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The Hamiltonian Connectivity of Rectangular Supergrid Graphs

Ruo-Wei Hung*, Chien-Hui Hou, Hao-Yu Chih, Xiaoguang Li, and Bing Sun

***Abstract*—A Hamiltonian path of a graph is a simple path which visits each vertex of the graph exactly once. The Hamiltonian path problem is to determine whether a graph contains a Hamiltonian path. A graph is called Hamiltonian connected if there exists a Hamiltonian path between any two distinct vertices. In this paper, we will study the Hamiltonian connectivity of rectangular supergrid graphs. Supergrid graphs were first introduced by us and include grid graphs and triangular grid graphs as their subgraphs. The Hamiltonian path problem for grid graphs and triangular grid graphs was known to be NP-complete. Recently, we have proved that the Hamiltonian path problem for supergrid graphs is also NP-complete. The Hamiltonian paths on supergrid graphs can be applied to compute the stitching traces of computer sewing machines. Rectangular supergrid graphs form a popular subclass of supergrid graphs, and they have strong structure. In this paper, we will show that rectangular supergrid graphs are Hamiltonian connected except two trivial forbidden conditions.**

***Index Terms*—Hamiltonian connected property, supergrid graph, rectangular supergrid graph, computer sewing machine.**

I. INTRODUCTION

A *Hamiltonian path* in a graph is a simple path in which each vertex of the graph appears exactly once. A *Hamiltonian cycle* in a graph is a simple cycle with the same property. The *Hamiltonian path* (resp., *cycle*) *problem* involves deciding whether or not a graph contains a Hamiltonian path (resp., cycle). A graph is called *Hamiltonian* if it contains a Hamiltonian cycle, and is said to be *Hamiltonian connected* if for each pair of distinct vertices in it, there is a Hamiltonian path between them. Clearly, a Hamiltonian connected graph contains many Hamiltonian cycles, and, hence, the sufficient conditions of Hamiltonian connectivity are stronger than those of Hamiltonicity.

The Hamiltonian path and cycle problems have numerous applications in different areas, including establishing transport routes, production launching, the on-line optimization of flexible manufacturing systems [1], computing the perceptual boundaries of dot patterns [22], DNA physical mapping [8], and fault-tolerant routing for 3D network-on-chip architectures [4]. It is well known that the Hamiltonian path and cycle problems are NP-complete for general graphs [6], [16]. The same holds true for bipartite graphs [19], grid graphs [15], triangular grid graphs [7], and supergrid graphs [10]. In the literature, there are many studies for the Hamiltonian connectivity of interconnection networks [5], [9], [21]. The popular hypercubes are Hamiltonian but are not Hamiltonian connected. However, many variants of hypercubes have been shown to be Hamiltonian connected [9], [12], [13], [23]. In this paper, we will study the Hamiltonian connected property of rectangular supergrid graphs.

The *two-dimensional integer grid* G^∞ is an infinite graph whose vertex set consists of all points of the Euclidean plane with integer coordinates and in which two vertices are adjacent if and only if the (Euclidean) distance between them is equal to 1. The *two-dimensional triangular grid* T^∞ is an infinite graph obtained from G^∞ by adding all edges on the lines traced from up-left to down-right. A *grid graph* is a finite, vertex-induced subgraph of G^∞. For a node v in the plane with integer coordinates, let v_x and v_y represent the x and y *coordinates* of node v, respectively, denoted by $v = (v_x, v_y)$. If v is a vertex in a grid graph, then its possible adjacent vertices include $(v_x, v_y - 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, and $(v_x, v_y + 1)$. A *triangular grid graph* is a finite, vertex-induced subgraph of T^∞. If v is a vertex in a triangular grid graph, then its possible neighboring vertices include $(v_x, v_y - 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, $(v_x, v_y + 1)$, $(v_x - 1, v_y - 1)$, and $(v_x + 1, v_y + 1)$. Thus, triangular grid graphs contain grid graphs as subgraphs. Note that triangular grid graphs defined above are isomorphic to the original triangular grid graphs studied in the literature [7] but these graphs are different when considered as geometric graphs. By the same construction of triangular grid graphs from grid graphs, we have proposed a new class of graphs, namely *supergrid graphs*, in [10]. The *two-dimensional supergrid* S^∞ is an infinite graph obtained from T^∞ by adding all edges on the lines traced from up-right to down-left. A *supergrid graph* is a finite, vertex-induced subgraph of S^∞. The possible adjacent vertices of a vertex $v = (v_x, v_y)$ in a supergrid graph include $(v_x, v_y - 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, $(v_x, v_y + 1)$, $(v_x - 1, v_y - 1)$, $(v_x + 1, v_y + 1)$, $(v_x + 1, v_y - 1)$, and $(v_x - 1, v_y + 1)$. Then, supergrid graphs contain grid graphs and triangular grid graphs as subgraphs. Notice that grid and triangular grid graphs are not subclasses of supergrid graphs, and the converse is also true: these classes of graphs have common elements (points) but in general they are distinct since the edge sets of these graphs are different. Obviously, all grid graphs are bipartite [15] but triangular grid graphs and supergrid graphs are not bipartite. The Hamiltonian cycle and path problems for grid graphs and triangular grid graphs were known to be NP-complete [7], [15]. Recently, we showed that the Hamiltonian path and cycle problems on supergrid graphs are also NP-complete [10]. Rectangular supergrid graphs first appeared in [10], in which we solved the Hamiltonian cycle problem. The rectangular supergrid graph $R(m, n)$ is a subgraph of S^∞ induced by vertex set $V(R(m, n)) = \{v = (v_x, v_y) | 1 \leqslant v_x \leqslant m$ and $1 \leqslant v_y \leqslant n\}$. In this paper, we will show that rectangular

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supergrid graphs are always Hamiltonian connected except two trivial cases.

