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# Designs of Minimal-Order State Observer and Servo Controller for a Robot Arm Using Piecewise Bilinear Models

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***Abstract*—This paper proposes a servo control system based on a minimal-order state observer for nonlinear systems approximated by piecewise bilinear (PB) models. The design method is capable of designing the state observer and the servo controller of nonlinear systems separately. The approximated system is found to be fully parametric. The input-output (I/O) feedback linearization is applied to stabilize PB control systems. Although the controller is simpler than the conventional I/O feedback linearization controller, the control performance based on PB model is the same as the conventional one. The PB models with feedback linearization are a very powerful tool for the analysis and synthesis of nonlinear control systems. We apply the control method to a robot arm model. An example confirms the feasibility of the our proposals.**

***Index Terms*—nonlinear control, piecewise bilinear model, input-output linearization, robot arm, minimal-order state observer**

## I. Introduction

PIECEWISE linear (PL) systems which are fully parametric have been intensively studied in connection with nonlinear systems [1], [2], [3], [4]. We are interested in the parametric piecewise approximation of nonlinear control systems based on the original idea of PL approximation. The PL approximation has general approximation capability for nonlinear functions with a given precision.

One of the authors suggested to use the piecewise bilinear (PB) approximation [5]. PB approximation also has general approximation capability for nonlinear functions with a given precision. We note that a bilinear function as a basis of PB approximation is, as a nonlinear function, the second simplest one after a linear function. The PB model has the following features. 1) The PB model is derived from fuzzy if-then rules with singleton consequents. 2) It is built on piecewise hyper-cubes partitioned in the state space. 3) It has general approximation capability for nonlinear systems. 4) It is a piecewise nonlinear model, the second simplest after a PL model. 5) It is continuous and fully parametric. So far we have shown the necessary and sufficient conditions for the stability of PB systems with respect to Lyapunov functions in the two dimensional case [6], [7] where membership functions are fully taken into account. We derived the stabilizing conditions [8], [9] based on the feedback linearization, where [8] applies the input-output linearization and [9] applies the full-state linearization. Although the controllers are simpler than the conventional I/O feedback linearization controller, the control performance based on PB model is the same as the conventional one.

This paper proposes a servo control system based on a minimal-order state observer of nonlinear control systems approximated by PB models. The design method is capable of designing the state observer and the servo controller of nonlinear systems separately. Although the controller is simpler than the conventional I/O feedback linearization controller, the control performance based on PB model is the same as the conventional one. In addition, the performance of the observer-based PB controller is equivalent to the PB controller without the state observer.

This paper is organized as follows. Section II presents the canonical form of PB models. Section III presents PB controllers for nonlinear plants with PB modeling and I/O linearization. Section IV presents a servo control of PB models. Section V proposes a minimal-order state observer for PB models. Section VI applies the proposed method to a robot arm model and shows the feasibility of the proposed methods. Section VII gives conclusions.

## II. Canonical form of piecewise bilinear models

### A. Open-loop systems

In this section, we introduce the PB models suggested in [5]. We deal with the two dimensional case without loss of generality. Define a vector $d(\sigma,\tau)$ and a rectangle $R_{\sigma\tau}$ in the two-dimensional space as, respectively,

$$d(\sigma,\tau) \equiv (d_1(\sigma), d_2(\tau))^T,$$
$$R_{\sigma\tau} \equiv [d_1(\sigma), d_1(\sigma+1)] \times [d_2(\tau), d_2(\tau+1)].$$

$\sigma$ and $\tau$ are integers: $-\infty < \sigma,\tau < \infty$ where $d_1(\sigma) < d_1(\sigma+1), d_2(\tau) < d_2(\tau+1)$ and $d(0,0) \equiv (d_1(0), d_2(0))^T$. The superscript $^T$ denotes *transpose* operation.

For $x \in R_{\sigma\tau}$, the PB system is expressed as

$$\begin{cases} \dot{x} = \displaystyle\sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) f(i,j), \\ x = \displaystyle\sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) d(i,j), \end{cases} \quad (1)$$

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where

$$\begin{cases} \omega_1^{\sigma}(x_1) = (d_1(\sigma+1) - x_1)/(d_1(\sigma+1) - d_1(\sigma)), \\ \omega_1^{\sigma+1}(x_1) = (x_1 - d_1(\sigma))/(d_1(\sigma+1) - d_1(\sigma)), \\ \omega_2^{\tau}(x_2) = (d_2(\tau+1) - x_2)/(d_2(\tau+1) - d_2(\tau)), \\ \omega_2^{\tau+1}(x_2) = (x_2 - d_2(\tau))/(d_2(\tau+1) - d_2(\tau)), \end{cases} \quad (2)$$

and $\omega_1^i, \omega_2^j \in [0,\ 1]$. In the above, we assume $f(0,0) = 0$ and $d(0,0) = 0$ to guarantee $\dot{x} = 0$ for $x = 0$.

