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Structural Aspects of Propositional SAT

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Abstract—**Some structural aspects in the area of propositional satisfiability of CNF formulas are discussed. We consider decompositions of formulas and introduce a decomposition of the class of unsatisfiable formulas. Further we investigate specific base hypergraphs of formulas recalling the fibre perspective.**

Keywords: CNF-formula, monotonicity, satisfiability, NP-completeness, hypergraph

1 Introduction

A fundamental problem in mathematics is the NP versus P problem which is attacked within the theory of NP-completeness. The genuine and one of the most important NP-complete problems is the propositional satisfiability problem (SAT) for conjunctive normal form (CNF) formulas [5]. More precisely, SAT is the natural NP-complete problem and thus lies at the heart of computational complexity theory. Moreover, SAT plays a fundamental role in the theory of designing exact algorithms, and it has a wide range of applications because many problems can be encoded as a SAT problem via reduction [11, 9] due to the rich expressiveness of the CNF language. Important areas where SAT plays a vital role are formal verification [18], bounded model checking [4], and artificial intelligence. In industrial applications most often the modelling CNF formulas are of a specific structure. And therefore it would be desirable to have fast algorithms for such instances. The applicational area is pushed by the fact that meanwhile several powerful solvers for SAT have been developed (cf. e.g. [13, 17] and references therein). Also from a theoretical point of view one is interested in classes for which SAT can be solved in polynomial time. There are known several classes, for which SAT can be tested efficiently, such as quadratic formulas, Horn formulas, matching formulas, nested formulas etc. [1, 3, 7, 12, 14, 10, 16, 19]. So, the structure of the set of all CNF formulas is of great importance. Therefore, in the present paper several structural approaches regarding CNF-SAT are proposed and discussed. The intention is to find new directions for SAT solving. More concretely we consider several decompositions of formulas and subclasses defined through that. Moreover we define the monotonicity index yielding a decomposition of the class of unsatisfiable formulas. Moreover, we recall the fibre view on clause sets and investigate the structure of specific base hypergraphs in this framework.

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2 Preliminaries

A Boolean variable x taking values from $\{0,1\}$ induces a positive literal (variable x) or a negative literal (negated variable $\overline{x}$). A clause c is a disjunction of different literals, and is represented as a set $c = \{l_1, \ldots, l_{|c|}\}$. Setting a literal to 1 means to set the corresponding variable accordingly. A formula C is a conjunction of different clauses and is considered as a set of its clauses $C = \{c_1, \ldots, c_{|C|}\}$. Let CNF denote the set of formulas (free of duplicate clauses) in conjunctive normal form over. For a formula C (clause c), by ($V(C)$ ($V(c)$) denote the set of variables occurring in C (c). For convenience we allow the empty set to be a formula $\varnothing \in$ CNF which is satisfiable in any sense. A clause containing no negated literal is called *positive monotone*. A clause containing only negated variables is called *negative monotone*. Let CNF_+ (CNF_-) denote the collection of all *positive (negative) monotone* clause sets, which are defined as those containing only positive (negative) monotone clauses. For a finite set M, let 2^M denote its powerset. Given $C \in$ CNF, SAT asks whether there is a truth assignment $t : V(C) \to \{0,1\}$ such that there is no $c \in C$ all literals of which are set to 0. If such an assignment exists it is called a *model* of C, and $M(C)$ denotes the collection of all models of C. For any t and $X \in 2^{V(C)}$, let t^X denote the assignment obtained from t by $t^X(x) := 1 - t(x)$ for every $x \in X$ and $t^X(x) := t(x)$ for the remaining variables. Let SAT $\subseteq$ CNF denote the collection of all clause sets for which there is a model, and UNSAT := CNF $\setminus$ SAT. Clauses containing a complemented pair of literals are always satisfied. Hence, it is assumed throughout that clauses only contain literals over different variables. Exactly those clauses c of a formula $C \in$ CNF which all have the same variable set $b = V(c) \subseteq V(C)$ yield the *fibre* $C_b = \{c \in C : V(c) = b\}$ of C over b [15]. The *base hypergraph* $\mathcal{H}(C) = (V(C), B(C))$ of C is given by the hyperedge set $B(C) := \{V(c) : c \in C\} \in \mathrm{CNF}_+$. Conversely, we can start with a fixed arbitrary hypergraph $\mathcal{H} = (V, B)$ serving as a base hypergraph if its vertices $x \in V$ are regarded as Boolean variables such that for every $x \in V$ there is a $b \in B$ containing x. By $W_b := \{c : V(c) = b\}$ denote the collection of all possible clauses over a fixed $b \in B$. The set of all clauses over $\mathcal{H}$ is $K_{\mathcal{H}} := \bigcup_{b \in B} W_b$. A $\mathcal{H}$*-based* formula is a subset $C \subseteq K_{\mathcal{H}}$ such that $C_b := C \cap W_b \neq \emptyset$, for every $b \in B$.