The Hamiltonian related problems on supergrid graphs can be applied to control the stitching trace of a computerized sewing machine as stated in [10]. Previous related works are summarized as follows. Itai *et al.* [15] showed that the Hamiltonian path problem on grid graphs is NP-complete. They also gave necessary and sufficient conditions for a rectangular grid graph having a Hamiltonian path between two given vertices. Note that rectangular grid graphs are not Hamiltonian connected. Zamfirescu *et al.* [26] gave sufficient conditions for a grid graph having a Hamiltonian cycle, and proved that all grid graphs of positive width have Hamiltonian line graphs. Later, Chen *et al.* [3] improved the Hamiltonian path algorithm of [15] on rectangular grid graphs and presented a parallel algorithm for the Hamiltonian path problem with two given endpoints in rectangular grid graphs. Also there is a polynomial-time algorithm for finding Hamiltonian cycles in solid grid graphs [20]. In [25], Salman introduced alphabet grid graphs and determined classes of alphabet grid graphs which contain Hamiltonian cycles. Keshavarz-Kohjerdi and Bagheri [17] gave necessary and sufficient conditions for the existence of Hamiltonian paths in alphabet grid graphs, and presented linear-time algorithms for finding Hamiltonian paths with two given endpoints in these graphs. Recently, Keshavarz-Kohjerdi *et al.* [18] presented a linear-time algorithm for computing the longest path between two given vertices in rectangular grid graphs. Reay and Zamfirescu [24] proved that all 2-connected, linear-convex triangular grid graphs except one special case contain Hamiltonian cycles. The Hamiltonian cycle (path) on triangular grid graphs has been shown to be NP-complete [7]. They also proved that all connected, locally connected triangular grid graphs (with one exception) contain Hamiltonian cycles. In addition, the Hamiltonian cycle problem on hexagonal grid graphs has been shown to be NP-complete [14]. Recently, we prove that the Hamiltonian cycle and path problems on supergrid graphs are NP-complete [10]. We also showed that every rectangular supergrid graph always contains a Hamiltonian cycle. Very recently, we prove that linear-convex supergrid graphs, which form a subclass of supergrid graphs, always contain Hamiltonian cycles [11].

The rest of the paper is organized as follows. In Section II, some notations and observations are given. Section III discusses the Hamiltonian connectivity of smaller sized rectangular supergrid graphs. In Section IV, we prove that rectangular supergrid graphs are Hamiltonian connected except two forbidden conditions. That is, we can construct a Hamiltonian path between any two distinct vertices on a rectangular supergrid graph. Finally, we make some concluding remarks in Section V.

II. Terminologies and Background Results

In this section, we will introduce some terminologies and symbols used in the paper. Some observations and previously established result on the Hamiltonian cycle problem in rectangular supergrid graphs are also presented. For graph-theoretic terminology not defined in this paper, the reader is referred to [2].

Let $G = (V, E)$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. Let S be a subset of vertices in G, and let u, v be two vertices in G. We write $G - S$ for the subgraph induced by $V - S$. In general, we write $G - v$ instead of $G - \{v\}$. If (u, v) is an edge in G, we say that u is *adjacent* to v and u, v are *incident* to edge (u, v). The notation $u \sim v$ (resp., $u \nsim v$) means that vertices u and v are adjacent (resp., non-adjacent). Edge $e_1 = (u_1, v_1)$ is said to be *incident* to edge $e_2 = (u_2, v_2)$ if ($u_1 \sim u_2$ and $v_1 \sim v_2$) or ($u_1 \sim v_2$ and $v_1 \sim u_2$). The notation $e_1 \approx e_2$ means that edges e_1 and e_2 are incident. A *neighbor* of v in G is any vertex that is adjacent to v. We use $N_G(v)$ to denote the set of neighbors of v in G, and let $N_G[v] = N_G(v) \cup \{v\}$. A path P of length $|P|$ in G, denoted by $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{|P|-1} \rightarrow v_{|P|}$, is a sequence $(v_1, v_2, \cdots, v_{|P|-1}, v_{|P|})$ of vertices such that $(v_i, v_{i+1}) \in E$ for $1 \leqslant i < |P|$. The first and last vertices visited by P are called the *path-start* and *path-end* of P, denoted by $start(P)$ and $end(P)$, respectively. We will use $v_i \in P$ to denote "P visits vertex v_i" and use $(v_i, v_{i+1}) \in P$ to denote "P visits edge (v_i, v_{i+1})". A path from v_1 to v_k is denoted by (v_1, v_k)-path. In addition, we use P to refer to the set of vertices visited by path P if it is understood without ambiguity. A path P is a cycle if $|V(P)| \geqslant 3$ and $end(P) \sim start(P)$. Two paths (or cycles) P_1 and Q_2 of graph G are called vertex-disjoint if and only if $V(Q_1) \cap V(Q_2) = \emptyset$. Two vertex-disjoint paths P_1 and P_2 can be concatenated into a path, denoted by $P_1 \Rightarrow P_2$, if $end(P_1) \sim start(P_2)$.

Let S^∞ be the infinite graph whose vertex set consists of all points of the plane with integer coordinates and in which two vertices are adjacent if and only if the difference of their x or y coordinates is not larger than 1. A *supergrid graph* is a finite, vertex-induced subgraph of S^∞. For a vertex v in a supergrid graph, let v_x and v_y denote x and y coordinates of its corresponding point, respectively. We color vertex v to be *white* if $v_x + v_y \equiv 0 \pmod 2$; otherwise, v is colored to be *black*. Then there are eight possible neighbors of vertex v including four white vertices and four black vertices. Obviously, all grid graphs are bipartite [15] but supergrid graphs are not bipartite.

