A Viscosity Method for Solving a General System of Finite Variational Inequalities for Finite Accretive Operators

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***Abstract*—In this paper, we prove a strong convergence theorem for finding a common solution of a general system of finite variational inequalities for finite different inverse-strongly accretive operators and solutions of fixed point problems for a nonexpansive mapping in a Banach space by using the weak contraction. Moreover, the above results are applied to find the solutions of zeros of accretive operators and the class of k-strictly pseudocontractive mappings. Our results are extended and improved of some authors ' recent results of the literature works in involving this field.**

***Index Terms*—Inverse-strongly accretive operator, Fixed point, General system of finite variational inequalities, Sunny nonexpansive retraction, Weak contraction.**

I. Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and C be a nonempty closed convex subset of E. Let E^* be the dual space of E and $\langle\cdot,\cdot\rangle$ denote the pairing between E and E^*. First, we recall the basic concept of mappings as shown in the following:

- $T : C \to C$ is said a *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. We denote the set of fixed point of T by $F(T)$.
- $f : C \to C$ is called a *weakly contractive* if there exists $\varphi : [0,\infty) \to [0,\infty)$ a continuous and strictly increasing function such that φ is positive on $(0,\infty)$, $\varphi(0) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$ and $x, y \in C$
$$\|f(x) - f(y)\| \leq \|x - y\| - \varphi(\|x - y\|). \quad (1)$$
- For $q > 1$, the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by
$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$$
for all $x \in E$. In particular, if $q = 2$, the mapping J_2 is called the *normalized duality mapping*, and usually written $J_2 = J$.
- $A : C \to E$ is said an *accretive* if there exists $j(x-y) \in J(x - y)$ such that $\langle Ax - Ay, j(x - y)\rangle \geq 0$ for all $x, y \in C$.

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- $A : C \to E$ is said a β-*strongly accretive* if there exists a constant $\beta > 0$ such that $\langle Ax - Ay, j(x - y)\rangle \geq \beta\|x - y\|^2 \quad \forall x, y \in C$.
- $A : C \to E$ is said a β-*inverse strongly accretive* if, for any $\beta > 0$, $\langle Ax - Ay, j(x - y)\rangle \geq \beta\|Ax - Ay\|^2$ for all $x, y \in C$.

The *variational inequality problem* is employed to find a point $x \in C$ and is defined by
$$\langle Ax, j(y - x)\rangle \geq 0, \quad \forall y \in C \quad (2)$$
where $A : C \to E$ is an accretive operator. The *general system of variational inequalities* is used to find $(x^*, y^*) \in C \times C$ and is defined by
$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \\ \langle \mu Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \end{cases} \quad (3)$$
for all $x \in C$ where λ and μ are two positive real numbers and $A, B : C \to E$ are two operators. The general system of variational inequalities, we extend into the *general system of finite variational inequalities* is applied to find $(x_1^*, x_2^*, \ldots, x_M^*) \in C \times C \times \cdots \times C$ and is defined by
$$\begin{cases} \langle \lambda_M A_M x_M^* + x_1^* - x_M^*, j(x - x_1^*)\rangle \geq 0, \\ \langle \lambda_{M-1} A_{M-1} x_{M-1}^* + x_M^* - x_{M-1}^*, j(x - x_M^*)\rangle \geq 0, \\ \vdots \\ \langle \lambda_2 A_2 x_2^* + x_3^* - x_2^*, j(x - x_3^*)\rangle \geq 0, \\ \langle \lambda_1 A_1 x_1^* + x_2^* - x_1^*, j(x - x_2^*)\rangle \geq 0,, \end{cases} \quad (4)$$
for all $x \in C$ where $\{A_l\}_{l=1}^M : C \to E$ is a family of mappings, $\lambda_l \geq 0, l \in \{1, 2, \ldots, M\}$. The set of solution of (4) is denoted by $GSVI(C, A_l)$.

II. Preliminaries

We always assume that E is a real Banach space and C is a nonempty closed convex subset of E. Let D be a subset of C and $Q : C \to D$. So Q is said to *sunny* if
$$Q(Qx + t(x - Qx)) = Qx,$$
whenever $Qx + t(x - Qx) \in C$ for any $x \in C$ and $t \geq 0$. A subset D of C is said a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction Q of C onto D. A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all z is in the range of Q.

