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A Single Wholesaler and Two Retailers Inventory Policy with Quantity Discounts for a Deteriorating Item

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Abstract—**In Japanese large-scale super markets, there are two or more sections at which the same items are sold in the same store. In many cases, managers of these sections independently order the items since they are under competition and are evaluated separately by their supervisor. They can possibly reduce their costs if either of them purchases the items for two sections in cooperation with each other. This study considers the quantity discount problem between a single seller (wholesaler) and two buyers (retailers). The seller attempts to increase her profit by controlling the buyer's order quantity through a quantity discount strategy. The buyers try to maximize their profits by considering both whether to cooperate with each other and whether to accept the seller's offer. We formulate the above problem as a Stackelberg game between a single seller and two buyers to analyze the existence of the seller's optimal quantity discount pricing policy, which maximizes her total profit per unit of time. Numerical examples are presented to illustrate the theoretical underpinnings of the proposed model.**

Index Terms—**quantity discounts, deteriorating items, total profit, Stackelberg game.**

I. Introduction

In Japanese large-scale super-markets, in each store, there are two or more sections at which the same items are sold. For instance, assorted sushi boxes are sold both at the ready-made dish section (sozai corner) and the fish section. Managers of these sections independently order the fishes as ingredients for sushi since they are under competition and are evaluated separately by their supervisor. They can possibly reduce their costs if either of them purchases the items for two sections in cooperation with each other. Several super-markets have recently applied this strategy.

This study discusses the quantity discount problem[1], [2], [3], [4], [5], [6] between a single seller (wholesaler) and two buyers (retailers). The wholesaler purchases items from upper-leveled supplier and sells them to these two retailers. The wholesaler attempts to increase her profit by controlling the retailers' order quantities through a quantity discount strategy. The retailers try to maximize their profits by considering both whether to cooperate with each other and whether to accept the wholesaler's offer. We consider the case where the retailers deal in perishable items such as fresh fruits, fishes, sushi boxes and vegetables, and where both the wholesaler's and the retailers' inventory levels are continuously depleted due to the combined effects of its demand and deterioration. Yang [7] and Kawakatsu[8] have developed the model to determine an optimal pricing and a ordering policy for deteriorating items with quantity discounts. However, they focused on the quantity discount problem between a single seller and a single buyer.

This study formulates the above problem as a Stackelberg game between a single wholesaler and two retailers to analyze the existence of the wholesale's optimal quantity discount pricing policy, which maximizes her total profit per unit of time. Numerical examples are presented to illustrate the theoretical underpinnings of the proposed model.

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II. Notation and Assumptions

The wholesaler uses a quantity discount strategy in order to improve her/his profit. The wholesaler proposes, for the retailers, an order quantity per lot along with the corresponding discounted wholesale price, which induces the retailers to alter their replenishment policies. We consider the two options throughout the present study as follows:

Option V_1**:** The retailer i ($i = 1, 2$) does not adopt the quantity discount proposed by the wholesaler. When the retailer i chooses this option, she/he purchases the products from the wholesaler at an initial price in the absence of the discount, and she/he determines her/himself an optimal order quantity which maximizes her/his own total profit per unit of time.

Option V_2**:** The retailer i accepts the quantity discount proposed by the wholesaler.

The main notations used in this paper are listed below:

$Q_i^{(j)}$**:** the order quantity per lot for the retailer i under Option $V_j (i = 1, 2,\ j = 1, 2)$.

$T_i^{(j)}$**:** the length of the order cycle for the retailer i under Option V_j.

h_i**:** the inventory holding cost for the retailer i per item and unit of time.

a_i**:** the ordering costs per lot for the retailer i.

θ_i**:** the deterioration rate of the retailer i's inventory.

p_b**:** the retailer's unit selling price, i.e., unit purchasing price for her/his customers.

μ_i**:** the constant demand rate of the product for the retailer i.

c_s**:** the wholesaler's unit acquisition cost (unit purchasing cost from the upper-leveled manufacturer).

p_s**:** the wholesaler's initial unit selling price, i.e., the retailer's unit acquisition cost in the absence of the discount.