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3 Decomposing Formulas

First let us consider a mapping on CNF defined as follows. For $C \in \text{CNF}$ and a subset $X \subseteq V(C)$ let $C^X \in \text{CNF}$ denote the formula that is obtained from C by *flipping*, i.e. complementing, all variables in X. We say that C^X is the result of flipping C by X. Obviously, for $C = \{c_1, \ldots, c_r\}$ we have $C^X = \{c_1^X, \ldots, c_r^X\}$, where c^X is obtained from c by complementing all literals over variables in $X \cap V(c)$ Flipping C first by $X \in 2^{V(C)}$ and then independently by $Y \in 2^{V(C)}$ obviously yields $(C^X)^Y = C^{X \oplus Y}$, meaning that the composition is commutative. Clearly, the effect of flipping any formula by $\varnothing$ equals the identity: $C^\varnothing = C^{X \oplus X} = C$. For short we write $C^\gamma := C^{V(C)}$, in case that all variables are flipped, similarly $c^\gamma := c^{V(c)}$. Observe that a formula $C \in \text{CNF}$ can be split into a maximal *symmetric* $S(C)$ and a maximal *anti-symmetric* part $A(C)$, $C := A(C) \cup S(C)$. Here the components are defined as $A(C) := \{c \in C : c^\gamma \notin C\}$ and $S(C) := \{c \in C : c^\gamma \in C\}$ which is the complement clause set of $A(C)$ in C. Motivated hereby let $\mathcal{S} := \{C \in \text{CNF} : C = C^\gamma\}$ be the set of *symmetric formulas* and let $\mathcal{A} := \{C \in \text{CNF} : C = A(C)\}$ denote the set of *anti-symmetric* formulas in CNF. For convenience, we define $\varnothing \in \mathcal{S}$, $\varnothing \notin \mathcal{A}$. An arbitrary $C \in \text{CNF}$ can also be split into three disjoint parts (any or all of which might be empty) $C = C_+ \cup C_- \cup C_\pm$ where $C_+ \in \text{CNF}_+$ (resp. $C_- \in \text{CNF}_-$) is the collection of all positive (resp. negative) monotone clauses in C, and $C_\pm$ is the remaining subformula. Observe that C is satisfiable if $C_- = \varnothing$ or $C_+ = \varnothing$. Regarding the connection of both types of decompositions of a formula $C \in \text{CNF}$ a first observation is that $C = C_+$ or $C = C_-$ implies $C = A(C)$.

Lemma 1 *For $C \in \text{CNF}$ and $X \subseteq V(C)$ holds $C \in \text{SAT}$ if and only if $C^X \in \text{SAT}$. Moreover $M(C)$ is in 1:1-correspondence to $M(C^X)$.*

Proof. It suffices to prove that the set $M(C)$ of models of C is in 1:1-correspondence to the set $M(C^X)$ of models of C^X. Since this particularly implies $C \notin \text{SAT} \Leftrightarrow M(C) = \varnothing \Leftrightarrow M(C^X) = \varnothing \Leftrightarrow C^X \notin \text{SAT}$, the first assertion is implied by the second. To verify the second assertion, it is not hard to see that flipping the values of X in a model t of C yields the unique model t^X of C^X and vice versa. Hence the assertion follows. □

A characterization of the class SAT appears as follows.