Proposition II.1. *Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the*

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following are equivalent:
(i) Q is sunny and nonexpansive;
(ii) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \forall x, y \in E;$
(iii) $\langle x - Qx, J(y - Qx)\rangle \leq 0, \forall x \in E, y \in C.$

Proposition II.2. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let T be a nonexpansive mapping of C into itself with* $F(T) \neq \emptyset$. *Then the set F(T) is a sunny nonexpansive retract of C.*

Lemma II.3. *Let* E *be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E.$$

Lemma II.4. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma II.5. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two nonnegative real number sequences and* $\{\alpha_n\}$ *a positive real number sequence satisfying the conditions:* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \frac{b_n}{\alpha_n} = 0$. *Let the recursive inequality*

$$a_{n+1} \leq a_n - \alpha_n \varphi(a_n) + b_n, \ n \geq 0$$

where $\varphi(a)$ *is a continuous and strict increasing function for all* $a \geq 0$ *with* $\varphi(0) = 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma II.6. *Let* E *be a uniformly convex Banach space and* $B_r(0) := \{x \in E : \|x\| \leq r\}$ *be a closed ball of* E. *Then there exists a continuous strictly increasing convex function* $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0$ *such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|)$$

for all $x, y, z \in B_r(0)$ *and* $\lambda, \mu, \gamma \in [0, 1]$ *with* $\lambda + \mu + \gamma = 1$.

Lemma II.7. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be nonexpansive mapping of C into itself. If* $\{x_n\}$ *is a sequence of C such that* $x_n \to x$ *weakly and* $x_n - Tx_n \to 0$ *strongly, then x is s fixed point of T.*

Lemma II.8. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let the mapping* $A : C \to E$ *be* β*-inverse-strongly accretive. Then, we have*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + 2\lambda(\lambda K^2 - \beta)\|Ax - Ay\|^2.$$

If $\beta \geq \lambda K^2$, *then* $I - \lambda A$ *is nonexpansive.*

III. Main result

Lemma III.1. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let* Q_C *be the sunny nonexpansive retraction from E onto C. Let the mapping* $A_l : C \to H$ *be a* β_l*-inverse-strongly accretive such that* $\beta_l \geq \lambda_l K^2$ *where* $l \in \{1, 2, \ldots, M\}$. *If* $\mathcal{Q} : C \to C$ *be a mapping defined by*

$$\mathcal{Q}(x) = Q_C(I - \lambda_M A_M)\ldots Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_1 A_1)x,$$

for all $x \in C$, *then* $\mathcal{Q}$ *is nonexpansive.*

Proof. Taking $\mathcal{Q}_C^l = Q_C(I - \lambda_l A_l)\ldots Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_1 A_1), l \in \{1, 2, 3, \ldots, M\}$ and $\mathcal{Q}_C^0 = I$, where I is the identity mapping on H. Then we have $\mathcal{Q} = \mathcal{Q}_C^M$. For any $x, y \in C$, we have

$$\begin{aligned}
&\|\mathcal{Q}(x) - \mathcal{Q}(y)\| \\
&= \|\mathcal{Q}_C^M x - \mathcal{Q}_C^M y\| \\
&= \|Q_C(I - \lambda_M A_M)\mathcal{Q}_C^{M-1}x \\
&\quad - Q_C(I - \lambda_M A_M)\mathcal{Q}_C^{M-1}y\| \\
&\leq \|(I - \lambda_M A_M)\mathcal{Q}_C^{M-1}x - (I - \lambda_M A_M)\mathcal{Q}_C^{M-1}y\| \\
&\leq \|\mathcal{Q}_C^{M-1}x - \mathcal{Q}_C^{M-1}y\| \\
&\ \vdots \\
&\leq \|\mathcal{Q}_C^0 x - \mathcal{Q}_C^0 y\| \\
&= \|x - y\|.
\end{aligned}$$

Therefore $\mathcal{Q}$ is nonexpansive. □

Lemma III.2. *Let C be a nonempty closed convex subset of a real smooth Banach space E. Let* Q_C *be the sunny nonexpansive retraction from E onto C. Let* $A_l : C \to H$ *be nonlinear mapping, where* $l \in \{1, 2, \ldots, M\}$. *For* $x_l^* \in C, l \in \{1, 2, \ldots, M\}$, $(x_1^*, x_2^*, \ldots, x_M^*)$ *is a solution of problem* (4) *if and only if*

$$\begin{cases} x_1^* = Q_C(I - \lambda_M A_M)x_M^* \\ x_2^* = Q_C(I - \lambda_1 A_1)x_1^* \\ x_3^* = Q_C(I - \lambda_2 A_2)x_2^* \\ \vdots \\ x_M^* = Q_C(I - \lambda_{M-1} A_{M-1})x_{M-1}^*, \end{cases} \tag{5}$$

that is

$$x_1^* = Q_C(I - \lambda_M A_M)\ldots Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_1 A_1)x_1^*.$$

Proof. From (4), we rewrite as

$$\begin{cases} \langle x_1^* - (x_M^* - \lambda_M A_M x_M^*), j(x - x_1^*)\rangle \geq 0, \\ \langle x_M^* - (x_{M-1}^* - \lambda_{M-1} A_{M-1} x_{M-1}^*), j(x - x_M^*)\rangle \geq 0, \\ \vdots \\ \langle x_3^* - (x_2^* - \lambda_2 A_2 x_2^*), j(x - x_3^*)\rangle \geq 0, \\ \langle x_2^* - (x_1^* - \lambda_1 A_1 x_1^*), j(x - x_2^*)\rangle \geq 0. \end{cases} \tag{6}$$

for all $x \in C$. Using Proposition II.1 (iii), the system (6) equivalent to (5). □

Throughout this paper, the set of fixed points of the mapping $\mathcal{Q}$ is denoted by $F(\mathcal{Q})$.