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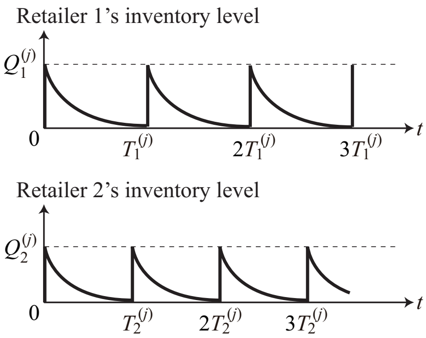

Fig. 1. Transition of Retailers' Inventory Level

y: the discount rate for the wholesale price proposed by the wholesaler. The wholesaler therefore offers a unit discounted price of $(1-y)p_s$ $(0 \le y < 1)$.

$S^{(j)}$: the wholesaler's order quantity per lot under Option V_j.

h_s: the wholesaler's inventory holding cost per item and unit of time.

a_s: the wholesaler's ordering cost per lot.

θ_s: the deterioration rate of the wholesaler's inventory.

The assumptions in this study are as follows:

1) The retailers' inventory levels are continuously depleted due to the combined effects of its demand and deterioration. In contrast, the wholesaler's inventory is only depleted by deterioration except when the wholesaler ships the products to the retailer.
2) The rate of replenishment is infinite and the delivery is instantaneous.
3) Backlogging and shortage are not allowed.
4) The quantity of the item can be treated as continuous for simplicity.
5) Both the wholesaler and the retailers are rational and use only pure strategies.

III. RETAILER'S TOTAL PROFIT

This section formulates the retailer i's total profit per unit of time for the Option V_1 and V_2 available to both the retailers.

Figure 1 shows two retailers' transitions of inventory level under Option V_j $(j = 1, 2)$.

A. Under Option V_1

If the retailer i chooses Option V_1, her/his order quantity per lot and her/his unit acquisition cost are respectively given by $Q_i^{(1)} = Q\left(T_i^{(1)}\right)$ and p_s, where p_s is the unit initial price in the absence of the discount. In this case, she/he determines her/himself the optimal order quantity $Q_i^{(1)} = Q_i^*$ which maximize her/his total profit per unit of time.

Since the inventory is depleted due to the combined effect of its demand and deterioration, the inventory level, $I_B(t)$, at time t during $[0, T_1)$ can be expressed by the following differential equation:

$$dI_B(t)/dt = -\theta_i I_B(t) - \mu_i. \tag{1}$$

By solving the differential equation in Eq. (1) with a boundary condition $I_B(T_i^{(1)}) = 0$, the retailer's inventory level at time t is given by

$$I_B(t) = \rho_i \left\{ e^{\theta_i [T_i^{(1)} - t]} - 1 \right\}, \tag{2}$$

where $\rho_i = \mu_i / \theta_i$.

Therefore, the initial inventory level, $I_B(0)$ $(= Q_i^{(1)} = Q\left(T_i^{(1)}\right))$, in the order cycle becomes

$$Q\left(T_i^{(1)}\right) = \rho_i \left[e^{\theta_{b_i} T_i^{(1)}} - 1 \right]. \tag{3}$$

On the other hand, the cumulative inventory, $A(T_1)$, held during $[0, T_1)$ is expressed by

$$A\left(T_i^{(1)}\right) = \int_0^{T_i^{(1)}} I_B(t)dt = \rho_i \left\{ \frac{\left[e^{\theta_i T_i^{(1)}} - 1 \right]}{\theta_i} - 1 \right\}. \tag{4}$$

Hence, the retailer i's total profit per unit of time under Option V_1 is given by

$$\pi_i^{(1)}\left(T_i^{(1)}\right) = \frac{p_b \int_0^{T_i^{(1)}} \mu_i dt - p_s Q\left(T_i^{(1)}\right) - h_i A\left(T_i^{(1)}\right) - a_i}{T_i^{(j)}} = \rho_i(p_b\theta_i + h_i) - \frac{\left(p_s + \frac{h_i}{\theta_i}\right) Q\left(T_i^{(1)}\right) + a_i}{T_i^{(1)}}. \tag{5}$$