A key point in the system is that the state variable $x$ is also expressed by a convex combination of $d(i,j)$ with respect to $\omega_1^i$ and $\omega_2^j$ just as in the case of $\dot{x}$. As is seen in Eq. (2), $x$ is located inside $R_{\sigma\tau}$ which is a rectangle: a hypercube in general. That is, the expression of $x$ is polytopic with four vertices $d(i,j)$. The model of $\dot{x} = f(x)$ is built on a rectangle including $x$ in the state space and it is also polytopic with four vertices $f(i,j)$. We call this form of the canonical model (1) parametric expression.

Representing $\dot{x}$ with $x$ in Eqs. (1) and (2), we can obtain the state space expression of the model which is found to be bilinear (bi-affine) [5]. Therefore, the derived PB model has simple nonlinearity. In the case of the PL approximation, a PL model is built on simplexes partitioned in the state space, triangles in the two dimensional case. Note that any three points in the three dimensional space are spanned with an affine plane: $y = a + bx_1 + cx_2$. A PL model is continuous. It is, however, difficult to handle simplexes in the rectangular coordinate system.

### B. Closed-loop systems

We consider a two-dimensional nonlinear control system.

$$\begin{cases} \dot{x} = f_o(x) + g_o(x)u(x), \\ y = h_o(x). \end{cases} \quad (3)$$

The PB model (4) can be constructed from the nonlinear system (3).

$$\begin{cases} \dot{x} = f(x) + g(x)u(x), \\ y = h(x), \end{cases} \quad (4)$$

where

$$\begin{cases} f(x) = \sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) f(i,j), \\ g(x) = \sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) g(i,j), \\ h(x) = \sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) h(i,j), \\ x = \sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) d(i,j). \end{cases} \quad (5)$$

The modeling procedure in the region $R_{\sigma\tau}$ is as follows.

*Algorithm 2.1:* Piecewise bilinear modeling procedure

1) Assign vertices $d(i,j)$ for $x_1 = d_1(\sigma)$,$d_1(\sigma+1)$, $x_2 = d_2(\tau)$,$d_2(\tau+1)$ of the state vector $x$, then the state space is partitioned into piecewise regions, see also Fig. 1.
2) Compute the vertices $f(i,j)$, $g(i,j)$ and $h(i,j)$ in Eqs. (5), by substituting the values of $x_1 = d_1(\sigma)$, $d_1(\sigma+1)$ and $x_2 = d_2(\tau)$, $d_2(\tau+1)$ into original nonlinear functions $f_o$, $g_o$ and $h_o$ in the system (3). Fig. 1 illustrates the expression of $f_1(x)$, where $f(x) = (f_1(x), f_2(x))^T$ and $x \in R_{\sigma\tau}$.

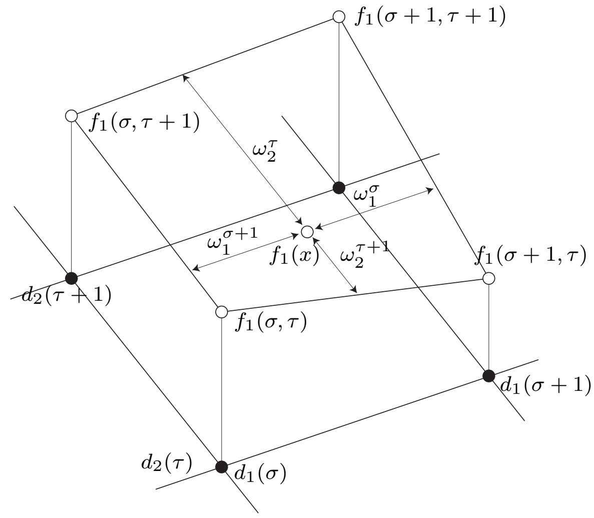


Fig. 1. Piecewise region ($f_1(x)$, $x \in R_{\sigma\tau}$)

The overall PB model can be obtained automatically when all the vertices are assigned. Note that $f(x)$, $g(x)$ and $h(x)$ in the PB model coincide with those in the original system at the vertices of all the regions.

## III. Design of PB controllers for nonlinear systems with PB modeling and I/O linearization

This section deals with the I/O linearization of nonlinear control systems approximated with PB models. We consider, in particular, nonlinear systems and show their I/O linearization based on PB models in detail. First we give a brief introduction to the I/O linearization of PB models [9], [10].

### A. I/O linearization

Consider the PB model (4) in the previous section. The derivative $\dot{y}$ is given by

$$\dot{y} = \frac{\partial h}{\partial x}(f(x) + g(x)u) = L_f h(x) + L_g h(x)u.$$