Theorem III.3. *Let* E *be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and C be a nonempty closed convex subset of E. Let* $S : C \to C$ *be a nonexpansive mapping and* Q_C *be a sunny nonexpansive retraction from E onto C. Let* $A_l : C \to E$ *be a* β_l*-inverse-strongly accretive such that* $\beta_l \geq \lambda_l K^2$ *where* $l \in \{1, 2, \ldots, M\}$ *and* K *be the best smooth constant. Let* f *be a weakly contractive of* C *into itself with function* φ. *Suppose* $\mathcal{F} := F(\mathcal{Q}) \cap F(S) \neq \emptyset$ *where* $\mathcal{Q}$ *is defined by Lemma III.1. For arbitrary given* $x_0 = x \in C$, *the sequence* $\{x_n\}$ *is generated by*

$$\begin{cases} y_n = Q_C(I - \lambda_M A_M)\ldots Q_C(I - \lambda_2 A_2)Q_C(I - \lambda_1 A_1)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S y_n. \end{cases} \tag{7}$$

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where the sequences $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $(0,1)$ *satisfy* $\alpha_n+\beta_n+\gamma_n = 1$, $n \geq 1$ *and* $\lambda_l, l = 1, 2, \ldots, M$ *are positive real numbers. The following conditions:*
(C1). $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(C2). $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$
are satisfied. Then $\{x_n\}$ *converges strongly to* $\bar{x}_1 = Q_{\mathcal{F}} f(\bar{x}_1)$ *and* $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_M)$ *is a solution of the problem* (4) *where* $Q_{\mathcal{F}}$ *is the sunny nonexpansive retraction of* C *onto* $\mathcal{F}$.

Proof. First, we prove that $\{x_n\}$ is bounded. Let $p \in \mathcal{F}$, and take

$$\mathcal{Q}_C^l = Q_C(I - \lambda_l A_l) \ldots Q_C(I - \lambda_2 A_2) Q_C(I - \lambda_1 A_1),$$

for $l \in \{1, 2, 3, \ldots, M\}$ and $\mathcal{Q}_C^0 = I$, where I is the identity mapping on E. Since Q_C is a nonexpansive, then $\mathcal{Q}_C^l, l \in \{1, 2, 3, \ldots, M\}$ also. We note that

$$\|y_n - p\| = \|\mathcal{Q}_C^l x_n - \mathcal{Q}_C^l p\| \leq \|x_n - p\|. \tag{8}$$

It follows from (7) and (8), we also have
$\|x_{n+1} - p\|$

$$\begin{aligned}
&= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S y_n - p\| \\
&\leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|S y_n - p\| \\
&\leq \alpha_n \big[\|x_n - p\| - \varphi(\|x_n - p\|)\big] + \alpha_n \|f(p) - p\| \\
&\quad + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \\
&\leq \|x_n - p\| - \alpha_n \varphi(\|x_n - p\|) + \alpha_n \|f(p) - p\| \\
&\leq \max\{\|x_1 - p\|, \varphi(\|x_1 - p\|), \|f(p) - p\|\}. \qquad (9)
\end{aligned}$$

This implies that $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{y_n\}$, and $\{Sy_n\}$.

Next, we show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Notice that
$\|y_{n+1} - y_n\|$

$$\begin{aligned}
&= \|\mathcal{Q}_C^M x_{n+1} - \mathcal{Q}_C^M x_n\| \\
&= \|Q_C(I - \lambda_M A_M)\mathcal{Q}_C^{M-1} x_{n+1} \\
&\quad - Q_C(I - \lambda_M A_M)\mathcal{Q}_C^{M-1} x_n\| \\
&\leq \|(I - \lambda_M A_M)\mathcal{Q}_C^{M-1} x_{n+1} - (I - \lambda_M A_M)\mathcal{Q}_C^{M-1} x_n\| \\
&\leq \|\mathcal{Q}_C^{M-1} x_{n+1} - \mathcal{Q}_C^{M-1} x_n\| \\
&\ \vdots \\
&\leq \|\mathcal{Q}_C^0 x_{n+1} - \mathcal{Q}_C^0 x_n\| \\
&= \|x_{n+1} - x_n\|.
\end{aligned}$$