Rectangular supergrid graphs first appeared in [10], in which we tried to solve the Hamiltonian cycle problem. Let $R(m, n)$ be the supergrid graph whose vertex set $V(R(m, n)) = \{v = (v_x, v_y) | 1 \leqslant v_x \leqslant m \text{ and } 1 \leqslant v_y \leqslant n\}$. That is, $R(m, n)$ contains m columns and n rows of vertices in S^∞. A *rectangular supergrid graph* is a supergrid graph which is isomorphic to $R(m, n)$ for some m and n. Then m and n, the *dimensions*, specify a rectangular supergrid graph up to isomorphism. The size of $R(m, n)$ is defined to be mn, and $R(m, n)$ is called n-rectangle. $R(m, n)$ is called *even-sized* if mn is even, and it is called *odd-sized* otherwise. In this paper, without loss of generality we assume that $m \geqslant n$.

Let $v = (v_x, v_y)$ be a vertex in $R(m, n)$. The vertex v is called the *upper-left* (resp., *upper-right*, *down-left*, *down-right*) *corner* of $R(m, n)$ if $w_x \geqslant v_x$ and $w_y \leqslant v_y$ (resp., $w_x \leqslant v_x$ and $w_y \leqslant v_y$, $w_x \geqslant v_x$ and $w_y \geqslant v_y$, $w_x \leqslant v_x$ and $w_y \geqslant v_y$) for any vertex $w = (w_x, w_y) \in R(m, n)$. The edge (u, v) is said to be *horizontal* (resp., *vertical*) if $u_y = v_y$ (resp., $u_x = v_x$), and is called *crossed* if it is neither a horizontal nor a vertical edge. In the figures we assume that $(1, 1)$ are coordinates of the down-left corner in a rectangular supergrid graph $R(m, n)$. There are four boundaries in a rectangular supergrid graph $R(m, n)$ with $m, n \geqslant 2$. The

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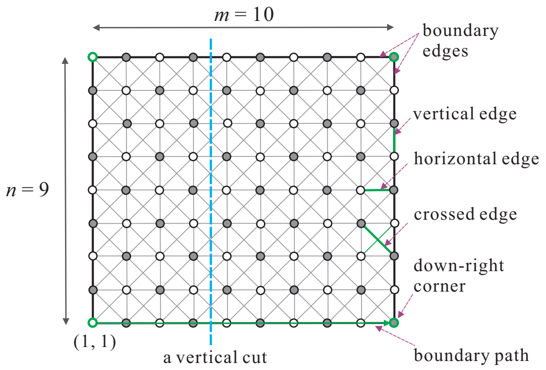

Fig. 1. A rectangular supergrid graph $R(10, 9)$.

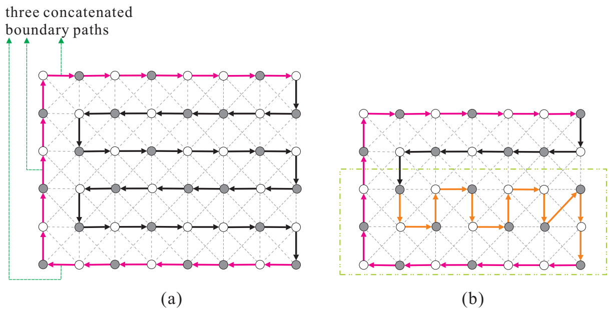

Fig. 2. A Hamiltonian cycle containing 3 concatenated boundary paths as a subpath for (a) $R(8, 6)$ and (b) $R(7, 5)$, where solid arrow lines indicate the edges in the cycles. Note that $R(8, 6)$ and $R(7, 5)$ are even-sized and odd-sized, respectively.

edge in the boundary of $R(m, n)$ is called *boundary edge*. The path is called *boundary* of $R(m, n)$ if it visits all vertices of the same boundary in $R(m, n)$. For example, Fig. 1 shows a rectangular supergrid graph $R(10, 9)$ which is called 9-rectangle and contains $2(9 + 8) = 34$ boundary edges. Fig. 1 also indicates the types of edges and corners.

In our algorithm, we need to partition a rectangular supergrid graph into two disjoint parts. The partition is defined as follows.

Definition 1. The *cut* operation of a rectangular supergrid graph $R(m, n)$ is a partition of $R(m, n)$ into two vertex disjoint rectangular supergrid graphs $R_1 = R(m_1, n_1)$ and $R_2 = R(m_2, n_2)$ such that ($m = m_1+m_2$ and $n_1 = n_2 = n$) or ($n = n_1+n_2$ and $m_1 = m_2 = m$). A cut is called *vertical* if $n_1 = n_2 = n$, and is called *horizontal* if $m_1 = m_2 = m$.

For example, the bold dashed line in Fig. 1 depicts a vertical cut of $R(10, 11)$ which partition it into $R(4, 11)$ and $R(6, 11)$.

In the past, we showed that rectangular supergrid graphs always contain Hamiltonian cycles except 1-rectangles. The following lemma states such a result concerning the Hamiltonicity of rectangular supergrid graphs.

Lemma 1. *(See [10]) Let $R(m, n)$ be a rectangular supergrid graph with $m, n \geqslant 2$. Then, $R(m, n)$ has a Hamiltonian cycle which contains at least three concatenated boundary paths as its subpath.*

Fig. 2 shows a Hamiltonian cycle for an even-sized or odd-sized rectangular supergrid graph found in Lemma 1. Each Hamiltonian cycle found by this lemma contains all

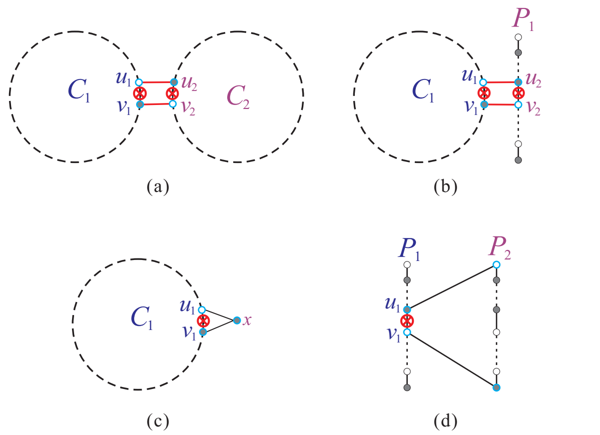

Fig. 3. A schematic diagram for (a) Proposition 2, (b) Proposition 3, (c) Proposition 4, and (d) Proposition 5, where ⊗ represents the destruction of an edge while constructing a cycle or a path.

the boundary edges on the three sides of the rectangular supergrid graph. This shows that for any rectangular supergrid graph $R(m, n)$ with $m, n \geqslant 2$, we can always construct a Hamiltonian cycle such that it contains all the boundary edges except one side of $R(m, n)$.

