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On the Optimality of Plug-In Optimal Control Systems

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Abstract—**We consider a plug-in control system via a concept of Implicit and Explicit Controls as the two stage controllers. We first show two conditions under which the plug-in control system can be optimized by a suitable design of explicit control in the 2nd stage, when an implicit control is given in the 1st stage. We then show another condition under which the resultant optimal cost can be minimized by a suitable design of the 1st stage implicit control, which is characterized as a feedback control that allocates all the closed-loop poles onto the imaginary axis. These two results clarify the system theoretic meaning of the implicit control from the viewpoint of inverse optimal control problem.**

Index Terms—**A plug-in control system, implicit and explicit controls, optimal control, inverse optimal control problem**

I. Introduction

A concept of "Implicit Control" and "Explicit Control" has been proposed recently in [1] by the second author.

To explain this concept, we consider for example a motor control system which consists of a motor and a servo-driver. As is often the case, the servo-driver has a certain built-in feedback control loop. Designers then regard the motor and the servo-driver as a controlled plant, and design a control law for the new controlled plant. We call the former inner control loop "Implicit control law" and the latter outer control loop "Explicit control law". Similar situations can be found in various control systems such as a brain nervous system of living things.

We can regard that the above two control systems are designed by a common design method. That is, firstly a certain feedback control law is constructed, and then an outer feedback control law is added. In this paper, we focus on such a design method and call the system as a plug-in control system. Especially, we adopt a concept of Implicit Control and Explicit Control. By using an inverse optimal control approach, we first show two equivalent *n.a.c.s.* conditions under which the plug-in control system can be optimized by a certain plug-in optimal control law. We then show *n.a.c.s.* conditions for a class of single input systems under which the resultant optimal cost can be minimized by some implicit control embedded in the 1st stage, which is characterized as a feedback control that allocates all the closed-loop poles onto the imaginary axis. These results together clarify the system theoretic meaning of implicit control characterized as above from the viewpoint of inverse optimal control problem.

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II. Problem Statement

A. Plug-In Control System

At first, we define a Plug-in Control System based on the concept of Implicit and Explicit Control as proposed by the second author [1] in the following.

Definition 1 (Plug-In Control System)

Suppose that a controllable plant

$$\boldsymbol{S_0}:\ \dot{x} = A_0 x + Bu \tag{1}$$

is constructed by a feedback control law named "Implicit Control Law" $u_I = -K_I x$ from the original controlled plant

$$\boldsymbol{S}:\ \dot{x} = Ax + Bv \tag{2}$$

as shown in Fig. 1 (a). That is,

$$\dot{x} = A_0 x + Bu = (A - BK_I)x + Bu\ . \tag{3}$$

Here, we regard that the Implicit Control Law appears due to the interaction between plant and field.