Setting $x_{n+1} = (1 - \beta_n) z_n + \beta_n x_n$ for all $n \geq 0$, we see that $z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$, then we have
$\|z_{n+1} - z_n\|$

$$\begin{aligned}
&= \Big\|\frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}\Big\| \\
&= \Big\|\frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} S y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n S y_n}{1 - \beta_n}\Big\| \\
&= \Big\|\frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} S y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_{n+1} f(x_n)}{1 - \beta_{n+1}} \\
&\quad + \frac{\alpha_{n+1} f(x_n)}{1 - \beta_{n+1}} - \frac{\gamma_{n+1} S y_n}{1 - \beta_{n+1}} + \frac{\gamma_{n+1} S y_n}{1 - \beta_{n+1}} \\
&\quad - \frac{\alpha_n f(x_n) + \gamma_n S y_n}{1 - \beta_n}\Big\| \\
&= \Big\|\frac{\alpha_{n+1}}{1 - \beta_{n+1}}(f(x_{n+1}) - f(x_n)) \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(S y_{n+1} - S y_n) \\
&\quad + \Big(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\Big) f(x_n) \\
&\quad + \Big(\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}\Big) S y_n\Big\| \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|f(x_{n+1}) - f(x_n)\| \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\|y_{n+1} - y_n\| \\
&\quad + \Big|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\Big| \|f(x_n)\| \\
&\quad + \Big|\frac{1 - \beta_{n+1} - \alpha_{n+1}}{1 - \beta_{n+1}} - \frac{1 - \beta_n - \alpha_n}{1 - \beta_n}\Big| \|S y_n\| \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\Big[\|x_{n+1} - x_n\| - \varphi(\|x_{n+1} - x_n\|)\Big] \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\|y_{n+1} - y_n\| \\
&\quad + \Big|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\Big| \|f(x_n)\| \\
&\quad + \Big|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\Big| \|S y_n\| \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|x_{n+1} - x_n\| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\|y_{n+1} - y_n\| \\
&\quad + \Big|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\Big| (\|f(x_n)\| + \|S y_n\|) \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \\
&\quad + \Big|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\Big| (\|f(x_n)\| + \|S y_n\|) \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|x_{n+1} - x_n\| + \|x_{n+1} - x_n\| \\
&\quad + \Big|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\Big| (\|f(x_n)\| + \|S y_n\|).
\end{aligned}$$

Therefore,
$\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|$

$$\begin{aligned}
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|x_{n+1} - x_n\| \\
&\quad + \Big|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\Big| (\|f(x_n)\| + \|S y_n\|).
\end{aligned}$$

It follows from the conditions (C1) and (C2), which imply that

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Applying Lemma II.4, we obtain $\lim_{n\to\infty} \|z_n - x_n\| = 0$ and also

$$\|x_{n+1} - x_n\| = (1 - \beta_n)\|z_n - x_n\| \to 0$$

as $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{10}$$

Next, we show that $\lim_{n\to\infty} \|S y_n - y_n\| = 0$. Since $p \in \mathcal{F}$, from Lemma II.6, we obtain
$\|x_{n+1} - p\|^2$

$$\begin{aligned}
&= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S y_n - p\|^2 \\
&\leq \alpha_n \|f(x_n) - p\|^2 + (1 - \alpha_n - \gamma_n)\|x_n - p\|^2 \\
&\quad + \gamma_n \|y_n - p\|^2
\end{aligned}$$

$$\begin{aligned}
&= \alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2 \\
&\quad -\gamma_n(\|x_n-p\|^2-\|y_n-p\|^2) \\
&= \alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2 \\
&\quad -\gamma_n(\|x_n-p\|-\|y_n-p\|)(\|x_n-p\|+\|y_n-p\|) \\
&\leq \alpha_n\|f(x_n)-p\|^2+\|x_n-p\|^2-\gamma_n\|x_n-y_n\|^2.
\end{aligned}$$

Therefore, we have
$\gamma_n\|x_n-y_n\|^2$

$$\begin{aligned}
&\leq \alpha_n\|f(x_n)-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\leq \alpha_n\|f(x_n)-p\|^2 \\
&\quad +(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|.
\end{aligned}$$

From the condition (C1) and (10), this implies that $\|x_n-y_n\|\to 0$ as $n\to\infty$. Now, we note that
$\|x_n-Sy_n\|$

$$\begin{aligned}
&\leq \|x_n-x_{n+1}\|+\|x_{n+1}-Sy_n\| \\
&= \|x_n-x_{n+1}\|+\|\alpha_n f(x_n)+\beta_n x_n+\gamma_n Sy_n-Sy_n\| \\
&= \|x_n-x_{n+1}\|+\|\alpha_n(f(x_n)-Sy_n)+\beta_n(x_n-Sy_n)\| \\
&\leq \|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-Sy_n\|+\beta_n\|x_n-Sy_n\|.
\end{aligned}$$

Therefore, we get
$\|x_n-Sy_n\|$

$$\leq \frac{1}{1-\beta_n}\|x_n-x_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|f(x_n)-Sy_n\|.$$

From the conditions (C1), (C2) and (10), which imply that $\|x_n-Sy_n\|\to 0$ as $n\to\infty$. Since

$$\|Sy_n-y_n\| \leq \|Sy_n-x_n\|+\|x_n-y_n\|,$$

it follows that $\lim_{n\to\infty}\|Sy_n-y_n\|=0$.