In the following, the results of analysis are briefly summarized:

There exists a unique finite $T_i^{(1)} = T_i^{(1)*}$ (> 0) which maximizes $\pi_i^{(1)}(T_i^{(1)})$ in Eq. (5). The optimal order quantity is therefore given by

$$Q_i^{(1)*} = \rho_i \left[e^{\theta_i T_i^{(1)*}} - 1 \right]. \tag{6}$$

The total profit per unit of time becomes

$$\pi_i^{(1)*} = \rho_i \left[(p_b\theta_i + h_i) - \theta_i \left(p_s + \frac{h_i}{\theta_i} \right) e^{\theta_i T_i^{(1)*}} \right]. \tag{7}$$

B. Under Option V_2

If the retailer i chooses Option V_2, the order quantity and unit discounted wholesale price are respectively given by $Q_i^{(2)} = Q\left(T_i^{(2)}\right) = \rho_i \left[e^{\theta_i T_i^{(2)}} - 1 \right]$ and $(1-y)p_s$. The retailer i's total profit per unit of time can therefore be expressed by

$$\pi_i^{(2)}\left(T_i^{(2)}, y\right) = \rho_i(p_b\theta_i + h_i) - \frac{\left[(1-y)p_s + \frac{h_i}{\theta_i} \right] Q\left(T_i^{(2)}\right) + a_i}{T_i^{(2)}}. \tag{8}$$

Let $p^{(1)}$ and $p^{(2)}$ be defined by $p^{(1)} = p_s$ and $p^{(2)} = (1-y)p_s$, respectively, then $\pi_i^{(1)}\left(T_i^{(1)}\right)$ in Eq. (5) and $\pi_i^{(2)}\left(T_i^{(2)}, y\right)$ in Eq. (8) can be rewritten as follows:

$$\pi_i^{(j)} = \rho_i(p_b\theta_i + h_i) - \frac{\left[p^{(j)} + \frac{h_i}{\theta_i} \right] Q\left(T_i^{(j)}\right) + a_i}{T_i^{(j)}}. \tag{9}$$

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IV. RETAILERS' OPTIMAL POLICY UNDER THE COOPERATIVE GAME

This section discusses a cooperative game between two retailers. In this study, we focus on the situation where there are two sections in the same store, and therefore we assume that the transportation cost of the product from one retailer to the other is zero. This signifies that the retailers can possibly reduce their costs by adopting the strategy that either of the retailers purchases the products from the wholesaler and stocks them, and then she/he distributes the products to the other retailer.

The joint profit function per unit of time can therefore be expressed by

$$J\left(T_i^{(j)}\right) = \frac{\mu_1+\mu_2}{\theta_i}(p_b\theta_i+h_i) - \frac{\left(p_s+\frac{h_i}{\theta_i}\right)Q\left(T_i^{(j)}\right)+a_i.}{T_i^{(j)}}. \tag{10}$$

A. Under Option V_1

Under Option V_1, we can prove that there exist a unique finite positive $T_i^{(1)} = T_i^{(1)^*}$, which maximizes $J\left(T_i^{(j)}\right)$ in Eq. (10), and the maximum joint profit becomes

$$J^{(1)^*} = \max_{i=1,2} \hat{J}_i^{(1)}, \tag{11}$$

where

$$\hat{J}_i^{(1)} = \frac{\mu_1+\mu_2}{\theta_i} \times \left[(p_b\theta_i+h_i) - (p_s\theta_i+h_i)\, e^{\theta_i T_i^{(1)^*}}\right]. \tag{12}$$

Equation (12) signifies a local maximum value of the joint profit when the retailer i is in charge of ordering and inventory management.