Let $(R(m, n), s, t)$ denote the rectangular supergrid graph $R(m, n)$ with two specified distinct vertices s and t. Without loss of generality, we will assume that $s_x \leqslant t_x$. We denote a Hamiltonian path between s and t in $R(m, n)$ by $HP(R(m, n), s, t)$. We say that $HP(R(m, n), s, t)$ exists if there is a Hamiltonian path between s and t in $R(m, n)$. From Lemma 1, we know that $HP(R(m, n), s, t)$ exists if $m, n \geqslant 2$ and (s, t) is an edge in the constructed Hamiltonian cycle of $R(m, n)$.

We next give some observations on the relations among cycle, path, and vertex. These propositions can be used in verifying our results. Let C_1 and C_2 be two vertex-disjoint cycles of a graph G. If there exist two edges $e_1 = (u_1, v_1) \in C_1$ and $e_2 = (u_2, v_2) \in C_2$ such that $e_1 \approx e_2$, then C_1 and C_2 can be concatenated into a cycle of G. Thus we have the following proposition.

Proposition 2. *Let C_1 and C_2 be two vertex-disjoint cycles of a graph G. If there exist two edges $e_1 \in C_1$ and $e_2 \in C_2$ such that $e_1 \approx e_2$, then C_1 and C_2 can be combined into a cycle of G. (see Fig. 3(a))*

Let C_1 be a cycle and let P_1 be a path in a graph G such that $V(C_1) \cap V(P_1) = \emptyset$. If there exist two edges $e_1 \in C_1$ and $e_2 \in P_1$ such that $e_1 \approx e_2$, then C_1 and P_1 can be combined into a path P of G with $start(P) = start(P_1)$ and $end(P) = end(P_1)$. Fig. 3(b) depicts such a construction, and hence the following proposition holds true.

Proposition 3. *Let C_1 and P_1 be a cycle and a path, respectively, of a graph G such that $V(C_1) \cap V(P_1) = \emptyset$. If there exist two edges $e_1 \in C_1$ and $e_2 \in P_1$ such that $e_1 \approx e_2$, then C_1 and P_1 can be combined into a path of G. (see Fig. 3(b))*

The above observation can be extended to a vertex x, where $P_1 = x$, as depicted in Fig. 3(c), and we then have the following proposition.

Proposition 4. *Let C_1 be a cycle (path) of a graph G and let*

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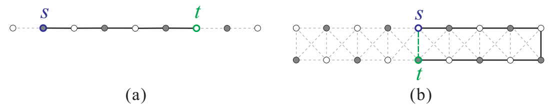

Fig. 4. Rectangular supergrid graph in which there is no Hamiltonian path between s and t, where the solid lines indicate the longest path between s and t.

x be a vertex in $G - V(C_1)$. If there exists an edge (u_1, v_1) in C_1 such that $u_1 \sim x$ and $v_1 \sim x$, then C_1 and x can be combined into a cycle (path) of G. (see Fig. 3(c))*

Let P_1 and P_2 be two vertex-disjoint paths of a graph G. If there exists one edges $(u_1, v_1) \in P_1$ such that $u_1 \sim start(P_2)$ and $v_1 \sim end(P_2)$, then P_1 and P_2 can be combined into a path P of G with $start(P) = start(P_1)$ and $end(P) = end(P_1)$. Hence, the following observation is true.

Proposition 5. *Let P_1 and P_2 be two paths of a graph G such that $V(P_1) \cap V(P_2) = \emptyset$. If there exists one edge $(u_1, v_1) \in P_1$ or $(u_2, v_2) \in P_2$ such that ($u_1 \sim start(P_2)$ and $v_1 \sim end(P_2)$) or ($u_2 \sim start(P_1)$ and $v_2 \sim end(P_1)$), then P_1 and P_2 can be combined into a path of G. (see Fig. 3(d))*

III. The Hamiltonian connected properties of 1-rectangle, 2-rectangle, and 3-rectangle

In this section, we will discuss the Hamiltonian connectivity of 1-, 2-, and 3-rectangles. We first observe two conditions for $HP(R(m, n), s, t)$ with $n = 1$ or $n = 2$ does not exist. We then prove that $HP(R(m, 3), s, t)$ does exist, i.e., any 3-rectangle always contains a Hamiltonian path between any two distinct vertices. Consider 1-rectangle $(R(m, 1), s, t)$. The following condition implies $HP(R(m, 1), s, t)$ does not exist.

(F1) $R(m, n)$ is a 1-rectangle, and either s or t is not a corner vertex (see Fig. 4(a)).

Obviously, if $(R(m, 1), s, t)$ does not satisfy condition (F1), then $HP(R(m, 1), s, t)$ does exist and hence the following lemma holds true.

Lemma 6. *Let $R(m, 1)$ be a 1-rectangle with $m \geqslant 2$ and let s, t be its two distinct vertices. Then, $HP(R(m, 1), s, t)$ does exist if $(R(m, 1), s, t)$ does not satisfy condition (F1).*

Next, we consider $(R(m, 2), s, t)$ with $m \geqslant 2$. By inspection, the following condition implies the graph $R(m, 2)$ has no Hamiltonian (s, t)-path.

(F2) $R(m, n)$ is a 2-rectangle, and (s, t) is a vertical edge but is not a boundary edge of $R(m, n)$ (see Fig. 4(b)).