Next, we prove that $z\in\mathcal{F}:=F(\mathcal{Q})\cap F(S)$.
(a) First, we show that $z\in F(S)$. To show this, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$. Since $\{y_{n_i}\}$ is bounded, we have a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ converging weakly to z. We may assume without loss of generality that $y_{n_i}\rightharpoonup z$. Since $\|Sy_n-y_n\|\to 0$, we obtain $Sy_{n_i}\rightharpoonup z$. Then, we can obtain $z\in\mathcal{F}$. Assuming that $z\notin F(S)$, we get $Sz\neq z$. From $y_{n_i}\rightharpoonup z$ and Opial's condition, we obtain
$\liminf_{i\to\infty}\|y_{n_i}-z\|$

$$\begin{aligned}
&< \liminf_{i\to\infty}\|y_{n_i}-Sz\| \\
&\leq \liminf_{i\to\infty}(\|y_{n_i}-Sy_{n_i}\|+\|Sy_{n_i}-Sz\|) \\
&\leq \liminf_{i\to\infty}\|y_{n_i}-z\|.
\end{aligned}$$

This is a contradiction. Thus, we have $z\in F(S)$.
(b) Next, we show that $z\in F(\mathcal{Q})$. From Lemma III.1, we know that $\mathcal{Q}=\mathcal{Q}_C^M$ is a nonexpansive, it follows that

$$\|y_n-\mathcal{Q}y_n\|=\|\mathcal{Q}_C^M x_n-\mathcal{Q}_C^M y_n\|\leq\|x_n-y_n\|.$$

Thus $\lim_{n\to\infty}\|y_n-\mathcal{Q}y_n\|=0$. Since $\mathcal{Q}$ is a nonexpansive, we get
$\|x_n-\mathcal{Q}x_n\|$

$$\begin{aligned}
&\leq \|x_n-y_n\|+\|y_n-\mathcal{Q}y_n\|+\|\mathcal{Q}y_n-\mathcal{Q}x_n\| \\
&\leq 2\|x_n-y_n\|+\|y_n-\mathcal{Q}y_n\|.
\end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty}\|x_n-\mathcal{Q}x_n\|=0. \tag{11}$$

By Lemma II.7 and (11), we have $z\in F(\mathcal{Q})$. Therefore $z\in\mathcal{F}$.

Next, we show that $\limsup_{n\to\infty}\langle(f-I)\bar{x}_1,J(x_n-\bar{x}_1)\rangle\leq 0$, where $\bar{x}_1=Q_{\mathcal{F}}f(\bar{x}_1)$. Since $\{x_n\}$ is bounded, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ which $x_{n_i}\rightharpoonup z$, such that $\limsup_{n\to\infty}\langle(f-I)\bar{x}_1,J(x_n-\bar{x}_1)\rangle$

$$=\lim_{i\to\infty}\langle(f-I)\bar{x}_1,J(x_{n_i}-\bar{x}_1)\rangle. \tag{12}$$

Now, from (12), Proposition II.1 (iii) and the weakly sequential continuity of the duality mapping *J*, we have
$\limsup_{n\to\infty}\langle(f-I)\bar{x}_1,J(x_n-\bar{x}_1)\rangle$

$$\begin{aligned}
&= \lim_{i\to\infty}\langle(f-I)\bar{x}_1,J(x_{n_i}-\bar{x}_1)\rangle \\
&= \langle(f-I)\bar{x}_1,J(z-\bar{x}_1)\rangle\leq 0. \qquad (13)
\end{aligned}$$

From (10), it follows that

$$\limsup_{n\to\infty}\langle(f-I)\bar{x}_1,J(x_{n+1}-\bar{x}_1)\rangle\leq 0. \tag{14}$$

Finally, we show that $\{x_n\}$ converges strongly to $\bar{x}_1=Q_{\mathcal{F}}f(\bar{x}_1)$. We compute that
$\|x_{n+1}-\bar{x}_1\|^2$