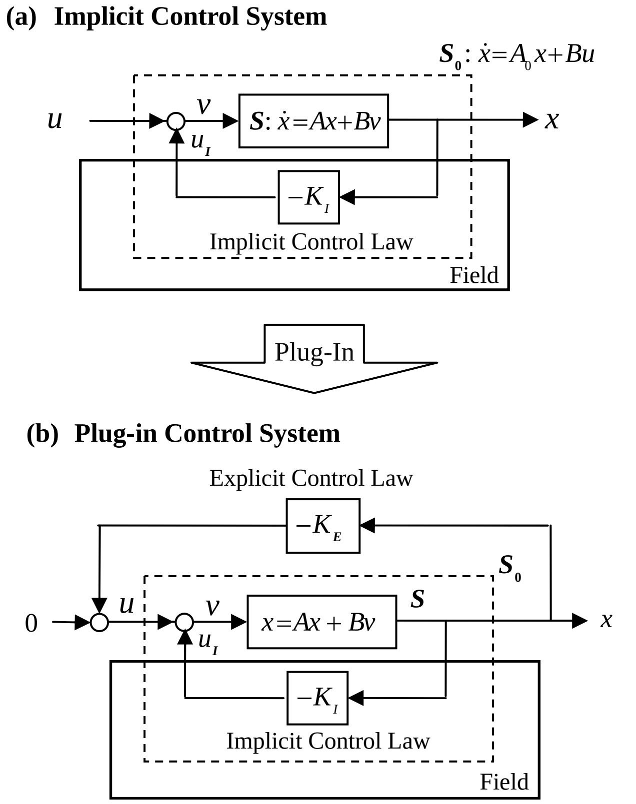

Fig. 1 Implicit Control and Plug-In Control Systems

We then consider an outer loop feedback control law

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$$u = u_E = -K_E x \quad (4)$$

as shown in Fig.1(b). We named this control law as Explicit Control Law in [1]. Similarly, we name this two stage design method shown in the figure as a plug-in design method, and name the resultant double loop control system as a Plug-In Control System.

B. Plug-In Optimal Control System

Consider the Plug-In Control System shown in Fig.1(b).

$$\boldsymbol{S}_{PI} : \dot{x} = (A - BK_I)x + Bu_E = Ax + Bv,\ v = u_I + u_E \quad (5)$$

where A, B and the Implicit Control Law $u_I = -K_I x$ are given, while the Explicit Control Law $u_E = -K_E x$ is unspecified. It is well known that we can always design u_E as an optimal control for $\boldsymbol{S}_0$ by LQ optimal control theory; we call the resultant optimal control system "Plug-In Optimal Control System." However, it is not trivial that the overall control $v = u_I + u_E$ also becomes an optimal control for the original system $\boldsymbol{S}$. In view of this, we first consider the following optimality problem of Plug-In Optimal Control System in the usual sense with the above setting.

Problem 1

Can you find conditions and design u_E such that the following two specifications are satisfied.

S1) The input $u = u_E$ is an optimal control for the system $\boldsymbol{S}_0$.

S2) The combined input $v = u_I + u_E$ is also an optimal control for the system $\boldsymbol{S}$.

Note that not every Plug-In Optimal Control System is optimal for $\boldsymbol{S}$. In other words, the specification S1 does not necessarily mean the specification S2, which is important for optimal design by the plug-in design method. To satisfy this specification, some constraints may be required on the quadratic weights used in LQ design of u_E. This problem is a kind of inverse problem of optimal control [2].

In Problem 1 the Implicit Control Law K_I is specified. So, there remains a possibility of strengthening the optimality of the Plug-In Optimal Control System, even if it satisfies S2, by proper choice of K_I. This observation allows us to consider the following optimality problem of Plug-In Optimal Control System in a stronger sense.

Problem 2

Consider the Plug-In Optimal Control System which is optimal in the sense of satisfying the specification S2. Can you find conditions and design K_I such that it minimizes the resultant optimal cost.

This problem is defined in more detail in the next section so that we can obtain a meaningful solution.

III. SOLUTIONS TO THE PROBLEMS

In this section we solve the above inverse optimal control problems. In order to understand Problem 1 intuitively, we first show a solution for a scalar system [3] in which A and B are scalars with $A = a,\ B = 1$.

A. Optimality of Plug-in Optimal Control System for a scalar system

Suppose that we design the Explicit Control Law $u_E = -k_E x$ for the system

$$\dot{x} = a_0 x + u_E, \quad a_0 = a - k_I \le 0 \quad (6)$$

as an optimal control law that minimizes the cost function.

$$J_E = \int_0^\infty \left(q_E x^2 + u_E^2 \right) dt \quad q_E \ge 0. \quad (7)$$

The resultant optimal feedback gain of the Explicit Control Law is given by

$$k_E = a_0 + \sqrt{a_0^2 + q_E} \quad (8)$$

Then, we can easily derive conditions and the weight q_V such that the combined input $v = u_I + u_E$ becomes an optimal control minimizing the cost function

$$J_V = \int_0^\infty \left(q_V x^2 + v^2 \right) dt \quad q_V \ge 0 \quad (9)$$

The answer to the question is the following.

