqslant 5$, there are two edge-disjoint Hamiltonian cycles in the n-dimensional twisted cube TQ_n.

Two cycles in a graph are said to be *equal* and *node-disjoint* if they contain the same number of nodes, there is no common node in them, and every node of the graph appears in one cycle exactly once. Finding two equal node-disjoint cycles in an interconnected network is equivalent to decompose the network into two disjoint sub-networks with the same number of nodes such that each sub-network contains a Hamiltonian cycle. Then, algorithms that require a ring structure can be preformed in the two sub-networks simultaneously. In this paper, we show that, for any odd integer $n \geqslant 3$, there exist two equal node-disjoint cycles in the n-dimensional twisted cube TQ_n.

Related areas of investigation are summarized as follows. The edge-disjoint Hamiltonian cycles in k-ary n-cubes and hypercubes has been constructed in [2]. Barth et al. [3] showed that the butterfly network contains two edge-disjoint Hamiltonian cycles. Petrovic et al. [23] characterized the number of edge-disjoint Hamiltonian cycles in hyper-tournaments. Hsieh et al. [14] constructed edge-disjoint spanning trees in locally twisted cubes. Hsieh et al. [15] investigated the edge-fault tolerant Hamiltonicity of an n-dimensional locally twisted cube. The existence of a Hamiltonian cycle in locally twisted cubes and twisted cubes has been shown in [26] and [16], respectively. However, there has been little work reported so far on edge-disjoint properties in locally twisted cubes and twisted cubes. In [17], we presented a linear time algorithm to construct two edge-disjoint Hamiltonian cycles in locally twisted cubes. In this paper, we show that there exist two edge-disjoint Hamiltonian cycles and two equal node-disjoint cycles in an n-dimensional twisted cube TQ_n. Note that for any TQ_n, n is always an odd integer.

The rest of the paper is organized as follows. In Section II, the structure of the twisted cube is introduced, and some definitions and notations used throughout this paper are given. Section III shows the construction of two edge-disjoint Hamiltonian cycles in the twisted cube. In Section IV, we construct two equal node-disjoint cycles in the twisted cube. Finally, we conclude this paper in Section V.

II. Preliminaries

We usually use a graph to represent the topology of an interconnection network. A graph $G = (V, E)$ is a pair of the node set V and the edge set E, where V is a finite set and E is a subset of $\{(u, v)|(u, v)$ is an unordered pair of $V\}$. We will use $V(G)$ and $E(G)$ to denote the node set and the edge set of G, respectively. If (u, v) is an edge in a graph G, we say that u *is adjacent to* v and u, v *are incident to* edge (u, v). A *neighbor* of a node v in a graph G is any node that is adjacent to v. Moreover, we use $N_G(v)$ to denote the set of neighbors of v in G. The subscript 'G' of $N_G(v)$ can be removed from the notation if it has no ambiguity.

Let $G = (V, E)$ be a graph with node set V and edge set E. A (simple) path P of length ℓ in G, denoted by $v_0 \to v_1 \to \cdots \to v_{\ell-1} \to v_\ell$, is a sequence $(v_0, v_1, \cdots, v_{\ell-1}, v_\ell)$ of nodes such that $(v_i, v_{i+1}) \in E$ for $0 \leqslant i \leqslant \ell - 1$. The first node v_0 and the last node v_ℓ visited by P are denoted by $start(P)$ and $end(P)$, respectively. Path $v_\ell \to v_{\ell-1} \to \cdots \to v_1 \to v_0$ is called the *reversed path*, denoted by P_{rev}, of path P. That is, path P_{rev} visits the nodes of path P from $end(P)$ to $start(P)$ sequentially. In addition, P is a cycle if $|V(P)| \geqslant 3$ and $end(P)$ is adjacent to $start(P)$. A path $P = v_0 \to v_1 \to \cdots \to v_{\ell-1} \to v_\ell$ may contain another subpath Q, denoted as $v_0 \to v_1 \to \cdots \to v_{i-1} \to Q \to v_{j+1} \to \cdots \to v_{\ell-1} \to v_\ell$, where $Q = v_i \to v_{i+1} \to \cdots \to v_j$ for $0 \leqslant i \leqslant j \leqslant \ell$. A path (or cycle) in G is called a *Hamiltonian path* (or *Hamiltonian cycle*) if it contains every node of G exactly once. Two paths (or cycles) P_1 and P_2 connecting a node u to a node v are said to be *edge-disjoint* if and only if $E(P_1) \cap E(P_2) = \emptyset$. Two paths (or cycles) Q_1 and Q_2 of graph G are called *node-disjoint* if and only if $V(Q_1) \cap V(Q_2) = \emptyset$. Two node-disjoint paths (or cycles) Q_1 and Q_2 of graph G are said to be *equal* if and only if $|V(Q_1)| = |V(Q_2)|$ and $V(Q_1) \cup V(Q_2) = V(G)$. Two node-disjoint paths Q_1 and Q_2 can be *concatenated* into a path, denoted by $Q_1 \Rightarrow Q_2$, if $end(Q_1)$ is adjacent to $start(Q_2)$.

