A New Structure and Design Method for Variable Fractional-Delay 2-D FIR Digital Filters

Jong-Jy Shyu, Soo-Chang Pei, Yun-Da Huang and Yu-Shiang Chen

***Abstract*—In this paper, a new structure is proposed for the computational design of variable fractional-delay (VFD) 2-D FIR digital filters. Based on the Taylor series expansion of the desired frequency response, a prefilter-subfilter cascaded structure can be derived. For the 1-D differentiating prefilters and the 2-D quadrantally symmetric subfilters, they can be designed simply by the least-squares method. Design example shows that the required number of independent coefficients of the proposed system is much less than that of the existing structure, while the performance of the designed VFD 2-D filters is still better under the cost of larger delays.**

***Index Terms*— Farrow structure, variable fractional-delay filter, 2-D FIR filter, least-squares method, 2-D quadrantally symmetric filter, subfilter.**

I. Introduction

Conventionally, the transfer function of a variable fractional-delay (VFD) 2-D FIR digital filter is given by

$$H(z_1,z_2,p_1,p_2)=\sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2} h_{n_1n_2}(p_{1,},p_2)\,z_1^{-n_1}z_2^{-n_2} \tag{1}$$

where

$$h_{n_1n_2}(p_1,p_2)=\sum_{m_1=0}^{M}\sum_{m_2=0}^{M} h(n_1,n_2,m_1,m_2)\,p_1^{m_1}p_2^{m_2}\ . \tag{2}$$

Hence, (1) can be represented by

$$H(z_1,z_2,p_1,p_2)=\sum_{m_1=0}^{M}\sum_{m_2=0}^{M}\hat{G}_{m_1m_2}(z_1,z_2)\ p_1^{m_1}p_2^{m_2} \tag{3}$$

where the 2-D subfilters

$$\hat{G}_{m_1m_2}(z_1,z_2)=\sum_{n_1=0}^{N_1}\sum_{n_2=0}^{N_2} h(n_1,n_2,m_1,m_2)\,z_1^{-n_1}z_2^{-n_2}, \tag{4}$$

and the system can be implemented by a 2-D Farrow structure [1].

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Jong-Jy Shyu and Yu-Shiang Chen are with Department of Electrical Engineering, National University of Kaohsiung, Kaohsiung, Taiwan (Phone:886-7-5919236; Fax:886-7-5919374; e-mail: jshyu@nuk.edu.tw, kyo1230147@yahoo.com.tw).

Soo-Chang Pei and Yun-Da Huang are with Graduate Institute of Communication Engineering, National Taiwan University, Taipei, Taiwan (e-mail: pei@cc.ee.ntu.edu.tw, d97942016@ntu.edu.tw).

VFD digital filters belong to the branch of variable digital filters which are applied to where frequency characteristics need to be adjusted online without redesigning the system. For the past decade, several works have been proposed for the design of variable digital filters [1]-[17] due to their wide applications in signal processing and communication systems. By the function, they are generally classified into two main categories. One is the filters with variable magnitude characteristics such as cutoff frequencies or magnitude responses [2]-[8], and the other is the filters with variable fractional delay [1][9]-[17].

In this paper, the design of VFD 2-D FIR digital filters will be investigated. Comparing with the conventional 2-D Farrow structure presented recently in [1], a prefilter-subfilter cascaded structure is proposed. The structure is developed based on the Taylor series expansion of the desired frequency response. In [1], there are four types of 2-D quadrantally symmetric/ antisymmetric filters [18][19] to be designed. But, only two 1-D differentiating prefilters and one type of 2-D quadrantally symmetric subfilters are needed to be designed in this paper. By the closed relationships among the proposed structure, the required number of independent coefficients of the designed sysem is much less than that in [1] while the performance of the designed filters is still better than that in [1] under the cost of larger delays. The phenomenon will become significant for wider band VFD 2-D filter design by using higher-order 1-D prefilters and lower-order 2-D subfilters.

This paper is organized as following. In Section II, the proposed prefilter-subfilter cascaded structure is derived from the Taylor series expansion of the desired frequency response. And the design of the mentioned prefilters and subfilters for even *M* is presented in Section III. For simplicity, the general least-squares method is applied, and design example will be presented to demonstrate the effectiveness of the presented method. Finally, the conclusions are given in Section IV.