When $(R(m, 2), s, t)$ does not satisfy condition (F2), we will prove $HP(R(m, 2), s, t)$ to be existent as the following lemma.

Lemma 7. *Let $R(m, 2)$ be a 2-rectangle with $m \geqslant 2$ and let s, t be two distinct vertices of it. Then, $HP(R(m, 2), s, t)$ does exist if $(R(m, 2), s, t)$ does not satisfy condition (F2).*

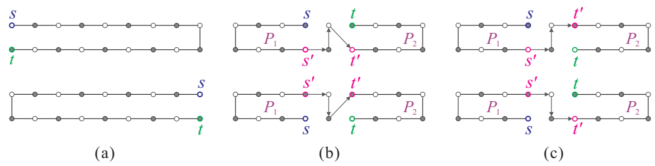

Fig. 5. The Hamiltonian path $HP(R(m, 2), s, t)$ when $(R(m, 2), s, t)$ does not satisfy condition (F2), where (a) $s_x = t_x$, (b) $s_x < t_x$ and $s_y = t_y$, and (c) $s_x < t_x$ and $s_y \neq t_y$.

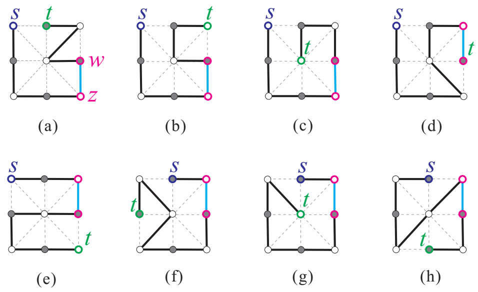

Fig. 6. The Hamiltonian path $HP(R(3, 3), s, t)$, where solid lines indicate the constructed Hamiltonian path.

Proof: Without loss of generality, assume that $s_x \leqslant t_x$. There are two cases:

Case 1: $s_x = t_x$. In this case, (s, t) is a vertical edge in $R(m, 2)$. Since $(R(m, 2), s, t)$ does not satisfy condition (F2), (s, t) is a boundary edge. Then, we can construct a Hamiltonian (s, t)-path as shown in Fig. 5(a).

Case 2: $s_x < t_x$. Suppose $s_y = t_y$. Let $s' \sim s$ and $t' \sim t$ such that $s'_y \neq s_y$ and $t'_y \neq t_y$. We can make a path P_1 from s to s' and a path P_2 from t to t' as shown in Fig. 5(b), and then connect s' to t' by a path P_3. Note that if $s \sim t$ then $s' \sim t'$. Then, $P_1 \Rightarrow P_3 \Rightarrow P_2$ is a $HP(R(m, 2), s, t)$. The case of $s_y \neq t_y$ can be proved similarly (see Fig. 5(c)).

It follows from the above two cases that $HP(R(m, 2), s, t)$ does exist when $(R(m, 2), s, t)$ does not satisfy condition (F2). Thus, the lemma holds true. ■

The conditions of (F1) and (F2) are called *forbidden* for $(R(m, n), s, t)$. In the rest of the paper, we will prove that $HP(R(m, n), s, t)$ exists if $(R(m, n), s, t)$ does not satisfy the forbidden conditions. We first consider $(R(m, 3), s, t)$ and prove that $HP(R(m, 3), s, t)$ does exist as follows.

Lemma 8. *Let $R(m, 3)$ be a 3-rectangle with $m \geqslant 3$ and let s, t be any two distinct vertices in $R(m, 3)$. Then, $HP(R(m, 3), s, t)$ does exist and it contains at least one boundary edge in each boundary of $R(m, 3)$.*

Proof: Let $w = (w_x, w_y)$ be a vertex in $R(m, 3)$ such that $w_x = m$ and $w_y = 2$. Note that the down-left corner of $R(m, 3)$ is coordinated as $(1, 1)$. We claim that there exists a Hamiltonian path P between s and t in $R(m, 3)$ such that $(w, z) \in P$, (w, z) is a vertical and boundary edge of $R(m, 3)$, and P visits at least one boundary edge in each boundary of $R(m, 3)$. Then, the lemma holds true.

We prove this claim by induction on m. Initially, let $m = 3$. By symmetry, the possible relative locations of s and t are shown in Fig. 6. Fig. 6 also depicts the desired Hamiltonian (s, t)-path of $R(3, 3)$. Thus, the claim holds when $m = 3$.

Now, assume that the claim holds true when $m = k$, $k \geqslant 3$. Then, $R(k, 3)$ contains a Hamiltonian path P' between any two vertices s' and t' such that $w' = (k, 2)$, $(w', z') \in P'$, (w', z') is a vertical and boundary edge of $R(k, 3)$, and P' visits at least one boundary edge in each boundary of $R(k, 3)$. Note that $z' = (k, 1)$ or $(k, 3)$. Consider that $m = k+1$. Let s, t be any two distinct vertices of $R(k+1, 3)$. Without loss of generality, assume that $s_x \leqslant t_x$. We first make a vertical cut on $R(k+1, 3)$ for partitioning it into $R_1 = R(k, 3)$ and $R_2 = R(1, 3)$. Depending on the locations of s and t, we consider the following three cases:

Case 1: $s, t \in R_1$. Let $w = (k+1, 2)$, $z = (k+1, 1)$, and $\tilde{z} = (k+1, 3)$ be the vertices of R_2. Let $P_2 = z \to w \to \tilde{z}$. Then, P_2 is the Hamiltonian path of R_2 with $start(P_2) = z$ and $end(P_2) = \tilde{z}$. By the induction hypothesis, there exists a Hamiltonian (s, t)-path P_1 of $R_1 = R(k, 3)$ such that $(w', z') \in P_1$ and P_1 visits at least one boundary edge in each boundary of $R(k, 3)$, where $w' = (k, 2)$ and $z' = (k, 1)$ or $(k, 3)$. By the structure of rectangular supergrid graphs, $z(= start(P_2)) \sim z'$ and $\tilde{z}(= end(P_2)) \sim w'$, or $z \sim w'$ and $\tilde{z} \sim z'$. By Proposition 5, P_1 and P_2 can be combined into a Hamiltonian (s, t)-path P of $R(k+1, 3)$ such that $(w, z) \in P$. Since P_1 visits at least one boundary edge in each boundary of $R(k, 3)$, P also visits at least one boundary edge in each boundary of $R(k+1, 3)$.