Now, we introduce twisted cubes. The node set of the n-dimensional twisted cube TQ_n is the set of all binary strings of length n. Note that due to the twisted edge property of a twisted cube, the dimension n of TQ_n is always defined as an odd integer. A binary string b of length n is denoted by $b_{n-1}b_{n-2}\cdots b_1b_0$, where b_{n-1} is the most significant bit. We denote the complement of bit b_i by $\overline{b}_i = 1 - b_i$. To define TQ_n, a i-th bit *parity function* $\mathcal{P}_i(b)$ is introduced. Let $b = b_{n-1}b_{n-2}\cdots b_1b_0$ be a binary string. For $0 \leqslant i \leqslant n-1$, $\mathcal{P}_i(b) = b_i \oplus b_{i-1} \oplus \cdots \oplus b_1 \oplus b_0$, where $\oplus$ is the exclusive-or operation. We then give the recursive definition of the n-dimensional twisted cube TQ_n, for any odd integer $n \geqslant 1$, as follows.

Definition 1. [13], [28] TQ_1 is the complete graph with two nodes labeled by 0 and 1, respectively. For an odd integer $n \geqslant 3$, TQ_n consists of four copies of TQ_{n-2}. We use TQ_{n-2}^{ij} to denote an $(n-2)$-dimensional twisted cube which is a subgraph of TQ_n induced by the nodes labeled by $ijb_{n-3}\cdots b_1b_0$, where $i, j \in \{0, 1\}$. Edges that connect these four sub-twisted cubes can be described as follows: Each node $b = b_{n-1}b_{n-2}\cdots b_1b_0 \in V(TQ_n)$ is adjacent to $\overline{b}_{n-1}b_{n-2}\cdots b_1b_0$ and $\overline{b}_{n-1}\overline{b}_{n-2}\cdots b_1b_0$ if $\mathcal{P}_{n-3}(b) = 0$; and to $\overline{b}_{n-1}b_{n-2}\cdots b_1b_0$ and $b_{n-1}\overline{b}_{n-2}\cdots b_1b_0$ if $\mathcal{P}_{n-3}(b) = 1$.

By the above definition, an n-dimensional twisted cube TQ_n is an n-regular graph with 2^n nodes and $n2^{n-1}$ edges, i.e., each node of TQ_n is adjacent to n nodes. The dimension n of TQ_n is always an odd integer. In addition, TQ_n can be decomposed into four sub-twisted cubes TQ_{n-2}^{00}, TQ_{n-2}^{10}, TQ_{n-2}^{01}, TQ_{n-2}^{11}, where TQ_{n-2}^{ij} consists of those nodes $b = b_{n-1}b_{n-2}\cdots b_1b_0$ with leading two bits $b_{n-1} = i$ and $b_{n-2} = j$. For each $ij \in \{00, 10, 01, 11\}$, TQ_{n-2}^{ij} is isomorphic to TQ_{n-2}. For example, Fig. 1 shows TQ_3 and Fig. 2 depicts TQ_5 containing four sub-twisted cubes TQ_3^{00}, TQ_3^{10}, TQ_3^{01}, TQ_3^{11}.

Let b be a binary string $b_{t-1}b_{t-2}\cdots b_1b_0$ of length t. We denote b^τ the new binary string obtained by repeating b string τ times. For instance, $(01)^3 = 010101$ and $0^4 = 0000$.