where

$$\begin{aligned} L_f h(x) = &\sum_{i_2=\sigma_2}^{\sigma_2+1} \omega_2^{i_2} \cdots \sum_{i_n=\sigma_n}^{\sigma_n+1} \omega_n^{i_n} \frac{h(\delta_1, i_2, \ldots, i_n)}{d_1(\delta_1)} f_1 + \cdots \\ &+ \sum_{i_1=\sigma_1}^{\sigma_1+1} \omega_1^{i_1} \cdots \sum_{i_{n-1}=\sigma_{n-1}}^{\sigma_{n-1}+1} \omega_{n-1}^{i_{n-1}} \frac{h(i_1, i_2, \ldots, \delta_n)}{d_n(\delta_n)} f_n, \\ L_g h(x) = &\sum_{i_2=\sigma_2}^{\sigma_2+1} \omega_2^{i_2} \cdots \sum_{i_n=\sigma_n}^{\sigma_n+1} \omega_n^{i_n} \frac{h(\delta_1, i_2, \ldots, i_n)}{d_1(\delta_1)} g_1 + \cdots \\ &+ \sum_{i_1=\sigma_1}^{\sigma_1+1} \omega_1^{i_1} \cdots \sum_{i_{n-1}=\sigma_{n-1}}^{\sigma_{n-1}+1} \omega_{n-1}^{i_{n-1}} \frac{h(i_1, i_2, \ldots, \delta_n)}{d_n(\delta_n)} g_n, \\ &d_1(\delta_1) = d_1(\sigma_1+1) - d_1(\sigma_1), \\ &d_2(\delta_2) = d_2(\sigma_2+1) - d_2(\sigma_2), \\ &d_n(\delta_n) = d_n(\sigma_n+1) - d_n(\sigma_n), \end{aligned}$$

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$$h(\delta_1, i_2, \dots, i_n) = h(\sigma_1+1, i_2, \dots, i_n) - h(\sigma_1, i_2, \dots, i_n),$$
$$h(i_1, \delta_2, \dots, i_n) = h(i_1, \sigma_2+1, \dots, i_n) - h(i_1, \sigma_2, \dots, i_n),$$
$$h(i_1, i_2, \dots, \delta_n) = h(i_1, i_2, \dots, \sigma_n+1) - h(i_1, i_2, \dots, \sigma_n).$$

If $L_g h(x) = 0$, then $\dot{y} = L_f h(x)$ is independent of $u$. We continue to calculate the second derivative of $y$, denoted by $y^{(2)}$ and then we obtain

$$y^{(2)} = \frac{\partial L_f h}{\partial x}(f(x) + g(x)u) = L_f^2 h(x) + L_g L_f h(x) u,$$

where

$$L_f^2 h(x) = \frac{\partial L_f h}{\partial x_1} f_1 + \cdots + \frac{\partial L_f h}{\partial x_n} f_n,$$
$$L_g L_f h(x) = \frac{\partial L_f h}{\partial x_1} g_1 + \cdots + \frac{\partial L_f h}{\partial x_n} g_n,$$
$$\begin{aligned}\frac{\partial L_f h}{\partial x_1} = &\sum_{i_3=\sigma_3}^{\sigma_3+1} \omega_3^{i_3} \cdots \sum_{i_n=\sigma_n}^{\sigma_n+1} \omega_n^{i_n} \frac{h(\delta_1, \delta_2, \dots, i_n)}{d_1(\delta_1) d_2(\delta_2)} f_2 + \cdots \\ &+ \sum_{i_2=\sigma_2}^{\sigma_2+1} \omega_2^{i_2} \cdots \sum_{i_{n-1}=\sigma_{n-1}}^{\sigma_{n-1}+1} \omega_{n-1}^{i_{n-1}} \frac{h(\delta_1, i_2, \dots, \delta_n)}{d_1(\delta_1) d_n(\delta_n)} f_n \\ &+ \sum_{i_2=\sigma_2}^{\sigma_2+1} \omega_2^{i_2} \cdots \sum_{i_n=\sigma_n}^{\sigma_n+1} \omega_n^{i_n} \frac{h(\delta_1, i_2, \dots, i_n)}{d_1(\delta_1)} \\ &\times \sum_{i_2=\sigma_2}^{\sigma_2+1} \omega_2^{i_2} \cdots \sum_{i_n=\sigma_n}^{\sigma_n+1} \omega_n^{i_n} \frac{f_1(\delta_1, i_2, \dots, i_n)}{d_1(\delta_1)} + \cdots \\ &+ \sum_{i_1=\sigma_1}^{\sigma_1+1} \omega_1^{i_1} \cdots \sum_{i_{n-1}=\sigma_{n-1}}^{\sigma_{n-1}+1} \omega_{n-1}^{i_{n-1}} \frac{h(i_1, i_2, \dots, \delta_n)}{d_n(\delta_n)} \\ &\times \sum_{i_2=\sigma_2}^{\sigma_2+1} \omega_2^{i_2} \cdots \sum_{i_n=\sigma_n}^{\sigma_n+1} \omega_n^{i_n} \frac{f_n(\delta_1, i_2, \dots, i_n)}{d_1(\delta_1)},\end{aligned}$$
$$\begin{aligned}\frac{\partial L_f h}{\partial x_n} = &\sum_{i_2=\sigma_2}^{\sigma_2+1} \omega_2^{i_2} \cdots \sum_{i_n=\sigma_{n-1}}^{\sigma_{n-1}+1} \omega_{n-1}^{i_{n-1}} \frac{h(\delta_1, i_2, \dots, \delta_n)}{d_1(\delta_1) d_n(\delta_n)} f_1 \\ &+ \cdots + \sum_{i_1=\sigma_1}^{\sigma_1+1} \omega_1^{i_1} \cdots \sum_{i_{n-2}=\sigma_{n-2}}^{\sigma_{n-2}+1} \omega_{n-2}^{i_{n-2}} \\ &\times \frac{h(i_1, \dots, \delta_{n-1}, \delta_n)}{d_{n-1}(\delta_{n-1}) d_n(\delta_n)} f_{n-1} \\ &+ \sum_{i_2=\sigma_2}^{\sigma_2+1} \omega_2^{i_2} \cdots \sum_{i_n=\sigma_n}^{\sigma_n+1} \omega_n^{i_n} \frac{h(\delta_1, i_2, \dots, i_n)}{d_1(\delta_1)} \\ &\times \sum_{i_1=\sigma_1}^{\sigma_1+1} \omega_1^{i_1} \cdots \sum_{i_{n-1}=\sigma_{n-1}}^{\sigma_{n-1}+1} \omega_{n-1}^{i_{n-1}} \frac{f_1(i_1, i_2, \dots, \delta_n)}{d_n(\delta_n)} + \cdots \\ &+ \sum_{i_1=\sigma_1}^{\sigma_1+1} \omega_1^{i_1} \cdots \sum_{i_{n-1}=\sigma_{n-1}}^{\sigma_{n-1}+1} \omega_{n-1}^{i_{n-1}} \frac{h(i_1, i_2, \dots, \delta_n)}{d_n(\delta_n)} \\ &\times \sum_{i_1=\sigma_1}^{\sigma_1+1} \omega_1^{i_1} \cdots \sum_{i_{n-1}=\sigma_{n-1}}^{\sigma_{n-1}+1} \omega_{n-1}^{i_{n-1}} \frac{f_n(i_1, i_2, \dots, \delta_n)}{d_n(\delta_n)}\end{aligned}$$