$$\begin{aligned}
&= \langle x_{n+1}-\bar{x}_1,J(x_{n+1}-\bar{x}_1)\rangle \\
&= \langle\alpha_n f(x_n)+\beta_n x_n+\gamma_n Sy_n-\bar{x}_1,J(x_{n+1}-\bar{x}_1)\rangle \\
&= \langle\alpha_n(f(x_n)-\bar{x}_1)+\beta_n(x_n-\bar{x}_1) \\
&\quad +\gamma_n(Sy_n-\bar{x}_1),J(x_{n+1}-\bar{x}_1)\rangle \\
&= \alpha_n\langle f(x_n)-f(\bar{x}_1),J(x_{n+1}-\bar{x}_1)\rangle \\
&\quad +\alpha_n\langle f(\bar{x}_1)-\bar{x}_1,J(x_{n+1}-\bar{x}_1)\rangle \\
&\quad +\beta_n\langle x_n-\bar{x}_1,J(x_{n+1}-\bar{x}_1)\rangle \\
&\quad +\gamma_n\langle Sy_n-\bar{x}_1,J(x_{n+1}-\bar{x}_1)\rangle \\
&\leq \alpha_n\Big[\|x_n-\bar{x}_1\|-\varphi(\|x_n-\bar{x}_1\|)\Big]\|x_{n+1}-\bar{x}_1\| \\
&\quad +\alpha_n\langle f(\bar{x}_1)-\bar{x}_1,J(x_{n+1}-\bar{x}_1)\rangle \\
&\quad +\beta_n\|x_n-\bar{x}_1\|\|x_{n+1}-\bar{x}_1\| \\
&\quad +\gamma_n\|y_n-\bar{x}_1\|\|x_{n+1}-\bar{x}_1\| \\
&\leq \alpha_n\|x_n-\bar{x}_1\|\|x_{n+1}-\bar{x}_1\| \\
&\quad -\alpha_n\varphi(\|x_n-\bar{x}_1\|)\|x_{n+1}-\bar{x}_1\| \\
&\quad +\alpha_n\langle f(\bar{x}_1)-\bar{x}_1,J(x_{n+1}-\bar{x}_1)\rangle \\
&\quad +\beta_n\|x_n-\bar{x}_1\|\|x_{n+1}-\bar{x}_1\| \\
&\quad +\gamma_n\|x_n-\bar{x}_1\|\|x_{n+1}-\bar{x}_1\| \\
&= \|x_n-\bar{x}_1\|\|x_{n+1}-\bar{x}_1\| \\
&\quad -\alpha_n\varphi(\|x_n-\bar{x}_1\|)\|x_{n+1}-\bar{x}_1\| \\
&\quad +\alpha_n\langle f(\bar{x}_1)-\bar{x}_1,J(x_{n+1}-\bar{x}_1)\rangle \\
&= \frac{1}{2}\Big(\|x_n-\bar{x}_1\|^2+\|x_{n+1}-\bar{x}_1\|^2\Big) \\
&\quad -\alpha_n\varphi(\|x_n-\bar{x}_1\|)\|x_{n+1}-\bar{x}_1\| \\
&\quad +\alpha_n\langle f(\bar{x}_1)-\bar{x}_1,J(x_{n+1}-\bar{x}_1)\rangle.
\end{aligned}$$

By (9) and $\{x_{n+1}-\bar{x}_1\}$ bounded, there exist $M>0$ such that $\|x_{n+1}-\bar{x}_1\|\leq M$, which imply that
$\|x_{n+1}-\bar{x}_1\|^2$

$$\begin{aligned}
&\leq \|x_n-\bar{x}_1\|^2-2\alpha_n M\varphi(\|x_n-\bar{x}_1\|) \\
&\quad +2\alpha_n\langle f(\bar{x}_1)-\bar{x}_1,J(x_{n+1}-\bar{x}_1)\rangle. \qquad (15)
\end{aligned}$$

Now, from (C1) and applying Lemma II.5 to (15), we get $\|x_n-\bar{x}_1\|\to 0$ as $n\to\infty$. This completes the proof. □

Corollary III.4. *Let E be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and C be a nonempty* *closed convex subset of E. Let $S : C \to C$ be a nonexpansive mapping and Q_C be a sunny nonexpansive retraction from E onto C. Let $A : C \to E$ be a β-inverse-strongly accretive such that $\beta \geq \lambda K^2$ where K be the best smooth constant. Let f be a weakly contractive of C into itself with function φ. Let the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$ and satisfy the conditions (C1) and (C2) in Theorem III.3. Suppose $\mathcal{F} := F(\mathcal{Q}) \cap F(S) \neq \emptyset$ where $\mathcal{Q}$ is defined by*

$$\mathcal{Q}(x) = Q_C(I-\lambda A)Q_C(I-\lambda A)\dots Q_C(I-\lambda A)x, \forall x \in C,$$

and λ be a positive real number. For arbitrary given $x_0 = x \in C$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = Q_C(I-\lambda A)Q_C(I-\lambda A)\dots Q_C(I-\lambda A)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S y_n. \end{cases} \tag{16}$$

Then $\{x_n\}$ converges strongly to $\bar{x}_1 = Q_{\mathcal{F}} f(\bar{x}_1)$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof. Putting $A = A_M = A_{M-1} = \dots = A_2 = A_1$, $\beta = \beta_M = \beta_{M-1} = \dots = \beta_2 = \beta_1$ and $\lambda = \lambda_M = \lambda_{M-1} = \dots = \lambda_2 = \lambda_1$ in Theorem III.3, we can conclude the desired conclusion easily. This completes the proof. □