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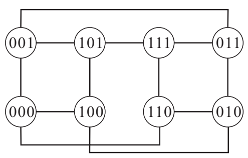

Fig. 1. The 3-dimensional twisted cube TQ_3

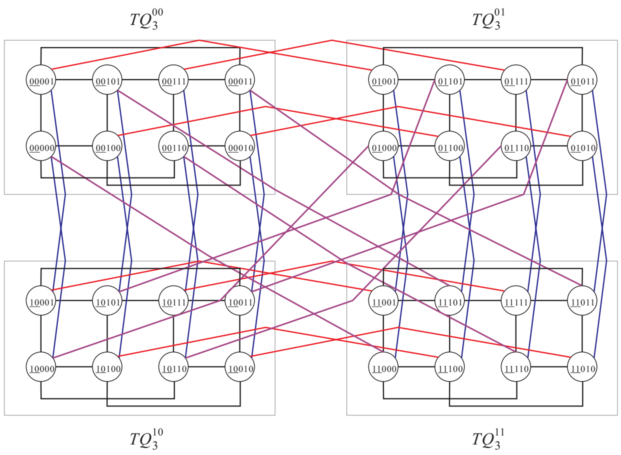

Fig. 2. The 5-dimensional twisted cube TQ_5 containing TQ_3^{00}, TQ_3^{10}, TQ_3^{01}, TQ_3^{11}, where the leading two bits of nodes are underlined

III. Two Edge-Disjoint Hamiltonian Cycles

In this section, we first show the existence of two edge-disjoint Hamiltonian cycles of an n-dimensional twisted cube TQ_n with odd integer $n \geqslant 5$. We then present a linear time algorithm to construct two such edge-disjoint Hamiltonian cycles of TQ_n. Obviously, a 3-dimensional twisted cube TQ_3 contains no two edge-disjoint Hamiltonian cycles since each node is incident to three edges. Note that the dimension of TQ_n is always odd. We will prove the existence of two edge-disjoint Hamiltonian cycles in TQ_n, with $n \geqslant 5$, by induction on n, the dimension of the twisted cube. For any odd integer $n \geqslant 5$, we show that there exist two edge-disjoint Hamiltonian paths P and Q in TQ_n such that $start(P) = 00(0)^{n-5}000$, $end(P) = 11(0)^{n-5}000$, $start(Q) = 00(0)^{n-5}100$, and $end(Q) = 01(0)^{n-5}100$. By the definition of parity function $\mathcal{P}_i(\cdot)$, $\mathcal{P}_{n-3}(end(P)) = \mathcal{P}_{n-3}(11(0)^{n-5}000) = 0$ and $\mathcal{P}_{n-3}(end(Q)) = \mathcal{P}_{n-3}(01(0)^{n-5}100) = 1$. By Definition 1, $start(P) \in N(end(P))$ and $start(Q) \in N(end(Q))$. Thus, P and Q are two edge-disjoint Hamiltonian cycles in TQ_n for any odd integer $n \geqslant 5$. In the following, we will show how to construct two such edge-disjoint Hamiltonian cycles. We first show that TQ_5 contains two edge-disjoint Hamiltonian paths as follows.

Lemma 1. *There are two edge-disjoint Hamiltonian paths P and Q in TQ_5 such that $start(P) = 00000$, $end(P) = 11000$, $start(Q) = 00100$, and $end(Q) = 01100$.*

Proof: We prove this lemma by constructing two such paths P and Q. Let
$P = 00000 \to 00001 \to 00101 \to 00100 \to 10100 \to 10101 \to 10001 \to 10000 \to 10110 \to 10010 \to 00010 \to 00011 \to 10011 \to 10111 \to 00111 \to 00110 \to 11110 \to 11010 \to 01010 \to 01011 \to 11011 \to 11111 \to 01111 \to 01110 \to 01000 \to 01001 \to 01101 \to 01100 \to 11100 \to 11101 \to 11001 \to 11000$, and let
$Q = 00100 \to 00000 \to 10000 \to 10100 \to 10010 \to 10011 \to 10001 \to 00001 \to 00011 \to 00111 \to 00101 \to 10101 \to 10111 \to 10110 \to 00110 \to 00010 \to 01010 \to 01110 \to 11110 \to 11111 \to 11101 \to 01101 \to 01111 \to 01011 \to 01001 \to 11001 \to 11011 \to 11010 \to 11100 \to 11000 \to 01000 \to 01100$.