II. The Proposed Structure

For designing a VFD 2-D FIR filter, the desired frequency response is given by

$$H_d(\omega_1,\omega_2,p_1,p_2)=M(\omega_1,\omega_2)\,e^{-j\left[\omega_1(I_1+p_1)+\omega_2(I_2+p_2)\right]} \tag{5}$$

where $M(\omega_1,\omega_2)$ is the desired magnitude response, I_1 and

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I_2 are the prescribed group-delays with respect to ω_1-axis and ω_2-axis, respectively, and $p_1, p_2 \in [-0.5, 0.5]$. For simplicity, only quadrantally symmetric magnitude response $M(\omega_1, \omega_2)$ is considered in this paper. By Taylor series expansion,

$$
\begin{aligned}
e^{-j(\omega_1 p_1 + \omega_2 p_2)} &= \sum_{m_1=0}^{\infty} \frac{(-j\omega_1 p_1)^{m_1}}{m_1!} \cdot \sum_{m_2=0}^{\infty} \frac{(-j\omega_2 p_2)^{m_2}}{m_2!} \\
&\approx \sum_{m_1=0}^{M} \frac{(-j\omega_1 p_1)^{m_1}}{m_1!} \cdot \sum_{m_2=0}^{M} \frac{(-j\omega_2 p_2)^{m_2}}{m_2!}
\end{aligned} \tag{6}
$$

for sufficiently large M. In this paper, the case for odd M is considered first, and the case for even M will be discussed in Section IV. Let $M = 2\hat{M} + 1$, then (6) becomes

$$
\begin{aligned}
& e^{-j(\omega_1 p_1 + \omega_2 p_2)} \\
&\approx \left[\sum_{m_1=0}^{\hat{M}} (-1)^{m_1} \frac{(\omega_1 p_1)^{2m_1}}{(2m_1)!} + (-j\omega_1) p_1 \sum_{m_1=0}^{\hat{M}} \frac{(-1)^{m_1}}{2m_1+1} \frac{(\omega_1 p_1)^{2m_1}}{(2m_1)!} \right] \times \\
&\quad \left[\sum_{m_2=0}^{\hat{M}} (-1)^{m_2} \frac{(\omega_2 p_2)^{2m_2}}{(2m_2)!} + (-j\omega_2) p_2 \sum_{m_2=0}^{\hat{M}} \frac{(-1)^{m_2}}{2m_2+1} \frac{(\omega_2 p_2)^{2m_2}}{(2m_2)!} \right] \\
&= \sum_{m_1=0}^{\hat{M}} \sum_{m_2=0}^{\hat{M}} (-1)^{m_1+m_2} \frac{(\omega_1 p_1)^{2m_1} (\omega_2 p_2)^{2m_2}}{(2m_1)!(2m_2)!} \\
&\quad + (-j\omega_1) p_1 \sum_{m_1=0}^{\hat{M}} \sum_{m_2=0}^{\hat{M}} \frac{(-1)^{m_1+m_2}}{2m_1+1} \frac{(\omega_1 p_1)^{2m_1} (\omega_2 p_2)^{2m_2}}{(2m_1)!(2m_2)!} \\
&\quad + (-j\omega_2) p_2 \sum_{m_1=0}^{\hat{M}} \sum_{m_2=0}^{\hat{M}} \frac{(-1)^{m_1+m_2}}{2m_2+1} \frac{(\omega_1 p_1)^{2m_1} (\omega_2 p_2)^{2m_2}}{(2m_1)!(2m_2)!} \\
&\quad + (-j\omega_1)(-j\omega_2) p_1 p_2 \times \\
&\qquad \sum_{m_1=0}^{\hat{M}} \sum_{m_2=0}^{\hat{M}} \frac{(-1)^{m_1+m_2}}{(2m_1+1)(2m_2+1)} \frac{(\omega_1 p_1)^{2m_1} (\omega_2 p_2)^{2m_2}}{(2m_1)!(2m_2)!}
\end{aligned} \tag{7}
$$

By (5) and (7), the transfer function of the VFD 2-D FIR filter is represented by

$$
\begin{aligned}
& H(z_1, z_2, p_1, p_2) \\
&= z_1^{-\frac{N_{d1}}{2}} z_2^{-\frac{N_{d2}}{2}} \sum_{m_1=0}^{\hat{M}} \sum_{m_2=0}^{\hat{M}} G_{2m_1,2m_2}(z_1, z_2)\, p_1^{2m_1} p_2^{2m_2} \\
&\quad + z_2^{-\frac{N_{d2}}{2}} D_1(z_1) \sum_{m_1=0}^{\hat{M}} \sum_{m_2=0}^{\hat{M}} \frac{1}{2m_1+1} G_{2m_1,2m_2}(z_1, z_2)\, p_1^{2m_1+1} p_2^{2m_2} \\
&\quad + z_1^{-\frac{N_{d1}}{2}} D_2(z_2) \sum_{m_1=0}^{\hat{M}} \sum_{m_2=0}^{\hat{M}} \frac{1}{2m_2+1} G_{2m_1,2m_2}(z_1, z_2)\, p_1^{2m_1} p_2^{2m_2+1} \\
&\quad + D_1(z_1) D_2(z_2) \times \\
&\qquad \sum_{m_1=0}^{\hat{M}} \sum_{m_2=0}^{\hat{M}} \frac{1}{(2m_1+1)(2m_2+1)} G_{2m_1,2m_2}(z_1, z_2)\, p_1^{2m_1+1} p_2^{2m_2+1}
\end{aligned} \tag{8}
$$

and the proposed structure is shown in Fig. 1. In (8), the quadrantally symmetric subfilters $G_{2m_1,2m_2}(z_1, z_2)$ are characterized by

$$
G_{2m_1,2m_2}(z_1, z_2) = \sum_{n_1=0}^{N_g} \sum_{n_2=0}^{N_g} g_{m_1 m_2}(n_1, n_2) z_1^{-n_1} z_2^{-n_2} \tag{9}
$$

where N_g is assumed to be even while the Type III linear-phase prefilters $D_i(z_i)$, $i = 1, 2,$ are characterized by

$$
D_i(z_i) = \sum_{n=0}^{N_{di}} d_i(n) z_i^{-n}, \quad N_{di}: \text{even}, \quad i = 1, 2. \tag{10}
$$