Theorem 1 *$C \in \text{CNF}$ is satisfiable if and only if there is an $X \subseteq V(C)$ (which may be empty) such that $C_+^X = \varnothing$ or $C_-^X = \varnothing$.*

Proof. If there is an X with $C^X \in \text{SAT}$ then also $C \in \text{SAT}$ according to Lemma 1. For the other direction assume that $C \in \text{CNF}$ is satisfiable with model t, and let X_α containing those variables that are assigned by t to α, for $\alpha \in \{0, 1\}$. Assume that $C_\varepsilon \neq \varnothing$, for $\varepsilon \in \{+, -\}$, (otherwise we are done). It follows that $X_1 \neq \varnothing$, because otherwise $X_0 = V(C)$ meaning $C_- = \varnothing$. Now we claim that flipping the variables in X_1 leads to a formula with vanishing positive monotone part. Indeed, suppose that all variables in X_1 are flipped. Then clearly each $c \in C_+$ after flipping contains at least one negated variable. Let $c \in C_-$ then c is satisfied by a variable set to 0 by t hence c contains at least one literal that is not flipped after flipping the variables in X_1. Hence, each $c \in C_-$ contains after flipping at least one negated variable. Finally, let $c \in C_\pm$ and assume that c is satisfied by a variable set to 1 by t then this variable must be unnegated in c and thus is flipped. Hence, such a clause after flipping still contains at least one negated variable. In the remaining case, $c \in C_\pm$ is satisfied by t through a variable set to 0. Therefore this variable occurring negated is not addressed by the flipping process. In summary we have that $C_+^{X_1} = \varnothing$. □

Lemma 2 *$\mathcal{S}$ and $\mathcal{A}$ are invariant subsets of* CNF *more precisely*
(i) For $C \in \mathcal{S}$, holds $C^X \in \mathcal{S}, \forall X \subseteq V(C)$,
(ii) For $C \in \mathcal{A}$, holds $C^X \in \mathcal{A}, \forall X \subseteq V(C)$.

Proof. First observe that

$$(c^X)^\gamma = c^{X \oplus V(C)} = (c^\gamma)^X$$

Next, by definition we always have $c \in C$ iff $c^X \in C^X$, for every $X \subseteq V(C)$. Hence for $C \in \mathcal{S}$ we have that $c^X \in C^X$ implies $c \in C$ implies $c^\gamma \in C$ implies $(c^\gamma)^X = (c^X)^\gamma \in C^X$, thus (i). Similarly, for $C \in \mathcal{A}$, $c^X \in C^X$ implies $c \in C$ implies $c^\gamma \notin C$ implies $(c^\gamma)^X = (c^X)^\gamma \notin C^X$, thus (ii). □

Lemma 3 SAT *and* UNSAT *both are invariant when variables are flipped.*

Proof. Let $C \in \text{CNF}$ and $X \subseteq V(C)$ be arbitrary. For $C \in \text{SAT}$ we have $C^X \in \text{SAT}$ according to Lemma 1. For $C \in \text{UNSAT}$ we have $C \notin \text{SAT}$ implying $C^X \notin \text{SAT}$ according to Lemma 1 hence $C^X \in \text{UNSAT}$. □

We shall call a set X such that C^X contains no positive- or negative monotone part a *sat-flipping* set. For $C \in \text{CNF}$, let $\mathcal{E}_\varepsilon(C)$ be the collection of all sets $X \subseteq V(C)$ with $C_\varepsilon^X = \varnothing$, $\varepsilon \in \{+, -\}$, and $\mathcal{E}(C) = \mathcal{E}_-(C) \cup \mathcal{E}_+(C)$. Clearly, $C \in \text{UNSAT}$ iff $\mathcal{E}(C) = \varnothing$. Observe that in general $\mathcal{E}_-(C) \cap \mathcal{E}_+(C) \neq \varnothing$ because there can occur sat-flipping sets X such that C^X contains no monotone part at all.

Lemma 4 *For $C \in \mathcal{S}$ we have $\mathcal{E}_-(C) = \mathcal{E}_+(C)$.*

Proof. In case $C \in \text{UNSAT}$ we have $\mathcal{E}_-(C) = \varnothing = \mathcal{E}_+(C)$. Now let $C \in \text{SAT}$. For C symmetric and $X \in \mathcal{E}_+(C)$ meaning $C_+^X = \varnothing$, assume that $C_-^X \neq \varnothing$. Due to Lemma 2 we have $C^X \in \mathcal{S}$ therefore every $c \in C_-^X$ yields $c^\gamma \in C_+^X \subseteq C^X$. Hence $C_+^X \neq \varnothing$ implying a contradiction, so $X \in \mathcal{E}_-(C)$ and $\mathcal{E}_+(C) \subseteq \mathcal{E}_-(C)$. The converse inclusion follows likewise exchanging the roles of $\mathcal{E}_+(C), \mathcal{E}_-(C)$. □

Relaxing the assumption from above to arbitrary instances yields equality with respect to the cardinalities as stated next.

