Approximation for ARL of EWMA Control Chart with EAR(p) Process

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***Abstract*—Traditional statistical process control (SPC) techniques are based on the assumption that the observations are independent. In recent years, the autocorrelation is present in the data and can have an impact on the performance of the control chart. This paper considers the problem of monitoring the mean of a process in which observations can be modeled as autoregressive order p (AR(p)) process with exponential white noise. We propose EWMA control chart which is based on average run length by numerical integral equation method.**

***Index Terms*—ARL, autoregressive process, exponential white noise, EWMA, Fredholm integral equation**

I. Introduction

Statistical Process Control techniques are to detect the shift in mean or variance of the process, the assumption that the observations are independent and normal distribution. In reality, the data are often non-normal and are correlated. The performance used for comparing control chart is average run length (ARL) defined as the mean number of data taken from an in-control process until the control chart falsely signals out-of-control is denote by ARL_0, it is large enough to keep the level of false alarms at an acceptable level. Another hand is the mean number of data taken from an out-of-control process until the control chart signals that mean the process is out-of-control is denote by ARL_1, it should be small as possible.

In literatures are many research for study EWMA control chart with correlated data and non-normal distribution i.e., [1], [2], [3], [4], [5]. In this paper, we show ARL for EWMA control chart for exponential autoregressive process order p (EAR(p)) by approximation integral equation.

II. The Properties of Control Chart

The observations $a_t; t = 1,2,3,\ldots$ are independent random variables with distribution function $F(x,\alpha)$, the parameter $\alpha = \alpha_0$ before a change point time $\theta \le \infty$ that mean it is in control process and $\alpha > \alpha_0$ after the change point time θ that mean it is out-of-control process. Control chart is based on use of stopping times τ and the condition on choice of stopping times τ is the following

$$E_\infty(\tau) = T.$$

Where T is given and usually large, $E_\infty(\cdot)$ is the expectation under distribution function $F(x,\alpha_0)$. The measure $E_\infty(\tau)$ is called as average run length for in-control process, then by definition, $ARL_0 = E_\infty(\tau)$ and the typical applied limitation is

$$ARL_0 = T,$$

and another typical applied limitation in minimizing the measure

$$ARL_1 = E_\theta(\tau - \theta + 1),$$

where $E_\theta(\cdot)$ is the expectation under distribution function $F(x,\alpha_1)$. The measure $E_\theta(\tau)$ is called as average run length for out-of-control process and α_1 is the value of parameter after the change point, ARL_1 is close to minimal value.

III. EWMA Control Chart with EAR(p) Process

The EWMA statistics based on EAR(p) process is defined by the following recursion:

$$y_t = (1-\lambda)y_{t-1} + \lambda x_t,\ t = 1,2,3,\ldots$$

where y_t is EWMA statistics, x_t is AR(p) processes and the initial value is a constant u and the smoothing parameter $0 < \lambda < 1$. The AR(p) process can be written as:

$$x_t = \mu + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \phi_3 x_{t-3} + \ldots + \phi_p x_{t-p} + a_t$$

where a_t is a white noise supposed with exponential distribution. An autoregressive coefficient $-1 \le \phi_i \le 1$ and μ is a constant. We assume the initial value of AR(p) process is 1.

The first passage times for EWMA control chart can be written as:

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$$\tau = \inf\{t > 0 : y_t > H\}$$

where H is a control limit.

IV. THE INTEGRAL EQUATION METHOD

The ARL of EWMA control chart can be calculated in term of a Fredholm Integral Equation of the second kind was derived by [6]. [7] followed a similar method to develop Integral Equations for EAR(1) process. For the EAR(1) process, approximations for ARL_0 and ARL_1 can be found by solving the Integral Equations numerically. Consider an EWMA control chart where $x_t, t = 1,2,3,\ldots$ be AR(p) process with exponential white noise, $a_t; t = 1,2,3,\ldots$. Assume the lower and upper control limits for $x_t, t = 1,2,3,\ldots$ are $UCL = H$ and $LCL = 0$, correspondingly.

Let $L(u)$ be the ARL_0 of EWMA control chart with initial values u, i.e. $L(u) = E_\infty(\tau)$, and if EWMA and AR(p) statistics begin at u and 1 respectively. There are two cases where it will be after a single random variable, a_1. The first, if a_1 is out-of-control limit, i.e. $(1-\lambda)u + \lambda\left(\mu + \sum_{i=1}^{p}\phi_i + a_1\right) > H$, therefore the run length is 1. In another hand, if a_1 is in control limit, i.e. $0 < (1-\lambda)u + \lambda\left(\mu + \sum_{i=1}^{p}\phi_i + a_1\right) < H$, then a random variable will have been run and on average $L\left[(1-\lambda)u + \lambda\left(\mu + \sum_{i=1}^{p}\phi_i + a_1\right)\right]$ more random variables will be formed before a signal is given. This can be expressed in the form