Result 1 (Optimality Problem)

Firstly, set the weight q_V in the cost function J_V as

$$q_V = k_I^2 - 2ak_I + q_E = (k_I - a)^2 + q_E - a^2 \quad (10)$$

and define

$$D = q_E - a^2 \quad (11)$$

Then we have the following result.

***Case* 1** If $D > 0$ ($i.e.$, $q_E > a^2$), then for an arbitrary k_I, the overall input $v = -(k_I + k_E)x$ is always an optimal control for the cost J_V.

***Case* 2** If $D < 0$ ($i.e.$, $q_E \le a^2$), then for an arbitrary k_I satisfying $k_I \ge a + \sqrt{a^2 - q_E}$, the overall control $v = -(k_I + k_E)x$ is an optimal control for the cost J_V.

B. Optimality of Plug-in Optimal Control System for a general single input system

The above result shows that a certain condition must be satisfied by the weight in the cost J_E chosen in the 2nd stage and the gain K_I determined in the 1st stage, in order that the overall control $v = -(K_I + K_E)x$ is also an optimal control for the original system $\boldsymbol{S}$. We show this condition in the next theorem in two ways. One is given in the form of frequency domain condition, and the other in the form of Linear Matrix Inequality (LMI).

Theorem 1 Let K_E be an optimal feedback control law for the system $\boldsymbol{S}_0$ that minimizes the standard quadratic cost:

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$$J_E = \int_0^\infty \left(Q_E x^2 + u^2 \right) dt \quad Q_E \geq 0 \tag{12}$$

Then the combined feedback control law $v = -(K_I + K_E)x$ is an optimal control for the system $\boldsymbol{S}$ that minimizes the cost

$$J_v = \int_0^\infty \left(Q_v x^2 + v^2 \right) dt \quad Q_v \geq 0 \tag{13}$$

for some weight $Q_v \geq 0$ if and only if the weight Q_E and the gain K_I satisfies the following two equivalent conditions:

C1) $T(-j\omega)^T T(j\omega) \geq I - B^T (j\omega I - A^T)^{-1} Q_E (j\omega I - A)^{-1} B$
a.e. $\omega \in \mathrm{R}$ (14)

$$T(s) := I + K_I (sI - A)^{-1} B$$

C2) The following LMI has a real symmetric solution P.

$$\begin{bmatrix} PA + A^T P - K_I^T K_I - Q_E & PB - K_I^T \\ B^T P - K_I & 0 \end{bmatrix} \leq 0 \tag{15}$$

Moreover, if the condition C2 holds, the overall control $v = -(K_I + K_E)x$ minimizes the cost J_v for the weight Q_v given by

$$Q_v = Q_E - PA - A^T P + K_I^T K_I \tag{16}$$

(Proof) Since $A_0 - BK_E = A - B(K_I + K_E)$ is stable by the assumption on K_E, the overall control law $K_I + K_E$ is a stabilizing control law for $\boldsymbol{S}$. Thus it follows from the Inverse LQ theory [2] that the control law $K_I + K_E$ is an optimal control law for $\boldsymbol{S}$ if and only if the following LMI has a real symmetric solution $X \geq 0$.

$$\begin{bmatrix} XA + A^T X - (K_I + K_E)^T (K_I + K_E) & XB - (K_I + K_E)^T \\ B^T X - (K_I + K_E) & 0 \end{bmatrix} \leq 0 \tag{17}$$

Since K_E is an optimal control law for $\boldsymbol{S}$ that minimizes the cost (13), there exists some real symmetric $P_0 \geq 0$ such that

$$\begin{aligned} & P_0 A + A^T P_0 - P_0 B B^T P_0 + Q_E = 0 \\ & K_E = B^T P_0 \end{aligned} \tag{18}$$