After mathematical management, the frequency response of (8) can be represented by

$$
\begin{aligned}
& H(e^{j\omega_1}, e^{j\omega_2}, p_1, p_2) \\
&\quad = e^{-j\left(\frac{N_{d1}}{2} + \frac{N_g}{2}\right)\omega_1} e^{-j\left(\frac{N_{d2}}{2} + \frac{N_g}{2}\right)\omega_2} \hat{H}(\omega_1, \omega_2, p_1, p_2)
\end{aligned} \tag{11}
$$

where

$$
\begin{aligned}
& \hat{H}(\omega_1, \omega_2, p_1, p_2) \\
&= \sum_{m_1=0}^{\hat{M}} \sum_{m_2=0}^{\hat{M}} \hat{G}_{2m_1,2m_2}(\omega_1, \omega_2)\, p_1^{2m_1} p_2^{2m_2} \\
&\quad + j\hat{D}_1(\omega_1) \sum_{m_1=0}^{\hat{M}} \sum_{m_2=0}^{\hat{M}} \frac{1}{2m_1+1} \hat{G}_{2m_1,2m_2}(\omega_1, \omega_2)\, p_1^{2m_1+1} p_2^{2m_2} \\
&\quad + j\hat{D}_2(\omega_2) \sum_{m_1=0}^{\hat{M}} \sum_{m_2=0}^{\hat{M}} \frac{1}{2m_2+1} \hat{G}_{2m_1,2m_2}(\omega_1, \omega_2)\, p_1^{2m_1} p_2^{2m_2+1} \\
&\quad - \hat{D}_1(\omega_1) \hat{D}_2(\omega_2) \times \\
&\qquad \sum_{m_1=0}^{\hat{M}} \sum_{m_2=0}^{\hat{M}} \frac{1}{(2m_1+1)(2m_2+1)} \hat{G}_{2m_1,2m_2}(\omega_1, \omega_2)\, p_1^{2m_1+1} p_2^{2m_2+1},
\end{aligned} \tag{12a}
$$

$$
\hat{G}_{2m_1,2m_2}(\omega_1, \omega_2) = \sum_{n_1=0}^{\frac{N_g}{2}} \sum_{n_2=0}^{\frac{N_g}{2}} \hat{g}_{m_1 m_2}(n_1, n_2) \cos(n_1 \omega_1) \cos(n_2 \omega_2), \tag{12b}
$$

$$
\hat{D}_i(\omega_i) = \sum_{n=1}^{\frac{N_{di}}{2}} \hat{d}_i(n) \sin(n\omega_i), \quad i = 1, 2, \tag{12c}
$$

$$
\hat{g}_{m_1 m_2}(n_1, n_2) = \begin{cases}
g_{m_1 m_2}\left(\frac{N_g}{2}, \frac{N_g}{2}\right), & n_1 = n_2 = 0, \\
2g_{m_1 m_2}\left(\frac{N_g}{2} - n_1, \frac{N_g}{2}\right), & 1 \le n_1 \le \frac{N_g}{2}, \quad n_2 = 0, \\
2g_{m_1 m_2}\left(\frac{N_g}{2}, \frac{N_g}{2} - n_2\right), & n_1 = 0, \ 1 \le n_2 \le \frac{N_g}{2}, \\
4g_{m_1 m_2}\left(\frac{N_g}{2} - n_1, \frac{N_g}{2} - n_2\right), & 1 \le n_1, n_2 \le \frac{N_g}{2},
\end{cases} \tag{12d}
$$

$$
\hat{d}_i(n) = 2d_i\left(\frac{N_{di}}{2} - n\right), \quad i = 1, 2. \tag{12e}
$$

Obviously, the integers I_1 and I_2 in (5) can be set as $I_i = \frac{N_{di}}{2} + \frac{N_g}{2}$, $i = 1, 2.$