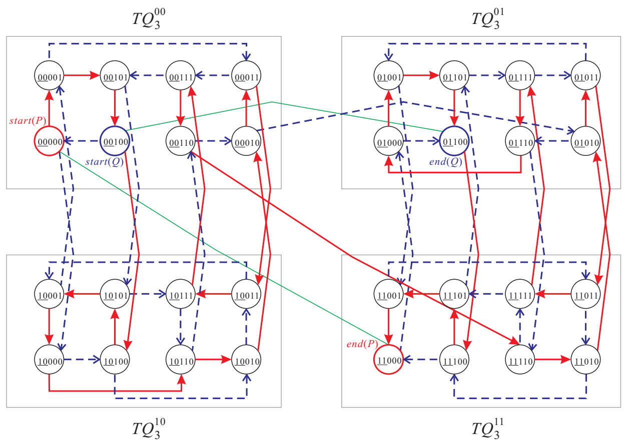

Fig. 3. Two edge-disjoint Hamiltonian paths in TQ_5, where solid arrow lines indicate a Hamiltonian path P, dashed arrow lines indicate the other edge-disjoint Hamiltonian path Q, and the leading two bits of nodes are underlined

Fig. 3 depicts the construction of P and Q. Clearly, P and Q form two edge-disjoint Hamiltonian paths in TQ_5. ■

Using Lemma 1, we prove the following lemma by induction.

Lemma 2. *For any odd integer $n \geqslant 5$, there are two edge-disjoint Hamiltonian paths P and Q in TQ_n such that $start(P) = 00(0)^{n-5}000$, $end(P) = 11(0)^{n-5}000$, $start(Q) = 00(0)^{n-5}100$, and $end(Q) = 01(0)^{n-5}100$.*

Proof: We prove this lemma by induction on n, the dimension of the twisted cube. By Lemma 1, the lemma holds true when $n = 5$. Assume that the lemma is true for any odd integer $n = k \geqslant 5$. We will prove that the lemma holds true for $n = k + 2$. We first partition TQ_{k+2} into four sub-twisted cubes TQ_k^{00}, TQ_k^{10}, TQ_k^{01}, TQ_k^{11}. By the induction hypothesis, there are two edge-disjoint Hamiltonian paths P^{ij} and Q^{ij} in TQ_k^{ij} for $i, j \in \{0, 1\}$ such that $start(P^{ij}) = ij00(0)^{k-5}000$, $end(P^{ij}) = ij11(0)^{k-5}000$, $start(Q^{ij}) = ij00(0)^{k-5}100$, and $end(Q^{ij}) = ij01(0)^{k-5}100$. By the definition of parity function $\mathcal{P}_i(\cdot)$, $\mathcal{P}_{k-1}(end(P^{ij})) = \mathcal{P}_{k-1}(start(P^{ij})) = 0$, $\mathcal{P}_{k-1}(end(Q^{ij})) = 0$, and $\mathcal{P}_{k-1}(start(Q^{ij})) = 1$. By Definition 1, we have that $end(P^{00}) \in N(end(P^{10}))$, $start(P^{10}) \in N(start(P^{01}))$, $end(P^{01}) \in N(end(P^{11}))$, $end(Q^{00}) \in N(end(Q^{10}))$, $start(Q^{10}) \in N(start(Q^{11}))$, and $end(Q^{11}) \in N(end(Q^{01}))$.

Let $P = P^{00} \Rightarrow P^{10}_{\text{rev}} \Rightarrow P^{01} \Rightarrow P^{11}_{\text{rev}}$ and let $Q = Q^{00} \Rightarrow Q^{10}_{\text{rev}} \Rightarrow Q^{11} \Rightarrow Q^{01}_{\text{rev}}$, where P^{10}_{rev}, P^{11}_{rev}, Q^{10}_{rev}, and Q^{01}_{rev} are the reversed paths of P^{10}, P^{11}, Q^{10}, and Q^{01}, respectively. Then, P and Q are two edge-disjoint Hamiltonian paths in TQ_{k+2} such that $start(P) = 00(0)^{k-3}000$, $end(P) = 11(0)^{k-3}000$, $start(Q) = 00(0)^{k-3}100$, and

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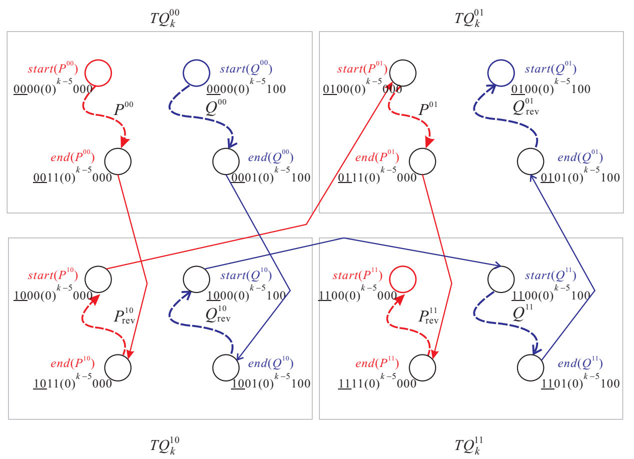