Lemma 5 *For $C \in \text{CNF}$ we have $|\mathcal{E}_-(C)| = |\mathcal{E}_+(C)|$.*

Proof. The claim follows from $X \in \mathcal{E}_-(C)$ if and only if $V(C) \setminus X \in \mathcal{E}_+(C)$. The latter relation is true because $X \in \mathcal{E}_-(C)$ iff

$$\varnothing = C_-^X = (C_+^X)^\gamma = C_+^{X \oplus V(C)} = C_+^{V(C) \setminus X}$$

which is equivalent with $V(C) \setminus X \in \mathcal{E}_+(C)$. □

From the proof of Theorem 1 we can deduce the connection between sat-flipping sets and models.

Theorem 2 *Let $C \in \text{SAT}$ then*
(i) for every $t \in M(C)$ one has $t^{-1}(1) \in \mathcal{E}_+(C)$ and $t^{-1}(0) \in \mathcal{E}_-(C)$,
(ii) every $X \in \mathcal{E}_+(C)$ defines $t \in M(C)$ through $t^{-1}(1) = X$ and $Y \in \mathcal{E}_-(C)$ defines $t \in M(C)$ $t^{-1}(0) = Y$.

Proof. From the proof of Theorem 1 it follows that $t^{-1}(1) \in \mathcal{E}_+(C)$, therefore as shown in the proof of Lemma 5 we have $V(C) \setminus t^{-1}(1) = t^{-1}(0) \in \mathcal{E}_-(C)$ yielding (i). Since $C \in \text{SAT}$, an $X \in \mathcal{E}_+(C)$ exists meaning $C_+^X = \varnothing$. Thus setting all variables to 0 satisfies C^X, let $s \in M(C^X)$ be this model. Hence due to Lemma 1, $t := s^X \in M(C)$ and $t^{-1}(1) = X$. The last part of (ii) can be deduced analogously. □

The *fibre-view* as introduced in [15] regards a clause set C as composed of fibres over a hypergraph as mentioned in the preliminaries. Actually this approach yields another decomposition of formulas, namely C appears as the disjoint union of all its fibres $C = \bigcup_{b \in B(C)} C_b$. We shall return to this decomposition in Section 5.

4 The Monotonicity Index

An equivalence relation on CNF is defined as follows. $C \sim C'$ if $\exists X \subseteq V(C)$ such that $C' = C^X$. Obviously we then have $V(C) = V(C')$ and $|C| = |C'|$. This indeed defines an equivalence relation since reflexivity is given through $X = \varnothing$. Symmetry can be concluded from

$$C \sim C' \Rightarrow C' = C^X \Rightarrow (C')^X = C^{X \oplus X} = C \Rightarrow C \sim C'$$

for an appropriate $X \in 2^{V(C)}$ as $V(C) = V(C')$ is guaranteed. Finally, transitivity is implied as follows, if $C \sim C'$ via X and $C' \sim C''$ via Y then we have $V(C) = V(C'')$ and $C \sim C''$ via $X \oplus Y$.

For $C \in \text{CNF}$ denote by $[C]$ its class corresponding to the above defined equivalence relation.

Definition 1 *Let $C \in \text{CNF}$ then $\mu(C) := \min\{\min\{|C'_+|, |C'_-|\} : C' \in [C]\}$ is called the* monotonicity index *of C.*

Theorem 3 *$C \in \text{SAT}$ if and only if $\mu(C) = 0$.*

Proof. By Theorem 1 $C \in \text{SAT}$ if and only if there is $C' \in [C]$ with $C'_+ = \varnothing$ or $C'_- = \varnothing$ which is equivalent to $\mu(C') = 0$ for an appropriate $C' \in [C]$. Since $\mu(C) \geq 0$ for every $C \in \text{CNF}$, the latter is equivalent to $\mu(C) = 0$ finishing the proof. □

As a direct consequence we obtain the following result.