Corollary III.5. *Let E be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and C be a nonempty closed convex subset of E. Let $S : C \to C$ be a nonexpansive mapping and Q_C be a sunny nonexpansive retraction from E onto C. Let $A_l : C \to E$ be a β_l-inverse-strongly accretive such that $\beta_l \geq \lambda_l K^2$ where $l \in \{1,2\}$ and K be the best smooth constant. Let f be a weakly contractive of C into itself with function φ. Let the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$ and satisfy the conditions (C1) and (C2) in Theorem III.3. Suppose $\mathcal{F} := F(\mathcal{Q}) \cap F(S) \neq \emptyset$ where $\mathcal{Q}$ is defined by*

$$\mathcal{Q}(x) = Q_C(I-\lambda_2 A_2)Q_C(I-\lambda_1 A_1)x, \forall x \in C,$$

and λ_1, λ_2 are positive real numbers. For arbitrary given $x_0 = x \in C$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = Q_C(I-\lambda_2 A_2)Q_C(I-\lambda_1 A_1)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S y_n. \end{cases} \tag{17}$$

Then $\{x_n\}$ converges strongly to $\bar{x}_1 = Q_{\mathcal{F}} f(\bar{x}_1)$ and $(\bar{x}_1, \bar{x}_2)$ is a solution of the problem (3), *where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of C onto $\mathcal{F}$.*

Proof. Taking $M = 2$ in Theorem III.3, we can conclude the desired conclusion easily. This completes the proof. □

IV. Some applications

(I) Application to finding zeros of accretive operators.

In Banach space E, we always assume that E is a uniformly convex and 2-uniformly smooth. Recall that, an accretive operator T is a m-accretive if $R(I + rT) = E$ for each $r > 0$. We assume that T is a m-accretive and has a zero (i.e., the inclusion $0 \in T(z)$ is solvable). The set of zeros of T is denoted by $T^{-1}(0)$, that

$$T^{-1}(0) = \{z \in D(T) : 0 \in T(z)\}.$$

The resolvent of T, i.e., $J_r^T = (I+rT)^{-1}$, for each $r > 0$. If T is a m-accretive, then $J_r^T : E \to E$ is a nonexpansive and $F(J_r^T) = T^{-1}(0), \forall r > 0$.

From the main result Theorem III.3, we can conclude the following result immediately.

Theorem IV.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let $A_l : C \to E$ be a β_l-inverse-strongly accretive such that $\beta_l \geq \lambda_l K^2$ where $l \in \{1,2,\dots,M\}$, K be the 2-uniformly smoothness constant of E and T be an m-accretive mapping. Let f be a weakly contractive of C into itself with function φ and suppose the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. Suppose $\mathcal{F} := T^{-1}(0) \cap \Big(\cap_{l=1}^M A_l^{-1}(0)\Big) \neq \emptyset$ and $\lambda_l, l = 1,2,\dots,M$, are positive real numbers. The following conditions:*
(i). $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^\infty \alpha_n = \infty$,
(ii). $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
are satisfied. The sequence $\{x_n\}$ is generated by $x_0 = x \in C$ and

$$\begin{cases} y_n = J_r^T(I-\lambda_M A_M)\dots(I-\lambda_2 A_2)(I-\lambda_1 A_1)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n. \end{cases} \tag{18}$$

Then $\{x_n\}$ converges strongly to $\bar{x}_1 = Q_{\mathcal{F}} f(\bar{x}_1)$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of E onto $\mathcal{F}$.

(II) Application to strictly pseudocontractive mappings

Let E be a Banach space and let C be a subset of E. Recall that, a mapping $T : C \to C$ is said a *k-stricly pseudocontractive* if there exist $k \in [0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx-Ty, j(x-y)\rangle \leq \|x-y\|^2 - \frac{1-k}{2}\|(I-T)x-(I-T)y\|^2 \tag{19}$$

for all $x, y \in C$. Then (19) can be written in the following form

$$\langle (I-T)x-(I-T)y, j(x-y)\rangle \geq \frac{1-k}{2}\|(I-T)x-(I-T)y\|^2. \tag{20}$$

We know that, A is a $\frac{1-k}{2}-$ inverse strongly monotone and $A^{-1}0 = F(T)$.