Let R denote the retailer who is in charge of ordering and inventory control and bargains with the wholesaler on behalf of two retailers, and then R is given by

$$R = \begin{cases} 1, & \text{if } \hat{J}_1^{(1)} \ge \hat{J}_2^{(1)}, \\ 2, & \text{if } \hat{J}_1^{(1)} < \hat{J}_2^{(1)}. \end{cases} \tag{13}$$

The analysis with respect to comparing $J_1^{(1)}$ with $J_2^{(1)}$ becomes considerably complicated since Eq. (12) includes the term $T_i^{(1)^*}$ which is determined by a nonlinear equation solution. Neglecting higher order terms of θ_i in the expansion of $e^{\theta_i T_i^{(1)}}$, we have $e^{\theta_i T_i^{(1)}} \approx 1+\theta_i T_i^{(1)} + [\theta_i T_i^{(1)}]^2/2$.

In this case, $J^{(1)^*}$ in Eq. (11) can be expressed as

$$J^{(1)^*} = \begin{cases} \hat{J}_1^{(1)}, & \text{if } a_1(p_s\theta_1+h_1) \le a_2(p_s\theta_2+h_2), \\ \hat{J}_2^{(1)}, & \text{if } a_1(p_s\theta_1+h_1) > a_2(p_s\theta_2+h_2). \end{cases} \tag{14}$$

It can also be shown in this case that $\hat{J}_R^{(1)} > \sum_{i=1}^{2} \pi_i^{(1)^*}$. We therefore focus on the case where $\hat{J}_R^{(1)} > \sum_{i=1}^{2} \pi_i^{(1)^*}$ in the following sections.

B. Under Option V_2

The wholesaler offers the quantity discount to the retailer R $(R=1,2)$ which is defined by the previous subsection.

Under Option V_2, the retailer R's joint profit per unit of time can be expressed by

$$J^{(2)}\left(T_R^{(2)}, y\right) = \frac{\mu_1+\mu_2}{\theta_R}(p_b\theta_R+h_R) - \frac{\left[(1-y)p_s+\frac{h_R}{\theta_R}\right]Q\left(T_R^{(2)}\right)+a_R}{T_R^{(2)}}. \tag{15}$$

V. RETAILERS' OPTIMAL RESPONSE AND SHAPLEY VALUE IMPUTATION

A. Retailers' optimal response

This subsection discusses the retailer R's optimal response. The retailer R prefers Option V_1 over Option V_2 if $J^{(1)^*} > J^{(2)}$, but when $J^{(1)^*} < J^{(2)}\left(T_R^{(2)}, y\right)$, she/he prefers V_2 to V_1. The retailer R is indifferent between the two options if $J^{(1)^*} = J^{(2)}\left(T_R^{(2)}, y\right)$, which is equivalent to

$$y = \frac{1}{p_s Q\left(T_R^{(2)}\right)} \times \left\{\left[Q\left(T_R^{(2)}\right) - \rho\theta_R T_R^{(2)} e^{\theta_R T_R^{(1)*}}\right] \times \left(p_s + \frac{h_R}{\theta_R}\right) + a_R\right\}. \tag{16}$$

Let us denote, by $\psi\left(T_R^{(2)}\right)$, the right-hand-side of Eq. (16). It can easily be shown from Eq. (16) that $\psi(T_2)$ is increasing in $T_R^{(2)}$ $(\ge T_R^{(1)*})$.

The maximum value of the joint profit is given by

$$J^* = \begin{cases} J^{(1)^*}, & \text{if } J^{(1)^*} \ge J^{(2)}\left(T_R^{(2)}, y\right), \\ J^{(2)}\left(T_R^{(2)}, y\right), & \text{if } J^{(1)^*} < J^{(2)}\left(T_R^{(2)}, y\right). \end{cases} \tag{17}$$

B. Shapley value imputation

We focus on the case where two retailers maximize their joint profit and share their cooperative profit according to the Shapley value[9], [10]. In this subsection, we determine the retailers' allocation of profit based on the concept of Shapley value. The Shapley value is one of the commonly used sharing mechanisms in static cooperation games with transferable payoff[9], [10].

Some additional notations used in this subsection are listed below.

x_i**:** the retailer i's allocation of the cooperative profit $(i = 1,2)$.

v**:** a characteristic function of the coalition, i.e., $v(1) = \pi_1^{(1)^*}$, $v(2) = \pi_2^{(1)^*}$ and $v(1,2) = J^*$.