Corollary 1 *$C \in \text{UNSAT}$ if and only if $\mu(C) > 0$.*

That means if there is no sat-flipping set for a formula C eliminating exactly one monotone part from C then $C \in \text{UNSAT}$. Since $\mu(C)$ must always be a non-negative integer we are lead to a classification as follows.

Definition 2 *Set $\text{UNSAT}_k := \{C \in \text{CNF} : \mu(C) = k\}$, for every fixed integer $k > 0$.*

Lemma 6 *A decomposition of the class of unsatisfiable formulas is given via $\text{UNSAT}_i \cap \text{UNSAT}_j = \varnothing$, for $i \neq j$ and $\text{UNSAT} = \bigcup_{k>0} \text{UNSAT}_k$. Moreover UNSAT_k is invariant when variables are flipped.*

Proof. Only the last assertion needs an argument. Let $C \in \text{UNSAT}_k$ and $X \subseteq V(C)$ then as $C^X \in [C]$ we have $\mu(C^X) = \mu(C) = k$ □

Consider formulas C such that $(*) : C^X = C$, for every $X \subseteq V(C)$. Specifically, such formulas are necessarily symmetric, because flipping all variables yields the same formula, too.

Definition 3 *A non-empty formula having property $(*)$ is called* perfectly symmetric*; the class of exactly such instances is denoted as $\mathcal{P} \subseteq \mathcal{S}$.*

A perfectly symmetric formula C obviously represents a unit class $[C] = \{C\}$.

Lemma 7 *$W_b \in \mathcal{P}$ and $\mu(W_b) = 1$, for every set b of variables. Moreover for $C, C' \in \mathcal{P}$ it holds that $C \cup C' \in \mathcal{P}$.*

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Proof. First observe that property $(*)$ for a formula C is equivalent with that $c \in C$ implies $c^X \in C$ for an arbitrarily chosen $X \subseteq V(C)$ and $c \in C$. Indeed, then we have $c \in C \Rightarrow c^X \in C \Rightarrow c \in C^X$ hence $C \subseteq C^X$. Then it follows that $C = C^X$ because $|C^X| = |C|$. The reverse direction is obvious. To prove the lemma, let $c \in W_b$ and $X \in 2^b$ be chosen arbitrarily. Since $V(c^X) = V(c)$ we have $c^X \in W_b$ implying $W_b = W_b^X$ yielding the first claim. Since $|W_{b+}| = |W_{b-}| = 1$ it follows that $\mu(W_b) = 1$. Next assume that $C, C' \in \mathcal{P}$ then $c \in C \cup C'$ means $c \in C$ implying $c^X \in C$ or it means $c \in C'$ implying $c^X \in C'$ hence $c^X \in C \cup C'$, for every $X \subseteq V(C)$. □

Theorem 4 *A non-empty formula $C \in$ CNF is perfectly symmetric if and only if $W_{V(c)} \subseteq C$ for every $c \in C$. Moreover one explicitly has $\mu(C) = |B(C)|$ for every $C \in \mathcal{P}$, where $\mathcal{H}(C) = (V(C), B(C))$ is the base hypergraph of C. Finally $\mathcal{P} \subset$ UNSAT.*

Proof. Let $C \in$ CNF and assume that $W_{V(c)} \subseteq C$ for every $c \in C$ then $C = \bigcup_{b \in B(C)} W_b$. According to Lemma 7 it follows that $C \in \mathcal{P}$. Conversely, let $C \in \mathcal{P}$ and assume there is $c \in C$ and $c' \in W_{V(c)} \setminus C$. Clearly $V(c) = V(c')$ hence there is $X \in 2^{V(c)}$ such that $c^X = c' \notin C$ yielding a contradiction to $(*)$ valid for C, thus one obtains $W_{V(c)} \subseteq C$.

By the previously shown result one has for $C \in \mathcal{P}$ that $C = \bigcup_{b \in B(C)} C_b$ as disjoint union where $C_b = W_b$. Since $\min\{|W_{b+}|, |W_{b-}|\} = 1$, for every $b \in B(C)$, one obtains $\min\{|C'_+|, |C'_-|\} = \mu(C) = |B(C)|$. The last part immediately follows from Corollary 1 and the previous claim because for $C \in \mathcal{P}$ we have $C \neq \emptyset$ hence $B(C) \neq \emptyset$ thus $\mu(C) = |B(C)| \geq 1$. □

A direct consequence of the characterization above is the next complexity result.