Fig. 4. The construction of two edge-disjoint Hamiltonian paths in TQ_{k+2}, with odd integer $k \geqslant 5$, where dashed arrow lines indicate the paths, solid arrow lines indicate concatenated edges, and the leading two bits of nodes are underlined

$end(Q) = 01(0)^{k-3}100$. Fig. 4 depicts the construction of two such edge-disjoint Hamiltonian paths in TQ_{k+2}. Thus, the lemma hods true when $n = k + 2$. By induction, the lemma holds true. ■

Let P and Q be two edge-disjoint Hamiltonian paths constructed in Lemma 2. By the definition of parity function, $\mathcal{P}_{n-3}(end(P)) = \mathcal{P}_{n-3}(11(0)^{n-5}000) = 0$ and $\mathcal{P}_{n-3}(end(Q)) = \mathcal{P}_{n-3}(01(0)^{n-5}100) = 1$. By Definition 1, node $start(P)$ is adjacent to node $end(P)$, and node $start(Q)$ is adjacent to node $end(Q)$. In addition, the two edges $(start(P), end(P))$ and $(start(Q), end(Q))$ are distinct. Thus the following theorem holds true.

Theorem 3. *For any odd integer $n \geqslant 5$, there exist two edge-disjoint Hamiltonian paths (cycles) in an n-dimensional twisted cube TQ_n .*

Based on the proofs of Lemmas 1 and 2, we design a recursive algorithm to construct two edge-disjoint Hamiltonian paths of an n-dimensional twisted cube. The algorithm typically follows a divide-and-conquer approach [7] and is sketched as follows. It is given by an n-dimensional twisted cube TQ_n with odd integer $n \geqslant 5$. If $n = 5$, then the algorithm constructs two edge-disjoint Hamiltonian paths according to the proof of Lemma 1. Suppose that $n > 5$. It first decomposes TQ_n into four sub-twisted cubes TQ^{00}_{n-2}, TQ^{10}_{n-2}, TQ^{01}_{n-2}, and TQ^{11}_{n-2}, where TQ^{ij}_{n-2} consists of those nodes $b = b_{n-1}b_{n-2}b_{n-3}\cdots b_1b_0$ with leading two bits $b_{n-1} = i$ and $b_{n-2} = j$. For each $ij \in \{00, 10, 01, 11\}$, TQ^{ij}_{n-2} is isomorphic to TQ_{n-2}. Then, the algorithm recursively computes two edge-disjoint Hamiltonian paths of TQ^{ij}_{n-2} for $ij \in \{00, 10, 01, 11\}$. Finally, it concatenates these eight paths into two edge-disjoint Hamiltonian paths of TQ_n according to the proof of Lemma 2, and outputs two such concatenated paths. The algorithm is formally presented as follows.

Algorithm CONSTRUCTING-2EDHP-TQ
Input: TQ_n, an n-dimensional twisted cube with odd integer $n \geqslant 5$.
Output: Two edge-disjoint Hamiltonian paths P and Q in TQ_n such that $start(P) = 00(0)^{n-5}000$, $end(P) = 11(0)^{n-5}000$, $start(Q) = 00(0)^{n-5}100$, and $end(Q) = 01(0)^{n-5}100$.
Method:

1. **if** $n = 5$, **then**
2. let $P = 00000 \to 00001 \to 00101 \to 00100 \to 10100 \to 10101 \to 10001 \to 10000 \to 10110 \to 10010 \to 00010 \to 00011 \to 10011 \to 10111 \to 00111 \to 00110 \to 11110 \to 11010 \to 01010 \to 01011 \to 11011 \to 11111 \to 01111 \to 01110 \to 01000 \to 01001 \to 01101 \to 01100 \to 11100 \to 11101 \to 11001 \to 11000$;
3. let $Q = 00100 \to 00000 \to 10000 \to 10100 \to 10010 \to 10011 \to 10001 \to 00001 \to 00011 \to 00111 \to 00101 \to 10101 \to 10111 \to 10110 \to 00110 \to 00010 \to 01010 \to 01110 \to 11110 \to 11111 \to 11101 \to 01101 \to 01111 \to 01011 \to 01001 \to 11001 \to 11011 \to 11010 \to 11100 \to 11000 \to 01000 \to 01100$;
4. **output** "P and Q" as two edge-disjoint Hamiltonian paths of TQ_5;
5. decompose TQ_n into four sub-twisted cubes TQ^{00}_{n-2}, TQ^{10}_{n-2}, TQ^{01}_{n-2}, and TQ^{11}_{n-2}, where TQ^{ij}_{n-2} consists of those nodes $b = b_{n-1}b_{n-2}b_{n-3}\cdots b_1b_0$ with leading two bits $b_{n-1} = i$ and $b_{n-2} = j$
6. **for** $ij \in \{00, 10, 01, 11\}$ **do**
7. call Algorithm CONSTRUCTING-2EDHP-TQ given TQ^{ij}_{n-2} to compute two edge-disjoint Hamiltonian paths P^{ij} and Q^{ij} of TQ^{ij}_{n-2}, where $start(P^{ij}) = ij00(0)^{n-7}000$, $end(P^{ij}) = ij11(0)^{n-7}000$, $start(Q^{ij}) = ij00(0)^{n-7}100$, and $end(Q^{ij}) = ij01(0)^{n-7}100$;
8. compute $P = P^{00} \Rightarrow P^{10}_{\text{rev}} \Rightarrow P^{01} \Rightarrow P^{11}_{\text{rev}}$ and $Q = Q^{00} \Rightarrow Q^{10}_{\text{rev}} \Rightarrow Q^{11} \Rightarrow Q^{01}_{\text{rev}}$, where P^{10}_{rev}, P^{11}_{rev}, Q^{10}_{rev}, and Q^{01}_{rev} are the reversed paths of P^{10}, P^{11}, Q^{10}, and Q^{01}, respectively;
9. **output** "P and Q" as two edge-disjoint Hamiltonian paths of TQ_n.

The correctness of Algorithm CONSTRUCTING-2EDHP-TQ follows from Lemmas 1 and 2. Now, we analyze its time complexity. Let m be the number of nodes in TQ_n. Then, $m = 2^n$. Let $T_Q(m)$ be the running time of Algorithm CONSTRUCTING-2EDHP-TQ given TQ_n. It is easy to verify from lines 2 and 3 that $T_Q(m) = O(1)$ if $n = 5$. Suppose that $n > 5$. By visiting every node of TQ_n once, decomposing TQ_n into TQ^{00}_{n-2}, TQ^{10}_{n-2}, TQ^{01}_{n-2} and TQ^{11}_{n-2} can be done in $O(m)$ time, where each node in TQ^{ij}_{n-2}, $ij \in \{00, 10, 01, 11\}$, is labeled with leading two bits ij. Thus, line 5 of the algorithm runs in $O(m)$ time. Then, our division of the problem yields four subproblems, each of which is $1/4$ the size of the original. It takes time $T_Q(\frac{m}{4})$ to solve one subproblem, and so it takes time $4 \cdot T_Q(\frac{m}{4})$ to solve the four subproblems. In addition, concatenating eight paths into two paths (line 8) can be easily done in $O(m)$ time. Thus, we get the following recurrence equation:

$$T_Q(m) = \begin{cases} O(1) & \text{, if } n = 5; \\ 4 \cdot T_Q(\frac{m}{4}) + O(m) & \text{, if } n > 5. \end{cases}$$

The solution of the above recurrence is $T_Q(m) =$

$O(m \log m) = O(n2^n)$. Thus, the running time of Algorithm CONSTRUCTING-2EDHP-TQ given TQ_n is $O(n2^n)$. Since an n-dimensional twisted cube TQ_n contains 2^n nodes and $n2^{n-1}$ edges, the algorithm is a linear time algorithm. Let P and Q be two edge-disjoint Hamiltonian paths output by Algorithm CONSTRUCTING-2EDHP-TQ given TQ_n. By Definition 1, $start(P) \in N(end(P))$ and $start(Q) \in N(end(Q))$. In addition, the edge connecting $start(P)$ with $end(P)$ is different from the edge connecting $start(Q)$ with $end(Q)$. Thus, P and Q are two edge-disjoint Hamiltonian cycles of TQ_n. We then conclude the following theorem.