Vector $\boldsymbol{x} = (x_1, x_2)$ is called an imputation if it satisfies the following two conditions:

(1) Individual rationality: $x_i \ge \pi_i^{(1)^*}$ $(i=1,2)$

(2) Group rationality: $x_1 + x_2 = J^*$

The Shapley value gives an imputation rule for retailer i $(i \in [1,2] \equiv K)$ described by Eq. (18).

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$$\phi_i = \sum_{S \subset K} \frac{(s-1)!(n-s)!}{n!} \left[v(S) - v(S \setminus \{i\}) \right], \quad (18)$$

where $|S| = s$. In this study, therefore, the imputation $x_1 = \phi_1$ and $x_2 = \phi_2$ are respectively given by

$$\phi_1 = \frac{\pi_1^{(1)^*} + J^* - \pi_2^{(1)^*}}{2}, \quad (19)$$

$$\phi_2 = \frac{J^* - \pi_1^{(1)^*} + \pi_2^{(1)^*}}{2}. \quad (20)$$

VI. WHOLESALER'S TOTAL PROFIT AND OPTIMAL POLICY

The retailers adopt the cooperative strategy to increase their profits as mentioned in Section IV. The wholesaler can therefore regard the retailers as a single retailer since either of the retailers is in charge of ordering and inventory management. In this case, the wholesaler's total profit per unit of time can be formulated in the same manner as our previous formulation[8]. For this reason, in the following we briefly summarize the results associated with the wholesaler's profits under Option V_1 and V_2 and her/his optimal policy.

The length of the wholesaler's order cycle is given by $N^{(j)} T_R^{(j)}$ under Option V_j ($j = 1, 2$), where $N^{(j)}$ is a positive integer. This is because the wholesaler can possibly improve her/his total profit by increasing the length of her/his order cycle from $T_R^{(j)}$ to $N^{(j)} T_R^{(j)}$.

The wholesaler's inventory is only depleted by deterioration except when the wholesaler ships the products to the retailer, as in assumption (1). The wholesaler's inventory level, $I_S(t)$, at time t can therefore be expressed by the following differential equation:

$$dI_S(t)/dt = -\theta_s I_S(t), \quad (21)$$

with a boundary condition $I_S(jT_R^{(j)}) = z_k(T_R^{(j)})$ under Option V_j, where $z_k(T_R^{(j)})$ denotes the remaining inventory at the end of the kth shipping cycle.

In this case, the wholesaler's total profit per unit of time under Option V_1 is given by

$$\begin{aligned} & P^{(1)} \left(N^{(1)}, T_R^{(1)^*} \right) \\ &= \frac{\left(p_s + \frac{h_s}{\theta_s} \right) Q \left(T_R^{(1)^*} \right)}{T_R^{(1)^*}} \\ &\quad - \frac{\left(c_s + \frac{h_s}{\theta_s} \right) S \left(N^{(1)}, T_R^{(1)^*} \right) + a_s}{N^{(1)} T_R^{(1)^*}}. \end{aligned} \quad (22)$$

In contrast, under Option V_2, the wholesaler's total profit per unit of time becomes

$$\begin{aligned} & P^{(2)} \left(N^{(2)}, T_R^{(2)}, y \right) \\ &= \frac{\left[(1-y) p_s + \frac{h_s}{\theta_s} \right] Q \left(T_R^{(2)} \right)}{T_R^{(2)}} \\ &\quad - \frac{\left(c_s + \frac{h_s}{\theta_s} \right) S \left(N^{(2)}, T_R^{(2)} \right) + a_s}{N^{(2)} T_R^{(2)}}, \end{aligned} \quad (23)$$