Theorem IV.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let $S : C \to C$ be a nonepansive mapping and $T_l : C \to C$ be a k_l-stricly pseudocontractive with $\lambda_l \leq \frac{(1-k_l)}{2K^2}$, $l \in \{1,2,\dots,M\}$. Let f be a weakly contractive of C into itself with function φ and suppose the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. Suppose $\mathcal{F} := F(S) \cap \Big(\cap_{l=1}^M F(T_l)\Big) \neq \emptyset$ and let $\lambda_l, l = 1,2,\dots,M$ are positive real numbers. The following conditions:*
(i). $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^\infty \alpha_n = \infty$,
(ii). $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
are satisfied. The sequence $\{x_n\}$ is generated by $x_0 = x \in C$ and

$$\begin{cases} y_n = \big((1-\lambda_M) + \lambda_M T_M\big) \\ \quad \dots\big((1-\lambda_2) + \lambda_2 T_2\big)\big((1-\lambda_1) + \lambda_1 T_1\big)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S y_n. \end{cases} \tag{21}$$

Then $\{x_n\}$ converges strongly to $Q_{\mathcal{F}}$, where $Q_{\mathcal{F}}$ is the sunny nonexpansive retraction of E onto $\mathcal{F}$.

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***Proof*.** Putting $A_l = I - T_l, l \in \{1, 2, \ldots, M\}$. Form (20), we get A_l is a $\frac{1-k_l}{2}$−inverse strongly accretive operator. It follows that $GSVI(C, A_l) = GSVI(C, I-T_l) = F(T_l) \neq \emptyset$ and $\Big(\cap_{l=1}^{M} GSVI(C, I-T_l)\Big) = F(\mathcal{Q}) \Leftrightarrow$ is the solution of problems (4).

$$\big((1-\lambda_1)+\lambda_1 T_1\big)x_n = Q_C\big((1-\lambda_1)+\lambda_1 T_1\big)x_n$$

$$\vdots$$

$\big((1-\lambda_M)+\lambda_M T_M\big)\ldots\big((1-\lambda_1)+\lambda_1 T_1\big)x_n$
$= Q_C\big((1-\lambda_M)+\lambda_M T_M\big)\ldots Q_C\big((1-\lambda_1)+\lambda_1 T_1\big)x_n.$
Therefore, by Theorem III.3, $\{x_n\}$ converges strongly to some element $\bar{x}_1$ of $\mathcal{F}$.

□

(III) Application to Hilbert spaces

In a real Hilbert spaces H, by Lemma III.2, we obtain the following Lemma:

Lemma IV.3. *$(x_1^*, x_2^*, \ldots, x_M^*)$ is a solution of*

$$\begin{cases} \langle \lambda_M A_M x_M^* + x_1^* - x_M^*, x - x_1^* \rangle \geq 0, \\ \langle \lambda_{M-1} A_{M-1} x_{M-1}^* + x_M^* - x_{M-1}^*, x - x_M^* \rangle \geq 0, \\ \vdots \\ \langle \lambda_2 A_2 x_2^* + x_3^* - x_2^*, x - x_3^* \rangle \geq 0, \\ \langle \lambda_1 A_1 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0, \end{cases} \tag{22}$$

for all $x \in C$ if and only if

$$x_1^* = P_C(I - \lambda_M A_M)\ldots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x_1^*$$

is a fixed point of the mapping $\mathcal{P} : C \to C$ which is defined by

$$\mathcal{P}(x) = P_C(I - \lambda_M A_M)\ldots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x$$

for all $x \in C$ where P_C is a metric projection H onto C.

It is well known that the smooth constant $K = \frac{\sqrt{2}}{2}$ in Hilbert spaces. From Theorem III.3, we can obtain the following result immediately.

Theorem IV.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A_l : C \to H$ be a β_l-inverse-strongly monotone mapping with $\lambda_l \in (0, 2\beta_l)$, $l \in \{1, 2, \ldots, M\}$. Let $S : C \to C$ be a nonepansive mapping and f be a weakly contractive of C into itself with function φ. Suppose the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. Assume that $\mathcal{F} := F(\mathcal{P}) \cap F(S) \neq \emptyset$ where $\mathcal{P}$ is defined by Lemma IV.3 and $\lambda_l, l = 1, 2, \ldots, M$, are positive real numbers. The following conditions:*
(i). $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,,
(ii). $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
are satisfied. For arbitrary given $x_0 = x \in C$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_C(I - \lambda_M A_M)\ldots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S y_n. \end{cases} \tag{23}$$

Then $\{x_n\}$ converges strongly to $\bar{x}_1 = P_{\mathcal{F}} f(\bar{x}_1)$ and $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_M)$ is a solution of the problem (22).

V. Conclusion

In this paper, motivated and inspired by the idea of Ceng et al. [1], Katchang and Kumam [2], Witthayarat et al. [3] and Yao et al. [4]. We introduce a new iterative scheme with weak contraction for finding solutions of a new general system of finite variational inequalities (4) for finite different inverse-strongly accretive operators and solutions of fixed point problems for nonexpansive mapping in a Banach space. Consequently, we obtain new strong convergence theorems for fixed point problems which solve the general system of variational inequalities (3). Moreover, using the above theorems, we can apply to find solutions of zeros of accretive operators and the class of k-strictly pseudocontractive mappings.

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