Once again, if $L_g L_f h(x) = 0$, then $y^{(2)} = L_f^2 h(x)$ is independent of $u$. Repeating this process, we see that if $h(x)$ satisfies

$$L_g L_f^i h(x) = 0, i = 0, 1, \dots, \rho - 2, \ L_g L_f^{\rho-1} h(x) \neq 0$$

then $u$ does not appear in the equations of $y$, $\dot{y}$, ..., $y^{(\rho-1)}$ and appears in the equation of $y^{(\rho)}$ with a nonzero coefficient:

$$y^{(\rho)} = L_f^\rho h(x) + L_g L_f^{\rho-1} h(x) u.$$

The foregoing equation shows clearly that the system is input-output linearizable, since the state feedback control

$$u = (-L_f^\rho h(x) + v) / L_g L_f^{\rho-1} h(x)$$

reduces the input-output map to $y^{(\rho)} = v$, which is a chain of $\rho$ integrators. In this case, the integer $\rho$ is called the relative degree of the system.

If $L_g L_f^{\rho-1} h(x_t) = 0$, the relative degree cannot be defined at $x = x_t$. In some cases the relative degree can be defined at the point because we can adjust a partition of the state space for PB modeling so that $L_g L_f^{\rho-1} h(x_t) \neq 0$.

*Definition 3.1:* The nonlinear system is said to have relative degree $\rho$, $1 \leq \rho \leq n$, in a region $D_0 \subset D$ if

$$L_g L_f^i h(x) = 0, \ i = 0, 1, \cdots, \rho - 2$$
$$L_g L_f^{\rho-1} h(x) \neq 0,$$

for all $x \in D_0$.

The input-output linearized system can be formulated as

$$\begin{cases} \dot{\xi} = A\xi + Bv, \\ y = C\xi, \end{cases} \tag{6}$$

where $\xi \in \Re^\rho$, $C = \begin{pmatrix} 1, & 0, & \cdots, & 0, & 0 \end{pmatrix}^T$,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Note that all the feedback linearizable PB systems (4) are transformed into the linear system (6). Therefore it is easy to design the stabilizing controller and analyze stability of the PB systems.

According to the relative degree, three cases of linearized systems (6) must be considered.

- Relative degree: $\rho = n$
  In this case, the state vector of the input-output linearized system is $z = \xi = (h(x), \ L_f h(x), \ \cdots, \ L_f^{\rho-1} h(x))^T$. The state vector $z$ is necessary to be a diffeomorphism.
- Relative degree: $\rho < n$
  There is unobservable state ($n - \rho$ dimensions). It is necessary to consider the zero dynamics of the unobservable state $\mu$. The state vector $z$ is necessary to be a diffeomorphism. $z = \begin{pmatrix} \xi, & \mu \end{pmatrix}^T$, $\xi \in \Re^\rho$, $\mu \in \Re^{n-\rho}$, $\dot{\mu}(\xi, \mu) = \zeta_1(\xi, \mu) + \zeta_2(\xi, \mu) v$. $\dot{\mu}(0, \mu)$ is characterized by zero dynamics.
- In the case of $L_g L_f^i h(x) = 0, \forall i$, the proposed approach cannot be applied.