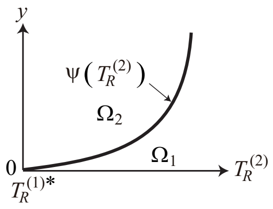

Fig. 2. Characterization of retailer's optimal responses

where

$$Q \left(T_R^{(j)} \right) = \frac{\mu_1 + \mu_2}{\theta_R} \left(e^{\theta_R T_R^{(j)}} - 1 \right), \quad (24)$$

$$S \left(N^{(j)}, T_R^{(j)} \right) = Q \left(T_R^{(j)} \right) \frac{e^{N^{(j)} \theta_s T_R^{(j)}} - 1}{e^{\theta_s T_R^{(j)}} - 1}. \quad (25)$$

The wholesaler's optimal values for $T_R^{(2)}$ and y can be obtained by maximizing her/his total profit per unit of time considering the retailer's optimal response which was discussed in Subsection V-A. Henceforth, let Ω_j ($j = 1, 2$) be defined by

$$\Omega_1 = \left\{ \left(T_R^{(2)}, y \right) \mid y \le \psi \left(T_R^{(2)} \right) \right\},$$

$$\Omega_2 = \left\{ \left(T_R^{(2)}, y \right) \mid y \ge \psi \left(T_R^{(2)} \right) \right\}.$$

Figure 2 depicts the region of Ω_j ($j = 1, 2$) on the $\left(T_R^{(2)}, y \right)$ plane.

A. Optimal Policy under Option V_1

If $\left(T_R^{(2)}, y \right) \in \Omega_1 \setminus \Omega_2$ in Fig. 2, the retailer will naturally select Option V_1. In this case, the wholesaler can maximize her/his total profit per unit of time independently of T_2 and y on the condition of $\left(T_R^{(2)}, y \right) \in \Omega_1 \setminus \Omega_2$. Hence, the wholesaler's locally maximum total profit per unit of time in $\Omega_1 \setminus \Omega_2$ becomes

$$P^{(1)^*} = \max_{N^{(1)} \in N} P^{(1)} \left(N^{(1)}, T_R^{(1)^*} \right), \quad (26)$$

where N signifies the set of positive integers.

B. Optimal Policy under Option V_2

On the other hand, if $\left(T_R^{(2)}, y \right) \in \Omega_2 \setminus \Omega_1$, the retailer's optimal response is to choose Option V_2. Then the wholesaler's locally maximum total profit per unit of time in $\Omega_2 \setminus \Omega_1$ is given by

$$P^{(2)^*} = \max_{N^{(2)} \in N} \hat{P}^{(2)} \left(N^{(2)} \right), \quad (27)$$

where

$$\begin{aligned} & \hat{P}^{(2)} \left(N^{(2)} \right) \\ &= \max_{\left(T_R^{(2)}, y \right) \in \Omega_2 \setminus \Omega_1} P^{(2)} \left(N^{(2)}, T_R^{(2)}, y \right). \end{aligned} \quad (28)$$

More precisely, we should use "sup" instead of "max" in Eq. (28).

For a given $N^{(2)}$, we show below the existence of the wholesaler's optimal quantity discount pricing policy $\left(T_R^{(2)}, y \right) = \left(T_R^{(2)^*}, y^* \right)$ which attains Eq. (28). It can easily

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be proven that $P^{(2)}\left(N^{(2)},T_R^{(2)},y\right)$ in Eq. (23) is strictly decreasing in y, and consequently the wholesaler can attain $\hat{P}^{(2)}\left(N^{(2)}\right)$ in Eq. (28) by letting $y\to\psi\left(T_R^{(2)}\right)+0$. By letting $y=\psi\left(T_R^{(2)}\right)$ in Eq. (23), the total profit per unit of time on $y=\psi\left(T_R^{(2)}\right)$ becomes

$$\begin{aligned}&P^{(2)}\left(N^{(2)},T_R^{(2)}\right)\\&=\rho\left(p_s+h_R/\theta_R\right)\theta_R e^{\theta_R T_R^{(1)^*}}\\&\quad-\frac{1}{N^{(2)}T_R^{(2)}}\cdot\left[C\cdot S\left(N^{(2)},T_R^{(2)}\right)\right.\\&\qquad\left.-H\left(N^{(2)}\right)Q\left(T_R^{(2)}\right)+a_bN^{(2)}+a_s\right],\quad(29)\end{aligned}$$