$$L(u) = \left(1 - P\left[\frac{-(1-\lambda)u - \lambda\left(\mu + \sum_{i=1}^{p}\phi_i\right)}{\lambda} < a_1 < \frac{H - (1-\lambda)u - \lambda\left(\mu + \sum_{i=1}^{p}\phi_i\right)}{\lambda}\right]\right)$$
$$+ \int_{\frac{(1-\lambda)u - \lambda\left(\mu + \sum_{i=1}^{p}\phi_i\right)}{\lambda}}^{\frac{H-(1-\lambda)u - \lambda\left(\mu + \sum_{i=1}^{p}\phi_i\right)}{\lambda}} \left(1 + L\left[(1-\lambda)u + \lambda\left(\mu + \sum_{i=1}^{p}\phi_i + y\right)\right]\right) f(y)\,dy$$
$$= 1 + \int_{\frac{(1-\lambda)u - \lambda\left(\mu + \sum_{i=1}^{p}\phi_i\right)}{\lambda}}^{\frac{H-(1-\lambda)u - \lambda\left(\mu + \sum_{i=1}^{p}\phi_i\right)}{\lambda}} L\left[(1-\lambda)u + \lambda\left(\mu + \sum_{i=1}^{p}\phi_i + y\right)\right] f(y)\,dy.$$

Setting $(1-\lambda)u + \lambda\left(\mu + \sum_{i=1}^{p}\phi_i + y\right) = z$, we obtain

$$L(u) = 1 + \frac{1}{\lambda}\int_0^H L(z)\, f\left(\frac{z - (1-\lambda)u - \lambda\left(\mu + \sum_{i=1}^{p}\phi_i\right)}{\lambda}\right) dz.$$

Where $f(\cdot)$ is the probability density function of a_t. A similar approach ARL_0 can be used to develop an integral equation for ARL_1, it is assumed that the out-of-control signal arises. Let $L(u,\alpha)$ as ARL_1 of the EWMA control chart, assumed that the parameter of change is equal to α and an initial value is u, i.e. $L(u,\alpha) = E_1(\tau)$.

V. NUMERICAL RESULTS

In this section, the numerical results for ARL_0 and ARL_1 were calculated as shown in Table 1 to Table 4. The parameter values for EWMA control chart was chosen by given desire ARL_0 = 100, 300, 500 and 1000, $\lambda = 0.25$, in-control parameter value $\alpha_0 = 1$ and out-of-control values $\alpha_1 = 1.01$ to 1.5 for EAR(2) process with $\phi_1 = \phi_2 = 0.1$ and $\phi_1 = \phi_2 = 0.2$.

TABLE I
SHOW ARL FROM NUMERICAL INTEGRAL EQUATION METHOD FOR GIVEN $\phi_1 = \phi_2 = 0.1$ AND ARL_0 = 100, 300

α	ARL	
	$H = 0.25365$	$H = 0.25552$
1.00	100.18310	300.16500
1.01	52.71044	80.76288
1.03	27.44358	33.34331
1.05	18.76375	21.28453
1.07	14.37413	15.77609
1.10	10.75784	11.50003
1.30	4.48546	4.58909
1.50	3.12500	3.16822

TABLE II
SHOW ARL FROM NUMERICAL INTEGRAL EQUATION METHOD FOR GIVEN $\phi_1 = \phi_2 = 0.1$ AND ARL_0 = 500, 1000

α	ARL	
	$H = 0.25589$	$H = 0.25617$
1.00	500.00940	1000.06732
1.01	90.28729	99.13964
1.03	34.82720	36.04162
1.05	21.86670	22.32910
1.07	16.08701	16.33067
1.10	11.65940	11.78302
1.30	4.61017	4.62626
1.50	3.17692	3.18354

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TABLE III
SHOW ARL FROM NUMERICAL INTEGRAL EQUATION METHOD FOR GIVEN $\phi_1 = \phi_2 = 0.2$ AND $\text{ARL}_0 = 100, 300$

α	ARL	
	$H = 0.22647$	$H = 0.22817$
1.00	100.25388	300.00345
1.01	53.21539	81.97706
1.03	27.78927	33.88344
1.05	19.019136	21.62403
1.07	14.57006	16.02013
1.10	10.90027	11.66847
1.30	4.52694	4.63426
1.50	3.14456	3.18929

TABLE IV
SHOW ARL FROM NUMERICAL INTEGRAL EQUATION METHOD FOR GIVEN $\phi_1 = \phi_2 = 0.2$ AND $\text{ARL}_0 = 500, 1000$

α	ARL	
	$H = 0.22851$	$H = 0.22876$
1.00	500.31352	1000.31521
1.01	91.92876	100.94300
1.03	35.43705	36.67399
1.05	22.23385	22.70484
1.07	16.34588	16.59407
1.10	11.83547	11.96138
1.30	4.65635	4.67273
1.50	3.19839	3.20512

VI. CONCLUSION

In this paper, an EWMA control chart is proposed for monitoring EAR(p) process. The ARL of EWMA control chart were calculated by approximation integral equation. Future researches may use other techniques to compare the performance of EWMA control chart with EAR(p) process i.e., Markov chain approach or explicit form.

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