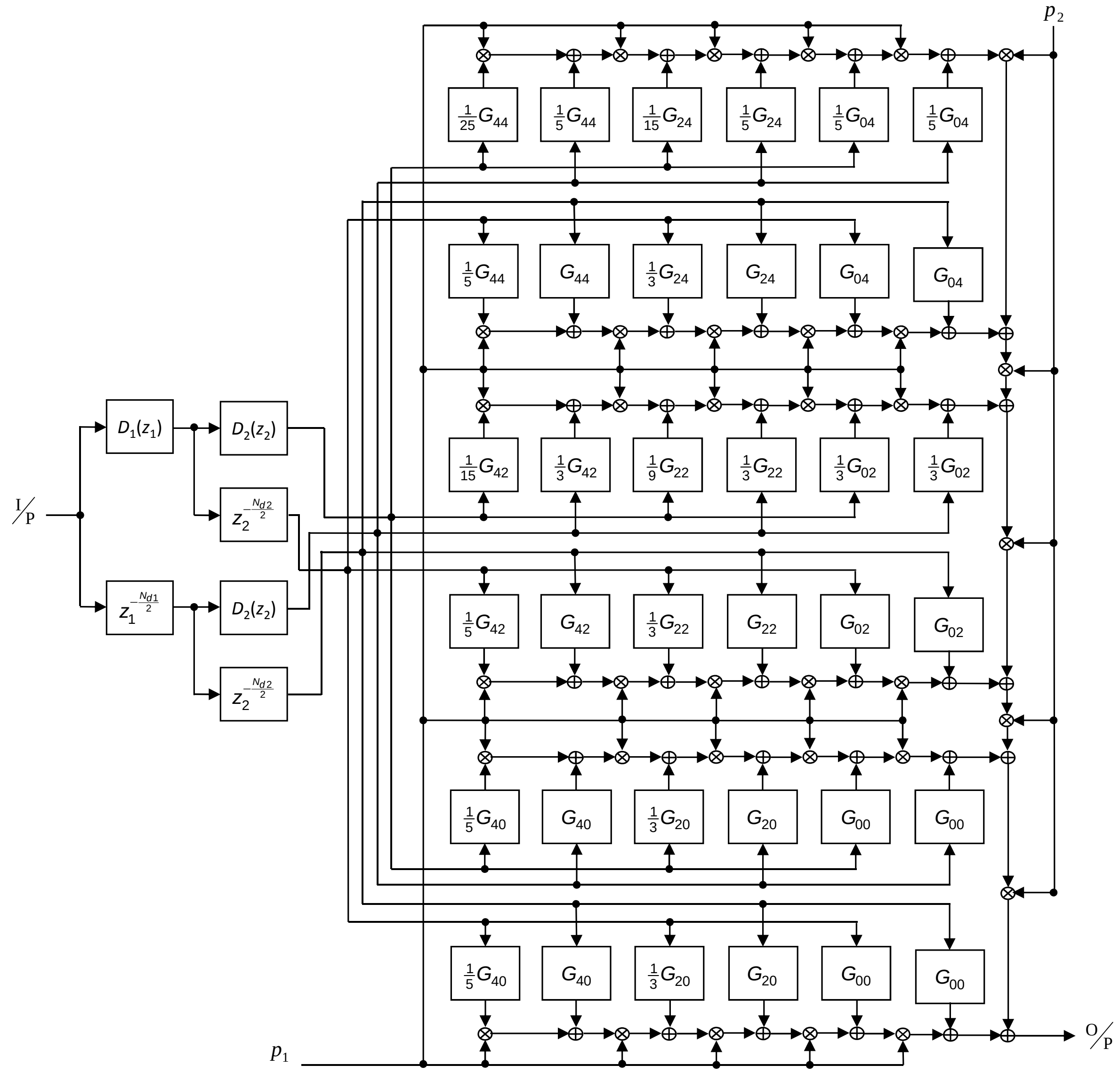

Fig. 1. The proposed structure of a VFD 2-D FIR digital filter. ($M = 5$)

III. Design of 2-D VFD FIR Digital Filters with Odd M

In this paper, it will be dealt with first for the design of the prefilters $D_1(z_1)$ and $D_2(z_2)$, and then they are applied for the design of the subfilters $G_{2m_1,2m_2}(z_1,z_2)$. Design examples will be given to demonstrate the effectiveness of the presented method.

A. *Design of the prefilters* $D_1(z_1)$ *and* $D_2(z_2)$

By (7) and (8), the prefilters $D_1(z_1)$ and $D_2(z_2)$ are used as differentiators with magnitudes $-\omega_1$ and $-\omega_2$, respectively, and their specifications depend on the magnitude response $M(\omega_1,\omega_2)$ in (5). For example, when the designed filter is an elliptically low-pass VFD filter with

$$M(\omega_1,\omega_2) = \begin{cases} 1, & \frac{\omega_1^2}{\omega_{p1}^2} + \frac{\omega_2^2}{\omega_{p2}^2} \le 1, \\ 0, & \frac{\omega_1^2}{\omega_{s1}^2} + \frac{\omega_2^2}{\omega_{s2}^2} \ge 1, \end{cases} \tag{13}$$

the prefilters $D_1(z_1)$ and $D_2(z_2)$ are designed with passband edges ω_{p1} and ω_{p2}, respectively, while their stopband edges are ω_{s1} and ω_{s2}, respectively.