Let $P := X - P_0$. Substituting $X = P + P_0$ into (17) and using (18) we see that the left side of (17) is the same as that of (15), and moreover applying some feedback transformation to (17) yields

$$X(A_0 - BK_E) + (A_0 - BK_E)^T X + (K_I + K_E)^T (K_I + K_E) \leq 0 \tag{19}$$

from which $X \geq 0$ follows by Lyapunov theorem as well as the stability of $A_0 - BK_E$ as stated above. Finally by a well-known result on LMI [4] two conditions C1 and C2 are equivalent. This completes the proof of Theorem 1 except the last part. Since (1,1) block of the left side matrix of (17) is equal to that of (15) and hence to $-Q_v$ by (16), the last part is obvious from LQ theory.

C. Strong optimality of the Plug-In Optimal Control System for a general single input system

In the previous section we have characterized the weighting matrix Q_v of those cost J_v that is minimized by the overall control law $K_I + K_E$ for a given K_I. In this section we minimize the resultant *optimal* cost $J_v = J_v^0$ further by proper choice of K_I. This minimization is equivalent to that of the weighting matrix Q_v, since the optimal cost J_v^0 reduces if so does the Q_v due to a well-known monotonicity property of maximal solutions of Riccati equations [4]. Here we are concerned only with the Q_v of *diagonal form*, and if all diagonal elements of Q_v is minimized by K_I, then we say that the associated Plug-In Optimal Control System is *strongly optimal*. With regard to this problem we are interested in those K_I for which all the eigenvalues of $A_0 = A - BK_I$ are on the imaginary axis, since the Implicit Control Law K_I is one of such control laws [1]. With these settings, we then define the following detailed version of Problem 2, and show necessary and sufficient conditions under which the optimal cost J_v^0 can be minimized by some implicit control law embedded in the 1st stage, thereby clarify a merit of the Implicit Control Law.

Problem 2 (Strong Optimality of Optimal S_{PI})

Consider the Plug-In *Optimal* Control System:

$$\dot{x} = A_0 x + Bu_E, \quad A_0 = A - BK_I, \quad \mathrm{Re}\,\lambda(A_0) \leq 0 \tag{20}$$

where $u_E = -K_E x$ is an optimal control law minimizing the cost:

$$J_E = \int_0^\infty \left(Q_E x^2 + u_E{}^2 \right) dt \quad Q_E = \mathrm{diag}\{q_i\} \geq 0 \tag{21}$$

and the overall control $v = -(K_I + K_E)x$ is also an *optimal* control minimizing the cost J_v with the weight Q_v given in Theorem 1. Can you find conditions and design K_I^0 such that the following three specifications are satisfied.

S1) The weighting matrix Q_v is diagonal for some symmetric solution P to the LMI of (15).

S2) The gain matrix $K_I = K_I^0$ minimizes the diagonal weighting matrix Q_v.

S3) All the eigenvalues of $A_0 = A - BK_I^0$ are on the imaginary axis.

To solve this problem, we consider the single input system $\boldsymbol{S}$ in the phase variable canonical form:

$$\boldsymbol{S}: \dot{x} = Ax + Bv$$

where

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{22}$$

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For such a system we can always obtain a diagonal weighting matrix Q_v for some real symmetric solution P of the LMI, and express it in terms of the elements of A, Q_E and K_I. However, these expressions are complicated in general, so we consider only the 2nd order system:

$$\boldsymbol{S}:\ \dot{x} = Ax + Bv,\quad A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{23}$$

Let $K_I = [k_1 \quad k_2]$. Then by (15) we have

$$B^T P = K_I = [k_1 \quad k_2] \tag{24}$$

and so by $B^T = [0 \quad 1]$ we obtain

$$P = \begin{bmatrix} p_1 & k_1 \\ k_1 & k_2 \end{bmatrix} \tag{25}$$

as a general expression for possible symmetric solutions to (15). Substituting this expression for P together with those for K_I and $Q_E = \mathrm{diag}\{q_1 \quad q_2\}$ as above into (16) yields

$$Q_v = Q_E - PA - A^T P + K_I^T K_I$$
$$= \begin{bmatrix} q_1 + k_1^2 - 2a_1 k_1 & * \\ k_1 k_2 - a_1 k_2 - a_2 k_1 - p_1 & q_2 + k_2^2 - 2a_2 k_2 - 2k_1 \end{bmatrix} \tag{26}$$