When the relative degree $\rho \leq n$, the input-output linearizing controller is $u = \alpha(x) + \beta(x) v$, where

$$\alpha(x) = -L_f^\rho h(x) / L_g L_f^{\rho-1} h(x), \ \beta(x) = 1 / L_g L_f^{\rho-1} h(x).$$

In the following, we assume the relative degree is $n$ (full). The stabilizing linear controller $v = -F\xi$ of the linearized system (6) can be obtained so that the transfer function $G = C(sI - A)^{-1}B$ is Hurwitz.

The linearizing controller is also characterized as the LUT (Look-Up-Table) controller, where the LUT-controller is widely used for industrial applications, in particular, for vehicle control because of simplicity and also visibility as a nonlinear controller. In the case of the LUT-controller, control inputs are calculated by interpolation based on the table. When bilinear piecewise interpolation is adopted, the LUT-controller is found to be exactly the PB system.

## IV. Servo Control

We apply a servo control [11] to nonlinear systems with the PB models based on I/O linearization. This is a two-degree of freedom servo control to nonlinear systems as shown in Fig. 2. The controller is designed to deal with disturbances and robustness. In this figure, $T$ and $\int$ show the coordinate transformation and the integrator. $F$, $K$, and $G$ are the feedback gains. $r$ ($\dot{r} = 0$) is the setpoint signal and $d_{ist}$ means a disturbance. Due to lack of space, we only discuss the nonlinear system with the relative degree $\rho = n$. The following approach can be also applied to the nonlinear systems with $\rho < n$. We consider the linearized system using PB models.

$$\begin{cases} \dot{z} = Az + Bv \\ y = Cz \end{cases} \tag{7}$$

where $z \in \Re^n$, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $v \in \Re^m$, $C \in \Re^{l\times n}$ and $y \in \Re^l$. The control system is of a following form.

$$\begin{cases} \dot{z} = Az + Bv, \\ \dot{\eta} = r - y, \end{cases} \tag{8}$$

where the controller

$$v = -Fz + K\eta + Gr$$

is designed to make $r - y \to 0$ as $t \to \infty$. We rewrite the system equations (8) as

$$\begin{pmatrix} \dot{z} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} A - BF & BK \\ -C & 0 \end{pmatrix} \begin{pmatrix} z \\ \eta \end{pmatrix} + \begin{pmatrix} BG \\ 1 \end{pmatrix} r \tag{9}$$

The gains of $F$ and $K$ are calculated such that the system (9) is stable. The gain $G$ can be obtained such that $\dot{z}(\infty) = 0$ and $y(\infty) = r$. Therefore the gain of $G$ can be chosen $G = -(C(A-BF)^{-1}B)^{-1}$. Finally, the two-degree freedom servo controller is designed as

$$u = \alpha(x) + \beta(x)v = \frac{-L_f^{\rho}h(x) - Fz + K\eta - Gr}{L_g L_f^{\rho-1}h(x)} \tag{10}$$

## V. Minimal-order state observer based on PB models

We proposed a full-order state observer of PB control system in [12]. In this paper, we propose an observer-based PB controller to estimate the minimal-order state $z \in \Re^{n-l}$ by using the output $y \in \Re^l$. Fig. 3 shows the minimal-order state observer for PB control system.

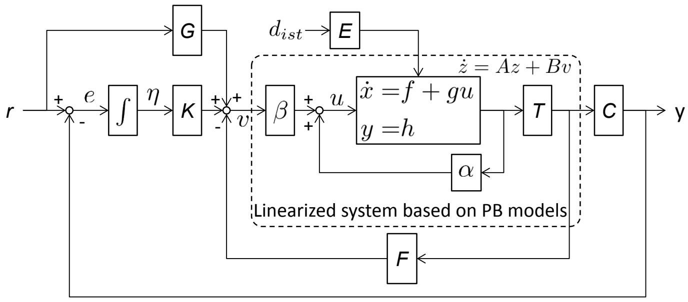


Fig. 2. Two-degree of freedom servo control system

The following system (11) is a minimal-order state observer, known as Gopinath observer [13].

$$\begin{cases} \dot{w} = \hat{A}w + \hat{B}v + Hy, \\ \hat{z} = \hat{C}w + \hat{D}y \end{cases} \tag{11}$$

where $w \in \Re^{n-l}$, $\hat{A} \in \Re^{n-l\times n-l}$, $\hat{B} \in \Re^{n-l\times m}$, $H \in \Re^{n-l\times l}$, $\hat{C} \in \Re^{n\times n-l}$, $\hat{D} \in \Re^{n\times l}$ and $\hat{z} \in \Re^{n-l}$.