Defining

$$\mathbf{d}_i = \left[\hat{d}_i(1), \hat{d}_i(2), \cdots, \hat{d}_i\left(\tfrac{N_{di}}{2}\right) \right]^T, \tag{14a}$$

$$\mathbf{s}_i(\omega_i) = \left[\sin(\omega_i), \sin(2\omega_i), \cdots, \sin\left(\tfrac{N_{di}}{2}\omega_i\right) \right]^T, \tag{14b}$$

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the magnitude responses $\hat{D}_i(\omega_i)$ of the prefilters can be represented by

$$\hat{D}_i(\omega_i) = \mathbf{d}_i^T \mathbf{s}_i(\omega_i), \quad i = 1, 2 \tag{15}$$

where the superscript T denotes a transpose operator. Hence, the objective error functions for designing the prefilters in least-squares sense can be defined by

$$e(\mathbf{d}_i) = \int_0^{\omega_{pi}} \left[-\omega_i - \hat{D}_i(\omega_i)\right]^2 d\omega_i + \int_{\omega_{si}}^{\pi} \left[\hat{D}_i(\omega_i)\right]^2 d\omega_i \tag{16}$$

$$= u_i + \mathbf{r}_i^T \mathbf{d}_i + \mathbf{d}_i^T \mathbf{Q}_i \mathbf{d}_i$$

where

$$u_i = \int_0^{\omega_{pi}} \omega_i^2 \, d\omega_i = \frac{\omega_{pi}^2}{3}, \tag{17a}$$

$$\mathbf{r}_i = 2\int_0^{\omega_{pi}} \omega_i \mathbf{s}_i(\omega_i) \, d\omega_i, \tag{17b}$$

$$\mathbf{Q}_i = \int_0^{\omega_{pi}} \mathbf{s}_i(\omega_i)\mathbf{s}_i^T(\omega_i) \, d\omega_i + \int_{\omega_{si}}^{\pi} \mathbf{s}_i(\omega_i)\mathbf{s}_i^T(\omega_i) \, d\omega_i, \tag{17c}$$

and the solutions are

$$\mathbf{d}_i = -\tfrac{1}{2}\mathbf{Q}_i^{-1}\mathbf{r}_i, \quad i = 1, 2. \tag{18}$$

B. Design of the subfilters $G_{2m_1,2m_2}(z_1, z_2)$

Similarly, by defining

$$\mathbf{g} = \Big[\hat{g}_{00}(0,0), \cdots, \hat{g}_{00}\left(\tfrac{N_g}{2}, \tfrac{N_g}{2}\right), \cdots, \hat{g}_{\hat{M}\hat{M}}(0,0), \cdots, \hat{g}_{\hat{M}\hat{M}}\left(\tfrac{N_g}{2}, \tfrac{N_g}{2}\right)\Big]^T, \tag{19a}$$

$$\mathbf{c}_{ee} = \Big[1, \cdots, \cos\left(\tfrac{N_g}{2}\omega_1\right)\cos\left(\tfrac{N_g}{2}\omega_2\right), \cdots, p_1^{2\hat{M}} p_2^{2\hat{M}}, \cdots, p_1^{2\hat{M}} p_2^{2\hat{M}} \cos\left(\tfrac{N_g}{2}\omega_1\right)\cos\left(\tfrac{N_g}{2}\omega_2\right)\Big]^T, \tag{19b}$$

$$\mathbf{c}_{oe} = \Big[p_1, \cdots, p_1\cos\left(\tfrac{N_g}{2}\omega_1\right)\cos\left(\tfrac{N_g}{2}\omega_2\right), \cdots, \tfrac{1}{M} p_1^{M} p_2^{2\hat{M}}, \cdots, \tfrac{1}{M} p_1^{M} p_2^{2\hat{M}} \cos\left(\tfrac{N_g}{2}\omega_1\right)\cos\left(\tfrac{N_g}{2}\omega_2\right)\Big]^T, \tag{19c}$$

$$\mathbf{c}_{eo} = \Big[p_2, \cdots, p_2\cos\left(\tfrac{N_g}{2}\omega_1\right)\cos\left(\tfrac{N_g}{2}\omega_2\right), \cdots, \tfrac{1}{M} p_1^{2\hat{M}} p_2^{M}, \cdots, \tfrac{1}{M} p_1^{2\hat{M}} p_2^{M} \cos\left(\tfrac{N_g}{2}\omega_1\right)\cos\left(\tfrac{N_g}{2}\omega_2\right)\Big]^T, \tag{19d}$$

$$\mathbf{c}_{oo} = \Big[p_1 p_2, \cdots, p_1 p_2\cos\left(\tfrac{N_g}{2}\omega_1\right)\cos\left(\tfrac{N_g}{2}\omega_2\right), \cdots, \tfrac{1}{M^2} p_1^{M} p_2^{M}, \cdots, \tfrac{1}{M^2} p_1^{M} p_2^{M} \cos\left(\tfrac{N_g}{2}\omega_1\right)\cos\left(\tfrac{N_g}{2}\omega_2\right)\Big]^T, \tag{19e}$$