Corollary 2 *It can be recognized in polynomial time in the size $\|C\|$ of C whether $C \in$ CNF is perfectly symmetric.*

Proof. For every $c \in C$ determine $V(c)$ and sort the results lexicographically. Thereby update a counter $N_{V(c)}$ for each c with the same $V(c)$. Finally check for each counter whether $N_{V(c)} = 2^{V(c)}$. If the latter is true for all counters output $C \in \mathcal{P}$ else $C \notin \mathcal{P}$. The correctness is given by Theorem 4. □

5 Specific Base Hypergraphs

Let $\mathcal{H} = (V, B)$ be a fixed but arbitrary base hypergraph with total clause set $K_{\mathcal{H}}$. As defined in [15] a *fibre-transversal* of $K_{\mathcal{H}}$ is a $\mathcal{H}$-based formula $F \subset K_{\mathcal{H}}$ such that $|F \cap W_b| = 1$, for every $b \in B$, this clause is denoted as $F(b)$. For $X \subseteq V$ we define F^X via $F^X(b) := (F(b))^X$, for every $b \in B$. A *compatible* fibre-transversal has the property that $\bigcup_{b \in B} F(b) \in W_V$. $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ is the set of all such fibre-transversals of $K_{\mathcal{H}}$. We can define a (compatible) fibre-transversal of a $\mathcal{H}$-based formula $C \subset K_{\mathcal{H}}$ as a (compatible) fibre-transversal $K_{\mathcal{H}}$ that is contained in C. A *diagonal* fibre-transversal is defined through the property that for each $F' \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ one has $F \cap F' \neq \emptyset$. Let $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ be the collection of all diagonal fibre-transversals of $K_{\mathcal{H}}$. As for the total clause set $K_{\mathcal{H}}$ we can define fibre-transversals for a $\mathcal{H}$-based formula $C \subset K_{\mathcal{H}}$ as follows. A fibre-transversal F of C contains exactly one clause of each fibre C_b of C. The collection of all fibre-transversals of C is denoted as $\mathcal{F}(C)$. We also define compatible and diagonal fibre-transversals of C via $\mathcal{F}_{\text{comp}}(C) := \mathcal{F}(C) \cap \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$, and $\mathcal{F}_{\text{diag}}(C) := \mathcal{F}(C) \cap \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$. Observe that every fibre-transversal $F \in \mathcal{F}(K_{\mathcal{H}})$ belongs to $\mathcal{A}$, one even obtains stronger results as follows.

Theorem 5 *For $\mathcal{H} = (V, B)$ with $V \neq \emptyset \neq B$, one has*
(i) $F \in \mathcal{F}(K_{\mathcal{H}})$ implies $F^X \in \mathcal{F}(K_{\mathcal{H}})$ for all $X \subseteq V$,
(ii) $F^X \neq F$ for every $F \in \mathcal{F}(K_{\mathcal{H}})$ and every $\emptyset \neq X \subseteq V$ specifically $\mathcal{F}(K_{\mathcal{H}}) \subseteq \mathcal{A}$,
(iii) $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}}) = [F]$, for any fixed $F \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$.

Proof. For proving (i) let $F \in \mathcal{F}(K_{\mathcal{H}})$ and $X \subseteq V$ then $F^X(b) = (F(b))^X \in W_b$ thus $1 = |F \cap W_b| = |F^X \cap W_b|$, for every $b \in B$, implying $F^X \in \mathcal{F}(K_{\mathcal{H}})$. Regarding (ii) suppose there are $F \in \mathcal{F}(K_{\mathcal{H}})$ and $X \subseteq V$ such that $F^X = F$ which makes sense because according to (i) $F^X \in \mathcal{F}(K_{\mathcal{H}})$ is guaranteed. Then we have $F^X(b) = (F(b))^X = F(b)$ for every $b \in B$. Thus the permutation induced by X is the identity map implying $X = \emptyset$. Considering (iii) let two members $F, F' \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ be chosen arbitrarily. We claim that there is $X \in 2^V$ such that $F^X = F'$ implying (iii). Indeed, by definition $\bigcup_{b \in B} F(b) =: c$ and $\bigcup_{b \in B} F'(b) =: c'$ both are clauses in $W_V \in \mathcal{P}$. Hence there must be $X \in 2^V$ with $c^X = c'$ implying that the restrictions to b of both are equal, thus $F^X(b) = c^X|_b = c'|_b = F'(b)$ for every $b \in B$ meaning $F^X = F'$. □