The observer is designed by using the following steps [13],

1) Set a transformation $T_1 = [C^T \ \ M^T]^T \in \Re^{n\times n}$ satisfying $\det T_1 \neq 0$, where $M \in \Re^{n-l\times n}$ is an arbitrary matrix.
2) $\bar{A}$ and $\bar{B}$ are divided as follows.
$$\bar{A} = T_1 A T_1^{-1} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix},\ \bar{B} = T_1 B = \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix}$$
where $\bar{A}_{11} \in \Re^l$, $\bar{A}_{12} \in \Re^{n-l\times l}$, $\bar{A}_{22} \in \Re^{n-l\times n-l}$, $\bar{B}_1 \in \Re^l$ and $\bar{B}_2 \in \Re^{n-l}$.
3) Derive $L \in \Re^{n-l\times l}$ so that $\hat{A} = \bar{A}_{22} - L\bar{A}_{12}$ is Hurwitz.
4) Calculate the following parameters by using $L$.
$$\hat{B} = -L\bar{B}_1 + \bar{B}_2,\ H = \hat{A}L + \bar{A}_{21} - L\bar{A}_{11},$$
$$\hat{C} = T_1^{-1}\begin{pmatrix} 0 \\ I_{n-l} \end{pmatrix},\ \hat{D} = T_1^{-1}\begin{pmatrix} I_l \\ L \end{pmatrix}.$$

The estimation $\hat{z}$ of (11) is substituted into the servo controller (10), then the observer-based PB controller is designed as

$$\begin{aligned} u &= \frac{-L_f^{\rho}h(x) - F\hat{z} + K\eta - Gr}{L_g L_f^{\rho-1}h(x)}, \\ \dot{w} &= \hat{A}w + \hat{B}v + Hy, \\ \hat{z} &= \hat{C}w + \hat{D}y. \end{aligned}$$

Note that we can design the state observer for all the PB control systems. Since the linearized systems of all the PB models are the same as the linear system (7) and the system (7) is observable. In addition, the design method is capable of designing the state observer and the servo controller of nonlinear systems separately.

## VI. Robot arm model

We consider a simple one-link robot arm [14]. The rotary motion is controlled by an elastically coupled actuator. The system is represented as

$$\begin{cases} \dot{x} = f_o(x) + g_o(x)u(x), \\ y = h_o(x) \end{cases} \tag{12}$$

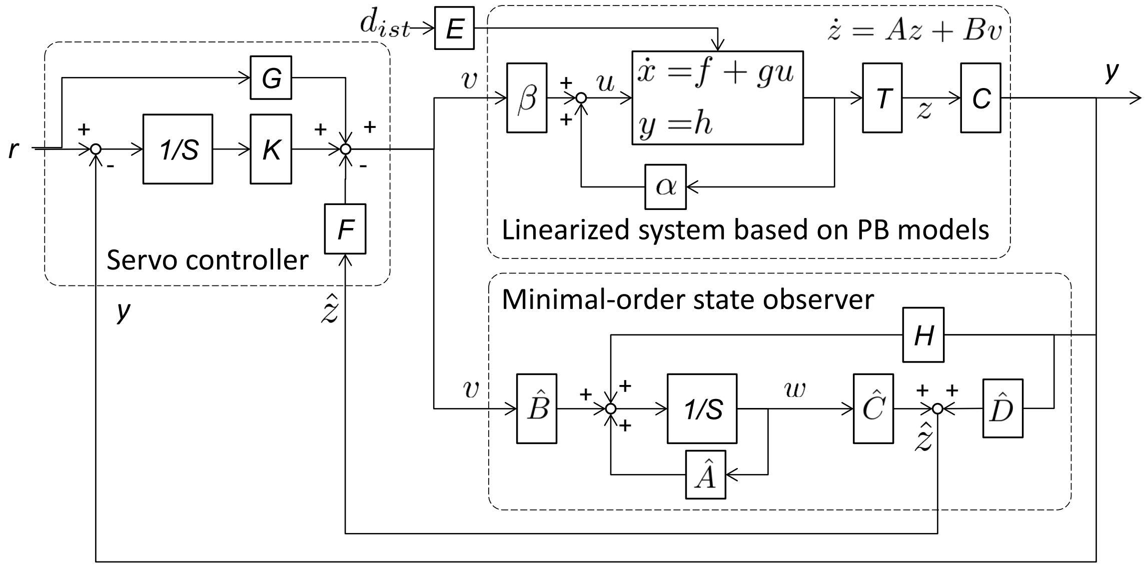


Fig. 3. Minimal-order state observer and servo control systems based on PB models

The vectors $x$, $f_o$ and $g_o$ are given by $x = (x_1, x_2, x_3, x_4)^T$,

$$f_o(x) = \begin{pmatrix} x_3 \\ x_4 \\ \frac{-K_1}{J_1 N^2} x_1 + \frac{K_1}{J_1 N} x_2 - \frac{F_1}{J_1} x_3 \\ \frac{K_1}{J_2 N} x_1 - \frac{K_1}{J_2} x_2 - \frac{mgd}{J_2} \cos x_2 - \frac{F_2}{J_2} x_4 \end{pmatrix},$$
$$g_o(x) = \begin{pmatrix} 0, & 0, & 1/J_1, & 0 \end{pmatrix}^T,$$

where $x_1$ is the angle of the torsional spring on the gear box and $x_2$ is the angle of the robot arm. $x_3$ and $x_4$ are the time derivatives of $x_1$ and $x_2$ respectively. $J_1$ and $J_2$ represent inertia, $F_1$ and $F_2$ are viscous friction constraints, $K_1$ represents the elastic coupling with the joint and $N$ is the transmission gear ratio. $m$ is the mass and $d$ is the position of the center of gravity of the link. $g$ is the acceleration due to gravity.