Theorem 4. *Algorithm* CONSTRUCTING-2EDHP-TQ *correctly constructs two edge-disjoint Hamiltonian cycles (paths) of an n-dimensional twisted cube TQ_n, with odd integer $n \geqslant 5$, in $O(n2^n)$-linear time.*

IV. TWO EQUAL NODE-DISJOINT CYCLES

In this section, we will construct two equal node-disjoint cycles P and Q in a n-dimensional twisted cube TQ_n, for any odd integer $n \geqslant 3$. Our method for constructing two equal node-disjoint cycles of TQ_n is also based on an inductive construction. For any odd integer $n \geqslant 3$, we will construct two equal node-disjoint paths P and Q in TQ_n such that $start(P) = 00(0)^{n-3}1$, $end(P) = 01(0)^{n-3}1$, $start(Q) = 00(0)^{n-3}0$, and $end(Q) = 11(0)^{n-3}0$. The basic idea is similar to that of constructing two edge-disjoint Hamiltonian paths and is described as follows. Initially, we construct two equal node-disjoint paths P and Q in TQ_3 such that $start(P) = 001$, $end(P) = 011$, $start(Q) = 000$, and $end(Q) = 110$. By Definition 1, P and Q are also node-disjoint cycles with the same length. Consider that n is an odd integer with $n \geqslant 5$. We first partition TQ_n into four subtwisted cubes $TQ^{00}_{n-2}, TQ^{10}_{n-2}, TQ^{01}_{n-2}, TQ^{11}_{n-2}$. Assume that P^{ij} and Q^{ij} are two equal node-disjoint paths in TQ^{ij}_{n-2}, for $i, j \in \{0, 1\}$, such that $start(P^{ij}) = ij00(0)^{n-5}1$, $end(P^{ij}) = ij01(0)^{n-5}1$, $start(Q^{ij}) = ij00(0)^{n-5}0$, and $end(Q^{ij}) = ij11(0)^{n-5}0$. We then concatenate them into two equal node-disjoint paths P and Q of TQ_n such that $start(P) = 00(0)^{n-3}1$, $end(P) = 01(0)^{n-3}1$, $start(Q) = 00(0)^{n-3}0$, and $end(Q) = 11(0)^{n-3}0$. By Definition 1, P and Q are also two equal node-disjoint cycles of TQ_n since $start(P) \in N(end(P))$ and $start(Q) \in N(end(Q))$. The concatenating process will be presented in Lemma 6.

For TQ_3, let $P = 001 \rightarrow 101 \rightarrow 111 \rightarrow 011$ and let $Q = 000 \rightarrow 100 \rightarrow 010 \rightarrow 110$. Then, P and Q are two equal node-disjoint paths in TQ_3. By Definition 1, $start(P) \in N(end(P))$ and $start(Q) \in N(end(Q))$. Thus, the following lemma holds true.

Lemma 5. *There are two equal node-disjoint paths (cycles) P and Q in TQ_3 such that $start(P) = 001$, $end(P) = 011$, $start(Q) = 000$, and $end(Q) = 110$.*

Based on Lemma 5, we prove the following lemma.

Lemma 6. *For any odd integer $n \geqslant 3$, there exist two equal node-disjoint paths P and Q in TQ_n such that $start(P) = 00(0)^{n-3}1$, $end(P) = 01(0)^{n-3}1$, $start(Q) = 00(0)^{n-3}0$, and $end(Q) = 11(0)^{n-3}0$.*

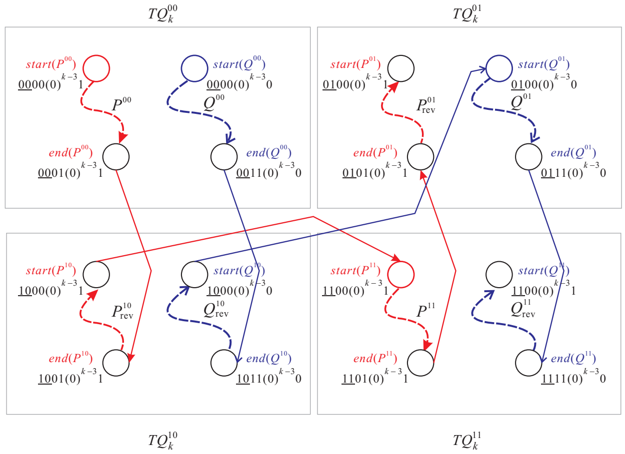

Fig. 5. The constructions of two equal node-disjoint paths in TQ_{k+2}, with $k \geqslant 3$, where dashed arrow lines indicate the paths and solid arrow lines indicate concatenated edges