To meet the specification S1 for Q_v, therefore, it is enough to choose p_1 uniquely as follows:

$$p_1 = k_1 k_2 - a_1 k_2 - a_2 k_1 = (k_1 - a_1)(k_2 - a_2) - a_1 a_2 \tag{27}$$

Then we can obtain the following diagonal weighting matrix

$$Q_v = \begin{bmatrix} (k_1 - a_1)^2 + q_1 - a_1^2 & 0 \\ 0 & (k_2 - a_2)^2 + q_2 - a_2^2 - 2k_1 \end{bmatrix} \tag{28}$$

This expression enables us to give a solution to this problem as follows:

Theorem 2

Let $n=2$, $Q_E = \mathrm{diag}\{q_1, q_2\}$ and $K_I^0 = [k_1^0 \quad k_2^0]$ in Problem 2. Then there exists a control law K_I^0 satisfying the specifications S2 and S3 if and only if

$$q_1 > 0,\quad q_2 \geq q_2^0 \equiv a_2^2 + 2a_1 + 2\sqrt{\max\left\{0,\ a_1^2 - q_1\right\}} \tag{29}$$

Under this condition the minimizing control law $K_I^0 = [k_1^0 \quad k_2^0]$ with $\mathrm{Re}\,\lambda(A_0) \leq 0$ is given by

$$k_1^0 = a_1 + \sqrt{\max\left\{0,\ a_1^2 - q_1\right\}},\quad k_2^0 = a_2 \tag{30}$$

and the minimized weighting matrix $Q_v = \mathrm{diag}\{Q_{11}^0,\ Q_{22}^0\}$ is given by

$$Q_{11}^0 = \sqrt{\max\left\{0,\ q_1 - a_1^2\right\}},\ Q_{22}^0 = q_2 - q_2^0. \tag{31}$$

Remark 1: By the phase variable canonical form of (23) it is easy to see that $k_2^0 = a_2$ means that $A_0 = A - BK_I^0$ has complex conjugate eigenvalues $\pm j\sqrt{k_1^0 - a_1}$, thereby satisfying the specification S3. We should note, however, that S3 does not imply S2 in general. In other words, not all K_I that allocates all the closed-loop poles onto the imaginary axis guarantees the strong optimality of the Plug–In Optimal Control System. This is obvious from the expression of the minimizing control law k_1^0 given by (30). Obviously this expression suggests the existence of both lower and upper bounds of k_1^0. For example, in the case of $a_1 = 2, a_2 = 1$, k_1^0 takes only the values between 2 and 4.

Remark 2: By Theorem 2 we can conclude that if we choose the weight q_2 alone larger than a certain value, then the resultant Plug-In Optimal Control System with $\mathrm{Re}\,\lambda(A_0) = 0$ becomes optimal for the original system $\boldsymbol{S}$. This observation clarifies an important role played by Implicit Control in the plug-in design method, in the sense that it achieves the strong optimality of Plug-In Optimal Control System.

IV. CONCLUSION

In this paper, we have considered a plug-in control system via a concept of Implicit Control Law and Explicit Control Law. First, we showed that the plug-in control system can be optimized by designing a plug-in optimal control law suitably if and only if the quadratic weight and the Implicit Control Law associated with the plug-in optimal control system satisfy a certain condition at a time. We then showed that the resultant optimal cost can be minimized further by choosing an Implicit Control Law suitably if and only if the quadratic weight alone satisfy a certain condition, thereby we clarified an important role which the Implicit Control Law plays in the plug-in design method. Although we treat only the 2nd order system for simplicity in the latter half, we can extend the results in the same way to a higher order system up to 5th order system.

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