Case 2: $s \in R_1$ and $t \in R_2$. Consider the position of t. There are two subcases:

Case 2.1: $t = (k+1, 2)$. Let $w = t$. Depending on the location of s, we have the following three subcases:

Case 2.1.1: $s = (k, 2)$. Let $t' = (k, 1)$, $z' = (k, 3)$, $p = (k+1, 3)$, and let $z = (k+1, 1)$. By the induction hypothesis, there exists a Hamiltonian (s, t')-path P_1 of $R(k, 3)$ such that $(s, z') \in P_1$ and P_1 visits at least one boundary edge in each boundary of $R(k, 3)$. Clearly, $(s, t') \notin P_1$. Since $p \sim s$, $p \sim z'$, and $(s, z') \in P_1$, by Proposition 4 P_1 and p can be combined into a path P'_1 with $start(P'_1) = s$ and $end(P'_1) = t'$. Let $P_2 = z \to t$. Since $t'(= end(P'_1)) \sim z(= start(P_2))$, $P'_1 \Rightarrow P$ forms a Hamiltonian (s, t)-path P of $R(k+1, 3)$ such that P visits at least one boundary edge in each boundary of $R(k+1, 3)$ and $(w, z) \in P$.

Case 2.1.2: $s = (k, 1)$ or $(k, 3)$. Without loss of generality, assume that $s = (k, 1)$. Let $t' = (k, 2)$, $z' = (k, 3)$, $p = (k+1, 3)$, and let $z = (k+1, 1)$. By the same construction in Case 2.1.1, we can construct a desired Hamiltonian (s, t)-path P of $R(k+1, 3)$.

Case 2.1.3: $s \notin \{(k, 1), (k, 2), (k, 3)\}$. Let $t' = (k, 2)$. By the induction hypothesis, there exists a Hamiltonian (s, t')-path P_1 of $R(k, 3)$ such that $(t', z') \in P_1$ and P_1 visits at least one boundary edge in each boundary of $R(k, 3)$, where $z' = (k, 1)$ or $(k, 3)$. Let p be the vertex of R_2 such that (p, z') is a horizontal edge in $R(k+1, 3)$, and let $z \in \{(k+1, 1), (k+1, 3)\} - \{p\}$. Then, the same construction in Case 2.1.1 can be used to obtain the desired Hamiltonian path of $R(k+1, 3)$.

Case 2.2: $t = (k+1, 1)$ or $(k+1, 3)$. Without loss of generality, assume that $t = (k+1, 1)$. Let $w = (k+1, 2)$, $z = (k+1, 3)$, and let $P_2 = z \to w \to t$. If $s = (k, 2)$, then let $t' = (k, 3)$; otherwise, let $t' = (k, 2)$. By the induction hypothesis, there exists a Hamiltonian (s, t')-path P_1 of $R(k, 3)$ such that P_1 visits at least one boundary edge in each boundary of $R(k, 3)$. Since $t'(= end(P_1)) \sim z(= start(P_2))$, $P_1 \Rightarrow P_2$ is the desired Hamiltonian (s, t)-path of $R(k+1, 3)$.

Case 3: $s, t \in R_2$. Depending on whether $s \sim t$, we have the following two subcases:

Case 3.1: $s \sim t$. In this subcase, $s, t \in \{(k+1, 1), (k+1, 2)\}$ or $s, t \in \{(k+1, 2), (k+1, 3)\}$. Consider that $s = w = (k+1, 2)$ and $t = (k+1, 3)$. Let $z = (k+1, 1)$, $s' = (k, 2)$, and let $t' = (k, 3)$. Then, $s' \sim z$ and $t' \sim t$. By the induction hypothesis, there exists a Hamiltonian (s', t')-path P_1 of $R(k, 3)$ such that P_1 visits at least one boundary edge in each boundary of $R(k, 3)$. Let $P = s \to z \to P_1 \to t$. Then, P is the desired Hamiltonian (s, t)-path of $R(k+1, 3)$. The other cases can be proved similarly.

Case 3.2: $s \nsim t$. In this subcase, $s, t \in \{(k+1, 1), (k+1, 3)\}$. Without loss of generality, assume that $s = z = (k+1, 1)$ and $t = (k+1, 3)$. Let $w = (k+1, 2)$, $s' = (k, 1)$, and let $t' = (k, 2)$. Then, $s' \sim w$ and $t' \sim t$. By the induction hypothesis, there exists a Hamiltonian (s', t')-path P_1 of $R(k, 3)$ such that P_1 visits at least one boundary edge in each boundary of $R(k, 3)$. Let $P = s \to w \to P_1 \to t$. Then, P is the desired Hamiltonian (s, t)-path of $R(k+1, 3)$.

We have considered any case for constructing a Hamiltonian (s, t)-path P of $R(k+1, 3)$ such that P visits at least one boundary edge in each boundary of $R(k+1, 3)$ and $(w, z) \in P$, where $w = (k+1, 2)$ and $z \in \{(k+1, 1), (k+1, 3)\}$. Thus, the claim holds true when $m = k+1$. By induction, the claim holds true for $m \geqslant 3$. This completes the proof of the lemma. ■

We have proved the Hamiltonian connectivity of 3-rectangles. In the next section, we will verify the Hamiltonian connectivity of $R(m, n)$ for $m, n > 3$.