where

$$C=(c_s+h_s/\theta_s),\quad(30)$$

$$H\left(N^{(2)}\right)=(h_s/\theta_s-h_R/\theta_R)N^{(2)}.\quad(31)$$

Let us now define $L\left(N^{(2)}\right)$ as follows:

$$\begin{aligned}&L\left(N^{(2)}\right)\\&\equiv\frac{C\theta_sT_R^{(2)}Q\left(T_R^{(2)}\right)}{\left(e^{\theta_sT_2}-1\right)^2}\\&\quad\times\left\{N^{(2)}e^{N^{(2)}\theta_sT_R^{(2)}}\left[e^{\theta_sT_R^{(2)}}-1\right]\right.\\&\qquad\left.-e^{\theta_sT_R^{(2)}}\left[e^{N^{(2)}\theta_sT_R^{(2)}}-1\right]\right\}\\&\quad+\left[\rho\theta_Re^{\theta_RT_R^{(2)}}T_R^{(2)}-Q\left(T_R^{(2)}\right)\right]\\&\quad\times\left[C\frac{e^{N^{(2)}\theta_sT_R^{(2)}}-1}{e^{\theta_sT_R^{(2)}}-1}-H\left(N^{(2)}\right)\right].\quad(32)\end{aligned}$$

We here summarize the results of analysis in relation to the optimal quantity discount policy which attains $\hat{P}^{(2)}\left(N^{(2)}\right)$ in Eq. (28) when N_2 is fixed to a suitable value.

1) $N_2=1$:
In this case, there exists a unique finite $\tilde{T}_R^{(2)}$ $(>T_R^{(1)^*})$ which maximizes $P^{(2)}\left(N^{(2)},T_R^{(2)}\right)$ in Eq. (29), and therefore $\left(T_R^{(2)^*},y^*\right)$ is given by

$$\left(T_R^{(2)^*},y^*\right)\to\left(\tilde{T}_R^{(2)},\tilde{y}\right),\quad(33)$$

where $\tilde{y}=\psi\left(\tilde{T}_R^{(2)}\right)$.
The wholesaler's total profit then becomes

$$\begin{aligned}\hat{P}^{(2)}\left(N^{(2)}\right)=&\frac{\mu_1+\mu_2}{\theta_R}\theta_R\big[(p_s+h_R/\theta_R)\,e^{\theta_RT_R^{(1)^*}}\\&-(c_s+h_R/\theta_R)\,e^{\theta_RT_R^{(2)^*}}\big].\quad(34)\end{aligned}$$

2) $N_2\geq 2$:
Let us define $\tilde{T}_R^{(2)}$ $(>T_R^{(1)^*})$ as the unique solution (if it exists) to

$$L(T_2)=a_RN_2+a_s.\quad(35)$$

In this case, the optimal quantity discount pricing policy is given by Eq. (33).

TABLE I
SENSITIVITY ANALYSIS

(a) Under Option V_1

a_s	$S^{(1)*}$	$N^{(1)*}$	$P^{(1)*}$	$x_1^{(1)}$	$x_2^{(1)}$
500	73.696	1	2216.637	1621.957	1309.561
1000	152.282	2	2140.522	1621.957	1309.561
2000	152.282	2	2062.686	1621.957	1309.561
3000	152.282	2	1984.851	1621.957	1309.561

(b) Under Option V_2

a_s	$S^{(2)*}$	$N^{(2)*}$	$P^{(2)*}$	$x_1^{(2)}$	$x_2^{(2)}$
500	127.866	1	2261.602	1621.957	1309.561
1000	145.928	1	2218.253	1621.957	1309.561
2000	176.955	1	2143.79	1621.957	1309.561
3000	203.659	1	2079.699	1621.957	1309.561

C. Optimal Policy under Option V_1 and V_2

In the case of $\left(T_R^{(2)},y\right)\in\Omega_1\cap\Omega_2$, the retailer is indifferent between Option V_1 and V_2. For this reason, this study confines itself to a situation where the wholesaler does not use a quantity discount policy $\left(T_R^{(2)},y\right)\in\Omega_1\cap\Omega_2$.