We choose the angle of the arm $x_2$ as the output, i.e.,

$$y = h_o(x) = x_2.$$

Now we divide the state-space of the robot arm model (12) as

$$x_1 \in \{-15, 15, 15\}, \; x_2 \in \{-5, 5, 5\}, \; x_4 \in \{-5, 5, 5\},$$
$$x_3 \in \{-4\pi, -39\pi/10, -38\pi/10, \ldots, 4\pi\},$$

then the PB model is constructed as

$$\dot{x} = f(x) + gu, \; y = h(x) = x_2$$

where

$$f_1 = \sum_{i_3=\sigma_3}^{\sigma_3+1} \omega_3^{i_3}(x_3) f_1(\cdot, \cdot, i_3, \cdot),$$
$$f_2 = \sum_{i_4=\sigma_4}^{\sigma_4+1} \omega_4^{i_4}(x_4) f_2(\cdot, \cdot, \cdot, i_4),$$
$$f_3 = \sum_{i_1=\sigma_1}^{\sigma_1+1} \sum_{i_2=\sigma_2}^{\sigma_2+1} \sum_{i_3=\sigma_3}^{\sigma_3+1} \omega_1^{i_1}(x_1) \omega_2^{i_2}(x_2) \omega_3^{i_3}(x_3) f_3(i_1, i_2, i_3, \cdot),$$
$$f_4 = \sum_{i_1=\sigma_1}^{\sigma_1+1} \sum_{i_2=\sigma_2}^{\sigma_2+1} \sum_{i_4=\sigma_4}^{\sigma_4+1} \omega_1^{i_1}(x_1) \omega_2^{i_2}(x_2) \omega_4^{i_4}(x_4) f_4(i_1, i_2, \cdot, i_4),$$

$$g = (0, 0, 0, 1/J_1)^T,$$
$$f_1(\cdot, \cdot, i_3, \cdot) = f_1(j_1, j_2, i_3, j_4),$$
$$j_1 = \sigma_1, \sigma_1 + 1, j_2 = \sigma_2, \sigma_2 + 1, j_4 = \sigma_4, \sigma_4 + 1,$$
$$f_2(\cdot, \cdot, \cdot, i_4) = f_2(j_1, j_2, j_3, i_4),$$
$$j_1 = \sigma_1, \sigma_1 + 1, j_2 = \sigma_2, \sigma_2 + 1, j_3 = \sigma_3, \sigma_3 + 1,$$
$$f_3(i_1, i_2, i_3, \cdot) = f_3(i_1, i_2, i_3, j_4), j_4 = \sigma_4, \sigma_4 + 1,$$
$$f_4(i_1, i_2, \cdot, i_4) = f_4(i_1, i_2, j_3, i_4), j_3 = \sigma_3, \sigma_3 + 1.$$

Here $f_1(\cdot, \cdot, i_3, \cdot)$ is independent of $i_1$, $i_2$ and $i_4$. It is also the same for $f_2(\cdot, \cdot, \cdot, i_3)$, $f_3(i_1, i_2, i_3, \cdot)$ and $f_4(i_1, i_2, \cdot, i_4)$.

In this case, the system has many local PB models. Note that the linearized systems of the all PB models are the same as the linear system (7).

We design the servo controller $u$

$$u = -L_f^4 h / L_g L_f^3 h + \nu / L_g L_f^3 h, \tag{13}$$

where

$$L_f h = f_2(x), \; L_f^2 h = f_4(x), \; L_g L_f^3 h = \frac{K_1}{J_1 J_2 N},$$

$$\begin{aligned} L_f^3 h =& \frac{K_1}{J_2 N} f_1(x) + \frac{f_2(\cdot, \sigma_2 + 1, \cdot, \cdot) - f_2(\cdot, \sigma_2, \cdot, \cdot)}{d_2(\sigma_2 + 1) - d_2(\sigma_2)} f_2(x) \\ &+ \frac{-F_2}{J_2} f_4(x), \\ L_f^4 h =& \frac{K_1}{J_2 N} f_3(x) + \frac{f_2(\cdot, \sigma_2 + 1, \cdot, \cdot) - f_2(\cdot, \sigma_2, \cdot, \cdot)}{d_2(\sigma_2 + 1) - d_2(\sigma_2)} f_4(x) \\ &+ \frac{-F_2}{J_2} \left( \frac{-K_1}{J_2 N} f_1(x) + \frac{-F_2}{J_2} f_4(x) \right. \\ &\left. + \frac{f_2(\cdot, \sigma_2 + 1, \cdot, \cdot) - f_2(\cdot, \sigma_2, \cdot, \cdot)}{d_2(\sigma_2 + 1) - d_2(\sigma_2)} f_2(x) \right), \\ \nu =& - Fz + K\eta + Gr \\ =& - (5.75, 8.22, 8.57, 2.57) z + 1.62\eta - 5.75r. \end{aligned}$$

Note that the controller (13) based on PB model is simpler than the conventional one since the nonlinear terms of controller (13) are not the original nonlinear terms (e.g., $\sin x_2$, $\cos x_2$) but the PB approximation models.