IV. The Hamiltonian connectivity of rectangular supergrid graphs

By Lemma 8, $HP(R(m, 3), s, t)$ does exist for $m \geqslant 3$ and any two distinct vertices s, t. In this section, we assume that a rectangular supergrid graph $R(m, n)$ satisfies $m \geqslant n > 3$. We will construct a Hamiltonian (s, t)-path of $R(m, n)$, and hence $HP(R(m, n), s, t)$ does exist. Following the technique used in [3], [15], [18], we will develop an algorithm for finding a Hamiltonian (s, t)-path of $R(m, n)$. The algorithm uses the divide-and-conquer technique and is outlined in the following steps:

- Step 1: Partition $R(m, n)$ into five disjoint rectangular supergrid subgraphs R_1–R_5 by a *peeling* operation such that $s, t \in R_5$, where a peeling operation consists of two vertical and two horizontal cuts on $R(m, n)$ and is defined later;
- Step 2: Construct Hamiltonian cycles of R_1–R_4;
- Step 3: Construct a Hamiltonian (s, t)-path or a longest (s, t)-path of R_5;
- Step 4: Combine all constructed Hamiltonian cycles and the Hamiltonian (longest) path into a Hamiltonian (s, t)-path of $R(m, n)$.

Before giving the peeling operation in Step 1, we first introduce the following notation:

Definition 2. Two vertices u and v in $R(m, n)$ are called *antipodes* if
(1) $\min\{u_x, v_x\} \leqslant 2$ and $\max\{u_x, v_x\} \geqslant m-1$, and
(2) $\min\{u_y, v_y\} \leqslant 2$ and $\max\{u_y, v_y\} \geqslant n-1$.

The peeling operation in Step 1 is then defined as follows:

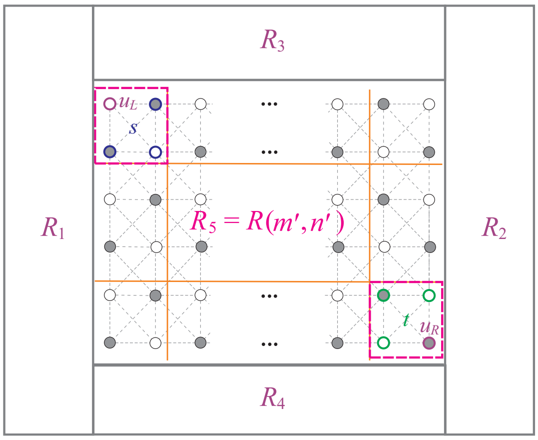

Fig. 7. A schematic diagram for a peeling operation on $(R(m,n),s,t)$ when $s_x \leqslant t_x$ and $s_y \leqslant t_y$.

Definition 3. For $(R(m,n),s,t)$, a *peeling* on $R(m,n)$ consists of two vertical and two horizontal cuts to partition $R(m,n)$ into five disjoint rectangular supergrid subgraphs R_1, R_2, R_3, R_4, R_5 as illustrated in Fig. 7, which satisfy
(1) $s,t \in R_5$ and are antipodes in R_5;
(2) each of R_1, R_2, R_3, R_4 is either a rectangular supergrid graph $R(m',n')$ with $m',n' > 1$, or empty;
(3) $R_1 = \emptyset$ (resp., $R_2 = \emptyset$, $R_3 = \emptyset$, $R_4 = \emptyset$) if and only if $1 \leqslant \min\{s_x,t_x\} \leqslant 2$ (resp., $m-1 \leqslant \max\{s_x,t_x\} \leqslant m$, $n-1 \leqslant \max\{s_y,t_y\} \leqslant n$, $1 \leqslant \min\{s_y,t_y\} \leqslant 2$).

Usually the two vertical cuts of a peeling operation are performed before the two horizontal cuts. Fig. 7 depicts a schematic diagram for the relations among R_1–R_5, s and t when $s_x \leqslant t_x$ and $s_y \geqslant t_y$. After performing a peeling operation on $(R(m,n),s,t)$, we construct four Hamiltonian cycles of R_1–R_4 by Lemma 1, one Hamiltonian or longest (s,t)-path of R_5, and then combine them into a Hamiltonian (s,t)-path of $R(m,n)$ by Propositions 2–5. In Step 4, we first combine the Hamiltonian cycles of R_1–R_4 to a larger cycle and then combine it with the Hamiltonian or longest path of R_5. For example, Fig. 8 depicts the construction of $HP(R(11,9),s,t)$, where $s=(4,6)$ and $t=(7,4)$. For the case of $t_y \geqslant s_y$, we can prove the Hamiltonian connectivity of $R(m,n)$ by the same arguments in proving the case of $s_y \geqslant t_y$. From now on, we assume that $s_x \leqslant t_x$ and $s_y \geqslant t_y$ for $(R(m,n),s,t)$. Let $u_{_L}$ and $u_{_R}$ be the upper-left corner and down-right corner of R_5, respectively. Since s and t are antipodes in R_5, $s \in N_{R_5}[u_{_L}]$ and $t \in N_{R_5}[u_{_R}]$.

By the definition of a peeling operation on $(R(m,n),s,t)$, $R_i = R(m_i,n_i)$, $1 \leqslant i \leqslant 4$, is a rectangular supergrid graph with $m_i, n_i \geqslant 2$, or empty. By Lemma 1, R_i has a Hamiltonian cycle C_i which contains three concatenated boundary paths as its subpath if $R_i \neq \emptyset$. We can make such a Hamiltonian cycle C_i of R_i such that the boundary paths of C_i are placed to face the neighboring rectangular supergrid subgraphs R_j's ($1 \leqslant j \leqslant 5$ and $j \neq i$). For example, Fig. 8 shows these Hamiltonian cycles C_i's of R_i's for $1 \leqslant i \leqslant 4$. Thus, we have the following lemma.