(12a) can be represented by

$$\hat{H}(\omega_1, \omega_2, p_1, p_2) = \mathbf{g}^T\mathbf{c}_{ee} + j\hat{D}_1(\omega_1)\mathbf{g}^T\mathbf{c}_{oe} + j\hat{D}_2(\omega_2)\mathbf{g}^T\mathbf{c}_{eo} - \hat{D}_1(\omega_1)\hat{D}_2(\omega_2)\mathbf{g}^T\mathbf{c}_{oo}. \tag{20}$$

Therefore, the objective error function for designing the subfilters $G_{2m_1,2m_2}(z_1, z_2)$ can be defined by

$$\begin{aligned} e(\mathbf{g}) &= \int_R \left| H_d(\omega_1,\omega_2,p_1,p_2) - H\left(e^{j\omega_1}, e^{j\omega_2}, p_1, p_2\right)\right|^2 d\boldsymbol{v} \\ &= \int_R \left| M(\omega_1,\omega_2) e^{-j(\omega_1 p_1 + \omega_2 p_2)} - \hat{H}(\omega_1,\omega_2,p_1,p_2)\right|^2 d\boldsymbol{v} \\ &= \int_R \Big| M(\omega_1,\omega_2)\cos(\omega_1 p_1 + \omega_2 p_2) - \mathbf{g}^T\mathbf{c}_{ee} + \hat{D}_1(\omega_1)\hat{D}_2(\omega_2)\mathbf{g}^T\mathbf{c}_{oo}\Big|^2 d\boldsymbol{v} \\ &\quad + \int_R \Big| -M(\omega_1,\omega_2)\sin(\omega_1 p_1 + \omega_2 p_2) - \hat{D}_1(\omega_1)\mathbf{g}^T\mathbf{c}_{oe} - \hat{D}_2(\omega_2)\mathbf{g}^T\mathbf{c}_{eo}\Big|^2 d\boldsymbol{v} \\ &= u + \mathbf{r}^T\mathbf{g} + \mathbf{g}^T\mathbf{Q}\mathbf{g} \end{aligned} \tag{21}$$

where

$$\int_R = \iint_{R_p} \iint_{R_\omega}, \tag{22a}$$

$$d\boldsymbol{v} = d\omega_1 d\omega_2 dp_1 dp_2, \tag{22b}$$

$$R = R_p \cup R_\omega = \{-0.5 \le p_1, p_2 \le 0.5\} \cup \{(\omega_1,\omega_2) \in \text{passbands and } (\omega_1,\omega_2) \in \text{stopbands}\} \tag{22c}$$

and

$$\begin{aligned} u &= \int_R \left| M(\omega_1,\omega_2)\cos(\omega_1 p_1 + \omega_2 p_2)\right|^2 d\boldsymbol{v} + \int_R \left| M(\omega_1,\omega_2)\sin(\omega_1 p_1 + \omega_2 p_2)\right|^2 d\boldsymbol{v} \\ &= \int_R \left| M(\omega_1,\omega_2)\right|^2 d\boldsymbol{v} \\ &= \iint_{R_\omega} \left| M(\omega_1,\omega_2)\right|^2 d\omega_1 d\omega_2, \end{aligned} \tag{23a}$$

$$\begin{aligned} \mathbf{r} &= -2\int_R M(\omega_1,\omega_2)\cos(\omega_1 p_1 + \omega_2 p_2)\left[\mathbf{c}_{ee} - \hat{D}_1(\omega_1)\hat{D}_2(\omega_2)\mathbf{c}_{oo}\right] d\boldsymbol{v} \\ &\quad + 2\int_R M(\omega_1,\omega_2)\sin(\omega_1 p_1 + \omega_2 p_2)\left[\hat{D}_1(\omega_1)\mathbf{c}_{oe} + \hat{D}_2(\omega_2)\mathbf{c}_{eo}\right] d\boldsymbol{v}, \end{aligned} \tag{23b}$$

$$\begin{aligned} \mathbf{Q} &= \int_R \left[\mathbf{c}_{ee} - \hat{D}_1(\omega_1)\hat{D}_2(\omega_2)\mathbf{c}_{oo}\right]\left[\mathbf{c}_{ee} - \hat{D}_1(\omega_1)\hat{D}_2(\omega_2)\mathbf{c}_{oo}\right]^T d\boldsymbol{v} \\ &\quad + \int_R \left[\hat{D}_1(\omega_1)\mathbf{c}_{oe} + \hat{D}_2(\omega_2)\mathbf{c}_{eo}\right]\left[\hat{D}_1(\omega_1)\mathbf{c}_{oe} + \hat{D}_2(\omega_2)\mathbf{c}_{eo}\right]^T d\boldsymbol{v}. \end{aligned} \tag{23c}$$