The initial condition is $x(0) = (0,0,0,0)^T$ and the setpoint signal is $r = \pi/2$. The disturbance signal $d_{ist} = 0.5$ is added to the state $x_2$ after 15 seconds. Fig. 4 shows the state response of $x_2$ using the PB servo controller with the disturbance $d_{ist}$.

We construct a minimal-order state observer of this robot arm model. In this example we set a transformation $T_1 = \text{diag}(1,1,1,1)$ and eigenvalues $\rho_1 = -200$, $\rho_2 = -201$, $\rho_3 = -202$ of $\hat{A}$. The parameters of the observer (11) are calculated as

$$L = \begin{pmatrix} 753, & 1.89 \times 10^5, & 1.58 \times 10^7 \end{pmatrix}^T,$$

$$\hat{A} = \begin{pmatrix} -753 & 1 & 0 \\ -1.89 \times 10^5 & 0 & 1 \\ -1.58 \times 10^7 & 0 & 0 \end{pmatrix}, \ \hat{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\hat{C} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \hat{D} = \begin{pmatrix} 1 \\ -753 \\ -1.89 \times 10^5 \\ -1.58 \times 10^7 \end{pmatrix}.$$

The estimation $\hat{z}$ of (11) is substituted into the servo controller (10), then the observer-based PB controller is obtain as

$$\begin{aligned} u =& \frac{-L_f^{\rho} h}{L_g L_f^{\rho-1} h} \\ &+ \frac{-(5.75, 8.22, 8.57, 2.57)\hat{z} + 1.62\eta - 5.75r}{L_g L_f^{\rho-1} h}, \\ \dot{w} =& \hat{A}w + \hat{B}v + Hy, \\ \hat{z} =& \hat{C}w + \hat{D}y. \end{aligned}$$

In Fig. 5, the upper graph shows the responses of the state $x_2(t)$ without the disturbance and the lower one shows the response with the disturbance. The results confirm the feasibility of the servo control and state observer. In addition, the results show that the controller has disturbance rejection feature. The performance of the observer-based PB controller is equivalent to the PB controller without the state observer.

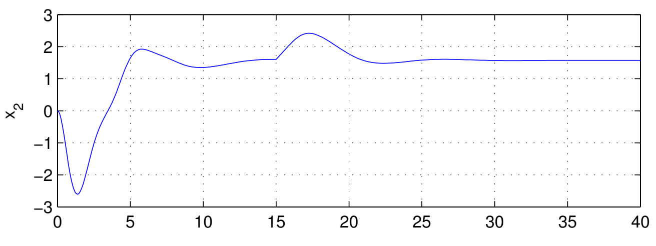


Fig. 4. State response $x_2$ using the PB servo controller with the disturbance

## VII. Conclusions

This paper has proposed a servo control system based on a minimal-order state observer for nonlinear systems approximated by piecewise bilinear (PB) models. The design method is capable of designing the state observer and the servo controller of nonlinear systems separately. The approximated system is found to be fully parametric. The input-output (I/O) feedback linearization is applied to stabilize PB control systems. The PB models with feedback linearization are a very powerful tool for the analysis and synthesis of nonlinear control systems. Although the controller is simpler than the conventional I/O feedback linearization controller, the control performance based on PB model is the same as the conventional one. We have applied the control method to a robot arm system. An example has confirmed the feasibility of our proposal.

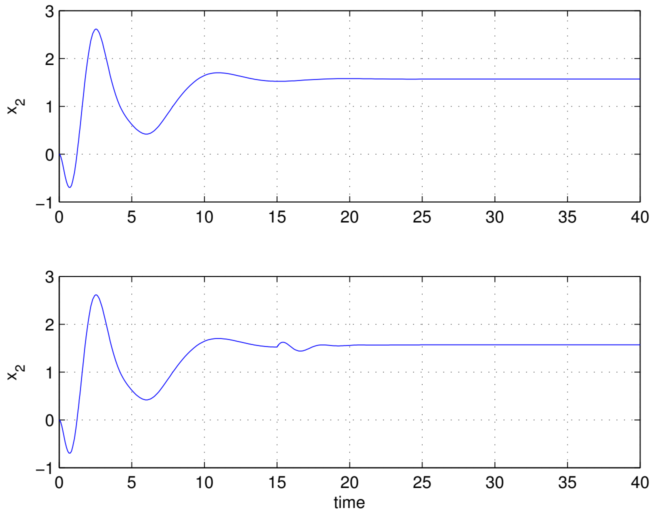


Fig. 5. State responses $x_2$ using the observer-based PB servo controller without/with the disturbance

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