VII. NUMERICAL EXAMPLES

Table I reveals the results of sensitively analysis in reference to $x_1^{(j)}$, $x_2^{(j)}$, $S^{(j)^*}(=S(N^{(j)*}T_R^{(j)*}))$ and $P^{(j)^*}(=P(N^{(j)*}T_R^{(j)*}))$ under Option V_j $(j=1,2)$ for $(p_b,p_s,c_s,h_s,\theta_s,a_s)=(600,300,100,1,0.01,1000)$, $(h_1,\theta_1,a_1,\mu_1)=(1.1,0.013,1200,6)$ and $(h_2,\theta_2,a_2,\mu_2)=(1.5,0.015,1300,5)$ when $a_s=500$, 1000, 2000 and 3000. In this case, we obtain $\pi_1^{(1)^*}=1526.521$ and $\pi_2^{(1)^*}=1214.124$, which are independent of a_s.

In Table I(a) indicates that both $S^{(1)^*}$ and $N^{(1)^*}$ are non-decreasing in a_s. As mentioned in Section II, under Option V_1, the retailer does not adopt the quantity discount offered by the wholesaler, which signifies that the wholesaler cannot control the retailer's ordering schedule. In this case, the wholesaler's cost associated with ordering should be reduced by increasing her/his own length of order cycle and lot size by means of increasing $N^{(1)}$. Table I(a) also implies $x_i^{(1)}>\pi_i^{(1)^*}$ $(i=1,2)$.

Table I(b) shows that, under Option V_2, S_2^* increases with a_s, in contrast, N_2^* takes a constant value, i.e., $N_2^*=1$. Under Option V_2, the retailer accepts the quantity discount proposed by the wholesaler. The wholesaler's lot size can therefore be increased by stimulating the retailer to alter her/his order quantity per lot through the quantity discount strategy. If the wholesaler increases N_2 one step, her/his lot size also significantly jumps up since N_2 takes a positive integer. Under this option, the wholesaler should increase her/his lot size using the quantity discount rather than increasing N_2 when a_s takes larger values. Table I reveals that we have $P_1^*<P_2^*$. This indicates that using the quantity discount strategy can increase the wholesaler's total profit per unit of time. We can notice in Table I that $x_i^{(1)}=x_i^{(2)}$ $(i=1,2)$ for each value of a_s. This signifies that the retailers' profits do not increase if they accept the quantity discount proposed by the wholesaler.

VIII. CONCLUSION

In this study, we have discussed a quantity discount problem between a single wholesaler and two retailers under circumstances where both the wholesaler's and the retailers' inventory levels of the product are depleted not only by demand but also by deterioration. In Japanese large-scale super markets, there are two or more sections at which the same items are sold in the same store. In many cases, managers of these sections independently order the items since they are under competition and are evaluated separately by their supervisor. They can possibly reduce their costs if either of them purchases the items for two sections in cooperation with each other.

The wholesaler is interested in increasing her/his profit by controlling the retailers' order quantity through the quantity discount strategy. The retailers attempt to maximize their profits by considering both whether to cooperate with each other and whether to accept the wholesaler's proposal. The analysis with respect to comparing the cooperative solution with non-cooperative one becomes considerably complicated since the local maximum values of the players' total profit per unit of time cannot be expressed as closed form expressions. For this reason, we have shown that the retailers can increase their profits by means of adopting the cooperative strategy in the case where higher order terms of the deterioration rate in the expansion of the exponential can be ignored. Focusing on such a situation, the wholesaler can regard the retailers as a single retailer since either of the retailers is in charge of ordering and inventory management. In this case, we can formulate the above problem as a Stackelberg game between the wholesaler and the retailers in the same manner as our previous formulation[8]. It should be pointed out that our results are obtained under the situation where the inventory holding cost is independent of the value of the item. The relaxation of such a restriction is an interesting extension.

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