The least-squares solution can be obtained by differentiating (21) with respect to the coefficient vector $\mathbf{g}$ and setting the result to zero, which yields

$$\mathbf{g} = -\tfrac{1}{2}\mathbf{Q}^{-1}\mathbf{r}. \tag{24}$$

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C. Design examples

In this subsection, design examples are presented and the results are compared with those of the conventional method [1]. To evaluate the performance, several measured criterions are defined as below:

$$\varepsilon_{m,rms} = \left[\frac{\int_R \left| H_d(\omega_1,\omega_2,p_1,p_2) - H(e^{j\omega_1}, e^{j\omega_2}, p_1, p_2) \right|^2 \boldsymbol{dv}}{\int_R \left| H_d(\omega_1,\omega_2,p_1,p_2) \right|^2 \boldsymbol{dv}} \right]^{1/2} \times 100\%, \tag{25a}$$

$$\varepsilon_{mp} = \max\Big\{ \left| H_d(\omega_1,\omega_2,p_1,p_2) - H(e^{j\omega_1}, e^{j\omega_2}, p_1, p_2) \right|, (\omega_1,\omega_2) \in \text{passbands}, \ -0.5 \le p_1, p_2 \le 0.5 \Big\} \tag{25b}$$

$$\varepsilon_{ms} = \max\Big\{ \left| H_d(\omega_1,\omega_2,p_1,p_2) - H(e^{j\omega_1}, e^{j\omega_2}, p_1, p_2) \right|, (\omega_1,\omega_2) \in \text{stopbands}, \ -0.5 \le p_1, p_2 \le 0.5 \Big\} \tag{25c}$$

$$\varepsilon_{\tau_1,rms} = \left[\frac{\int_R \left| \tau_{d1}(\omega_1,\omega_2,p_1,p_2) - \tau_1(\omega_1,\omega_2,p_1,p_2) \right|^2 \boldsymbol{dv}}{\int_R p_1^2 \boldsymbol{dv}} \right]^{1/2} \times 100\%, \tag{25d}$$

$$\varepsilon_{\tau_2,rms} = \left[\frac{\int_R \left| \tau_{d2}(\omega_1,\omega_2,p_1,p_2) - \tau_2(\omega_1,\omega_2,p_1,p_2) \right|^2 \boldsymbol{dv}}{\int_R p_2^2 \boldsymbol{dv}} \right]^{1/2} \times 100\%, \tag{25e}$$

$$\varepsilon_{\tau_1} = \max\Big\{ \left| \tau_{d1}(\omega_1,\omega_2,p_1,p_2) - \tau_1(\omega_1,\omega_2,p_1,p_2) \right|, (\omega_1,\omega_2) \in \text{passbands}, -0.5 \le p_1, p_2 \le 0.5 \Big\}, \tag{25f}$$

$$\varepsilon_{\tau_2} = \max\Big\{ \left| \tau_{d2}(\omega_1,\omega_2,p_1,p_2) - \tau_2(\omega_1,\omega_2,p_1,p_2) \right|, (\omega_1,\omega_2) \in \text{passbands}, -0.5 \le p_1, p_2 \le 0.5 \Big\} \tag{25g}$$

where $\tau_{di}(\omega_1,\omega_2,p_1,p_2)$ and $\tau_i(\omega_1,\omega_2,p_1,p_2)$ denote the desired and actual group delays, respectively, with respect to ω_i- direction , $i = 1, 2$. Meanwhile, the numbers of independent coefficients are also taken into account for comparison, which are computed as below:

Proposed method (including scale factors):

$$N_d + \left(\tfrac{N_g}{2}+1\right)^2 \left(\hat{M}+1\right)^2 + 4\hat{M} + 3\hat{M}^2 \tag{26a}$$

Conventional method [1]:

$$\left(\tfrac{N}{2}+1\right)^2 (M_c+1)^2 + \left(\tfrac{N}{2}\right)^2 M_s^2 + 2\left(\tfrac{N}{2}+1\right)\tfrac{N}{2}(M_c+1)M_s \tag{26b}$$

where

$$\begin{cases} M_c = M_s = \frac{M}{2}, & \text{for even } M, \\ M_c + 1 = M_s = \frac{M+1}{2}, & \text{for odd } M. \end{cases} \tag{27}$$

TABLE I
COMPARISONS FOR THE PROPOSED METHOD AND THE CONVENTIONAL METHOD [1].

Example	Example 1	
Method	Proposed	Conventional
Filter order	$N_{d1} = N_{d2} = 30$ $N_g = 20$	$N = 20$
Number of independent coefficients	1139	3969
Average delays	ω_1-direction: 25 ω_1-direction: 25	ω_1-direction: 10 ω_1-direction: 10
$\varepsilon_{m,rms}(\%)$	0.21486344	0.24878285
ε_{mp}	0.01013381	0.01140162
ε_{ms}	0.00844768	0.00992308
$\varepsilon_{\tau_1,rms}(\%)$	0.00141645	0.02759631
$\varepsilon_{\tau_2,rms}(\%)$	0.00241521	0.05946121
ε_{τ_1}	0.03280007	0.09599047
ε_{τ_2}	0.03488614	0.12677654

To compute the errors in (25), the frequencies ω_1 and ω_2 are uniformly sampled at step size $\pi/100$, and the variable parameters p_1 and p_2 are uniformly sampled at step size $1/50$.