Lemma 9. *Let $R_i = R(m_i,n_i)$, $1 \leqslant i \leqslant 4$, be a rectangular supergrid graph for a peeling operation on $(R(m,n),s,t)$. If $R_i \neq \emptyset$, then there exists a Hamiltonian cycle C_i of R_i such that the three boundary paths of C_i are placed to face the neighboring rectangular supergrid subgraphs R_j's ($1 \leqslant j \leqslant 5$ and $j \neq i$).*

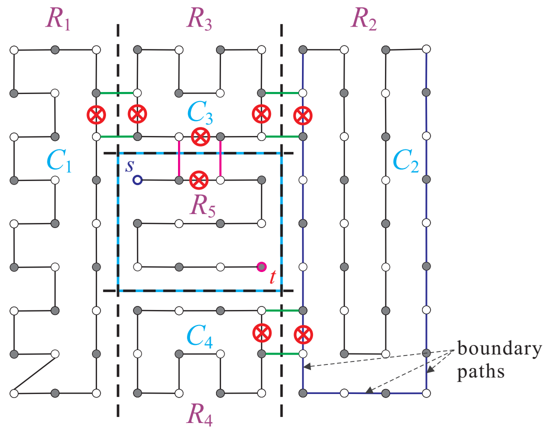

Fig. 8. A Hamiltonian (s,t)-path of $R(11,9)$ combined from all constructed Hamiltonian cycles of R_1–R_4 and the Hamiltonian (s,t)-path of R_5, where bold dashed lines indicate the cuts in a peeling operation, solid lines indicate the constructed Hamiltonian path, and ⊗ represents the destruction of an edge under construction.

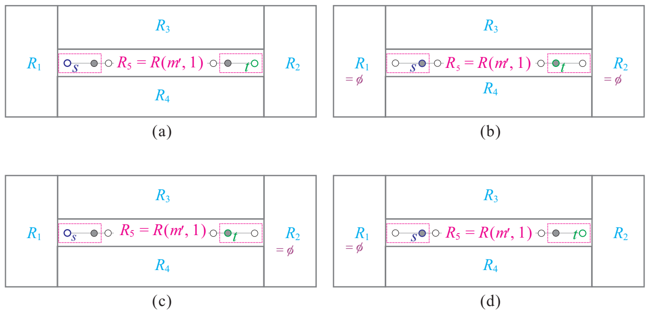

Fig. 9. The possible cases for $R_5 = R(m',1)$, where (a) s,t are corners of R_5, (b) s,t are not corners of R_5, (c) s is a corner but t is not a corner of R_5, and (d) t is a corner but s is not a corner of R_5.

In the following, we will show how to construct a Hamiltonian (s,t)-path of $R(m,n)$. For $(R(m,n),s,t)$ with $m \geqslant n \geqslant 4$, w.l.o.g., assume that $s_x \leqslant t_x$ and $s_y \geqslant t_y$. Let $R(m,n)$ be partitioned into five disjoint rectangular supergrid subgraphs R_1–R_5 by a peeling operation. Depending on the size of R_5, we make the following four claims:

Claim 1: If $R_5 = R(m',n')$ is a 1-rectangle, then $HP(R(m,n),s,t)$ does exist.
Claim 2: If $R_5 = R(m',n')$ is a 2-rectangle, then $HP(R(m,n),s,t)$ does exist.
Claim 3: If $R_5 = R(m',n')$ is a 3-rectangle, then $HP(R(m,n),s,t)$ does exist.
Claim 4: If $R_5 = R(m',n')$ satisfies $m',n' \geqslant 4$, then $HP(R(m,n),s,t)$ does exist.

We can prove the above four claims by constructing a Hamiltonian (s,t)-path of $R(m,n)$. For example, in proving Claim 1 we consider the possible cases as follows:
Case 1: s and t are corners of R_5.
Case 2: neither s nor t is a corner of R_5.
Case 3: either s or t is a corner of R_5.
The above cases are depicted in Fig. 9. We then construct $HP(R(m,n),s,t)$ for each case as shown in Fig. 10.
Due to the space limitation, we omit the proofs of these four claims. It immediately follows from Lemma 8 and

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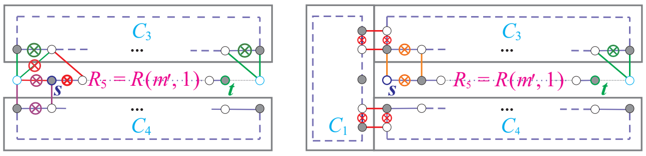

(a) (b)

Fig. 10. The construction of $HP(R(m,n),s,t)$ for $R_5 = R(m',1)$ under (a) s,t are not corners of R_5 and $R_3, R_4 \neq \emptyset$, and (b) s is a corner but t is not a corner of R_5 and $R_1, R_3, R_4 \neq \emptyset$, where C_i, $i = 1,3,4$, is a Hamiltonian cycle of R_i and $\otimes$ represents the destruction of an edge under the construction.

Claims 1–4 that the following theorem holds true.

Theorem 1. *Let $R(m,n)$ be a rectangular supergrid graph with $m,n \geqslant 3$, and let s,t be two distinct vertices in $R(m,n)$. Then, $HP(R(m,n),s,t)$ does exist.*

Combining Lemmas 6–7 with Theorem 1, we conclude the following theorem.

Theorem 2. *Let $R(m,n)$ be a rectangular supergrid graph with $mn \geqslant 2$, and let s,t be two distinct vertices in $R(m,n)$. If $(R(m,n),s,t)$ does not satisfy forbidden conditions (F1) and (F2), then $HP(R(m,n),s,t)$ does exist.*

V. Concluding Remarks

The Hamiltonian cycle and Hamiltonian path problems for supergrid graphs were known to be NP-complete. Deciding where a supergrid graph is Hamiltonian connected is hence a NP-complete problem. The Hamiltonian cycle and path problems for rectangular supergrid graphs are easy to solve. In this paper, we provide a constructive proof to show that rectangular supergrid graphs are Hamiltonian connected except two trivial forbidden conditions. It is interesting to see whether the Hamiltonian related problems for the other subclasses of supergrid graphs, including solid and locally connected, are polynomial solvable. We would like to post it as an open problem to interested readers.

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