Proof: We prove this lemma by induction on n, the dimension of TQ_n. By Lemma 5, the lemma holds true when $n = 3$. Assume that the lemma holds when $n = k \geqslant 3$. We will prove that the lemma holds true for $n = k + 2$. We first partition TQ_{k+2} into four subtwisted cubes TQ^{00}_k, TQ^{10}_k, TQ^{01}_k, TQ^{11}_k. By the induction hypothesis, there are two equal node-disjoint paths P^{ij} and Q^{ij}, for $i, j \in \{0, 1\}$, in TQ^{ij}_k such that $start(P^{ij}) = ij00(0)^{k-3}1$, $end(P^{ij}) = ij01(0)^{k-3}1$, $start(Q^{ij}) = ij00(0)^{k-3}0$, and $end(Q^{ij}) = ij11(0)^{k-3}0$. By the definition of parity function $\mathcal{P}_i(\cdot)$, $\mathcal{P}_{k-1}(end(P^{ij})) = 0$, $\mathcal{P}_{k-1}(start(P^{ij})) = 1$, and $\mathcal{P}_{k-1}(end(Q^{ij})) = \mathcal{P}_{k-1}(start(Q^{ij})) = 0$. According to Definition 1, we have that $end(P^{00}) \in N(end(P^{10}))$, $start(P^{10}) \in N(start(P^{11}))$, $end(P^{11}) \in N(end(P^{01}))$, $end(Q^{00}) \in N(end(Q^{10}))$, $start(Q^{10}) \in N(start(Q^{01}))$, and $end(Q^{01}) \in N(end(Q^{11}))$.

Let $P = P^{00} \Rightarrow P^{10}_{\text{rev}} \Rightarrow P^{11} \Rightarrow P^{01}_{\text{rev}}$ and let $Q = Q^{00} \Rightarrow Q^{10}_{\text{rev}} \Rightarrow Q^{01} \Rightarrow Q^{11}_{\text{rev}}$, where P^{10}_{rev}, P^{01}_{rev}, Q^{10}_{rev}, and Q^{11}_{rev} are the reversed paths of P^{10}, P^{01}, Q^{10}, and Q^{11}, respectively. Then, P and Q are two equal node-disjoint paths in TQ_{k+2} such that $start(P) = 00(0)^{k-1}1$, $end(P) = 01(0)^{k-1}1$, $start(Q) = 00(0)^{k-1}0$, and $end(Q) = 11(0)^{k-1}0$. Fig. 5 depicts the constructions of two such equal node-disjoint paths P and Q in TQ_{k+2}. Thus, the lemma hods true when $n = k + 2$. By induction, the lemma holds true. ■

By Definition 1, nodes $start(P) = 00(0)^{n-3}1$ and $end(P) = 01(0)^{n-3}1$ are adjacent, and nodes $start(Q) = 00(0)^{n-3}0$ and $end(Q) = 11(0)^{n-3}0$ are adjacent. It immediately follows from Lemma 6 that the following corollary holds true.

Corollary 7. *For any odd integer $n \geqslant 3$, there exist two equal node-disjoint cycles in TQ_n.*

By the same arguments and analysis in constructing two edge-disjoint Hamiltonian cycles of TQ_n, we can easily construct two equal node-disjoint cycles of TQ_n in linear time. We then conclude the following theorem.

Theorem 8. *There exists an algorithm such that it correctly constructs two equal node-disjoint cycles (paths) of an n-dimensional twisted cube TQ_n, with odd integer $n \geqslant 3$, in $O(n2^n)$-linear time.*

V. Concluding Remarks

In this paper, we construct two edge-disjoint Hamiltonian cycles (paths) of a n-dimensional twisted cubes TQ_n, for any odd integer $n \geqslant 5$. Furthermore, we construct two equal node-disjoint cycles (paths) of TQ_n, for any odd integer $n \geqslant 3$. Note that due to the twisted edge property of a twisted cube, the dimension n of TQ_n is always an odd integer. In the construction of two edge-disjoint Hamiltonian cycles (paths) of TQ_n, some edges are not used. It is interesting to see if there are more edge-disjoint Hamiltonian cycles of TQ_n with odd dimension $n \geqslant 7$. We would like to post this as an open problem to interested readers.

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