Example 1: In this example, an elliptically symmetric low-pass VFD FIR filter is designed and the desired magnitude response has been given in (13). When $\omega_{p1} = 0.45\pi$, $\omega_{p2} = 0.6\pi$, $\omega_{s1} = 0.7\pi$, $\omega_{s2} = 0.85\pi$, $N_{d1} = N_{d2} = 30$, $N_g = 20$, $M = 5$, the obtained magnitude responses for $(p_1, p_2) = (0,0)$, $(0.25, 0.25)$, $(0.5, 0.5)$, $(0.5, -0.5)$ are shown in Fig. 2(a), the group-delay responses at $(p_1, p_2) = (0.25, 0.25)$ and $(0.5, -0.5)$ are shown in Fig. 2(b) and (c), while the variable group-delay responses and magnitude responses for both $\omega_2 = 0$, $p_2 = 0$ and $\omega_1 = 0$, $p_1 = 0$ are shown in Fig. 2(d) and (e), respectively. The errors defined in (25) are tabulated in Table I, accompanying those of the conventional method with $N = 20$.

IV. CONCLUSION

In this paper, a prefilter-subfilter cascaded structure for the design of VFD 2-D FIR digital filters has been proposed, which is derived based on the Taylor series expansion of the desired frequency response. By the specified relationships among the presented structure, it has been shown that the required number of independent coefficients is much less than that of the existing structure, while the performance of the designed filters is still better. Design example hse been presented to demonstrate the effectiveness of the presented method.

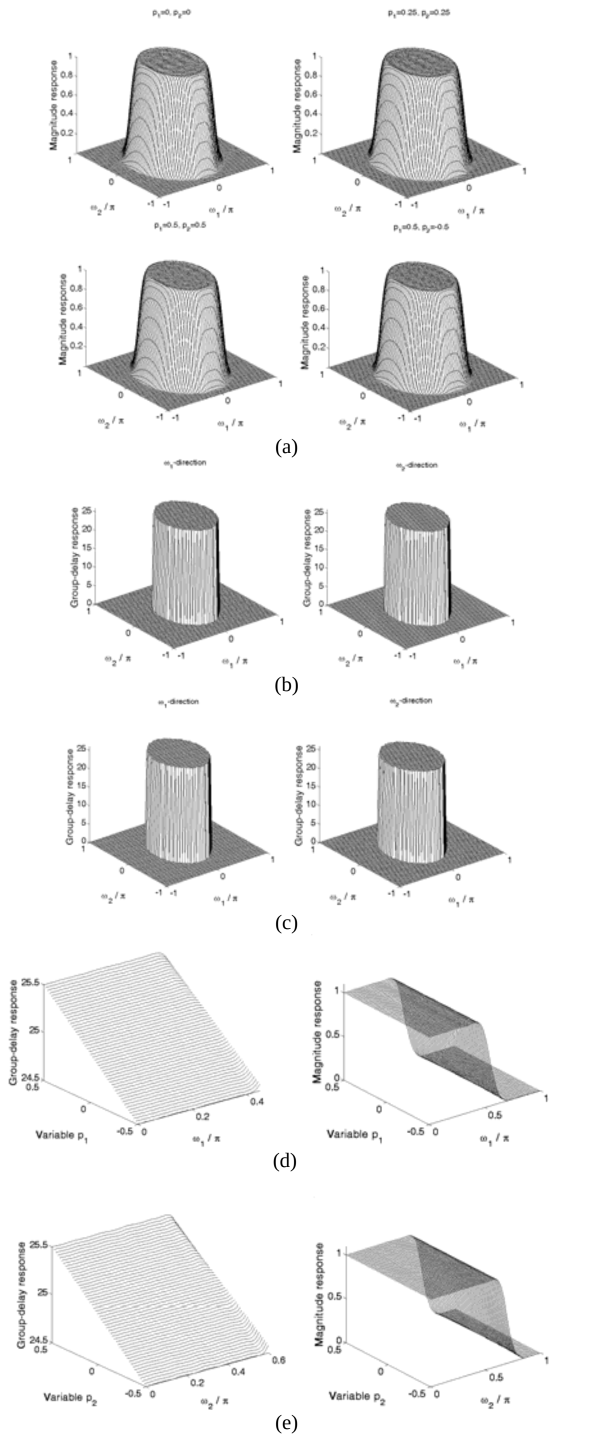

Fig. 2. Design of an elliptically symmetric low-pass VFD FIR filter. (a) Magnitude responses at $(p_1, p_2) = (0,0)$, $(0.25, 0.25)$, $(0.5, 0.5)$, $(0.5, -0.5)$. (b) ω_1-directional and ω_2-directional group-delay responses in the passband at $(p_1, p_2) = (0.25, 0.25)$. (c) ω_1-directional and ω_2-directional group-delay responses in the passband at $(p_1, p_2) = (0.5, -0.5)$. (d) Variable group-delay response in the passband and magnitude response at $\omega_2 = 0$, $p_2 = 0$. (e) Variable group-delay response in the passband and magnitude response at $\omega_1 = 0$, $p_1 = 0$.

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