The following result proved in [15] characterizes satisfiability of a formula C in terms of compatible fibre-transversals in its based complement formula $\bar{C} := K_{\mathcal{H}} \setminus C$.

Theorem 6 *[15] For $\mathcal{H} = (V, B)$, let $C \subset K_{\mathcal{H}}$ be a $\mathcal{H}$-based formula such that $\bar{C}$ is $\mathcal{H}$-based, too. Then C is satisfiable if and only if $\bar{C}$ admits a compatible fibre-transversal F. Moreover, the union of all clauses in F^γ is a model of C.*

One can establish the existence of formulas for which no diagonal fibre-transversal of the total clause set exists, but unsatisfiable formulas $C \subset K_{\mathcal{H}}$ can exist although.

So we conclude that in general $C \in \text{UNSAT}$ is not equivalent to $\mathcal{F}_{\text{diag}}(C) \neq \varnothing$. However, things may be different if $\mathcal{H}$ is structured such that $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) \neq \varnothing$. So, we next pose the question whether under this assumption holds $C \in \text{UNSAT}$ iff $\mathcal{F}_{\text{diag}}(C) \neq \varnothing$. Observe that the implication $\Leftarrow$ holds because if C admits a diagonal fibre-transversal then $\bar{C}$ cannot have a compatible fibre-transversal therefore $C \in \text{UNSAT}$ due to Theorem 6.

Definition 4 *Let $\mathcal{H} = (V, B)$ be a base hypergraph. $\mathcal{H}$ is called* strictly diagonal *if $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) \neq \varnothing$ and for every $C \subset K_{\mathcal{H}}$ such that $B(C) = B = B(\bar{C})$ it holds that*

$$C \in \text{UNSAT} \Leftrightarrow \mathcal{F}_{\text{diag}}(C) \neq \varnothing$$

We have the following characterization.

Theorem 7 *Let $\mathcal{H} = (V, B)$ be strictly diagonal. Then a fibre-transversal $F \in \mathcal{F}(K_{\mathcal{H}})$ of $\mathcal{H}$ is compatible if and only if it satisfies $F \cap F' \neq \varnothing$ for every diagonal fibre-transversal $F' \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ of $\mathcal{H}$.*

Proof. If $F \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ then by definition for every diagonal transversal F' there is $b \in B$ such that $F'(b) = F(b)$ yielding the claim. Conversely, let $F \in \mathcal{F}(K_{\mathcal{H}})$ meeting all members of $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$. First we claim that F cannot be diagonal which follows because then also F^{γ} was diagonal and there is no $b \in B$ such that F has a non-empty intersection with F^{γ}. So it remains to verify that F' diagonal implies that F'^{γ} is diagonal, a property obviously valid for compatible fibre-transversals. For $F' \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$, suppose there exists $\tilde{F} \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ with $\tilde{F}(b) \neq F'^{\gamma}(b)$, for all $b \in B$. The latter is equivalent with $\tilde{F}^{\gamma}(b) \neq F'(b)$, for all $b \in B$, contradicting that F' is diagonal, therefore F'^{γ} is diagonal, too. Next, if F is compatible we are done. Finally, assume that F is neither compatible nor diagonal, then specifically $\bar{F} \in \text{UNSAT}$ according to Theorem 6. Since $\mathcal{H}$ is strictly diagonal it is implied that $\mathcal{F}_{\text{diag}}(\bar{F}) \neq \varnothing$ meaning there is $F' \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ such that $F \cap F' = \varnothing$. $\square$

6 Open Problems

A specific class of formulas is given by $C \in \mathcal{S}$ such that $C_{\pm} = \varnothing$. SAT restricted to such instances remains NP-complete which can be shown by reduction from SET SPLITTING [8] respectively hypergraph bicolorability [2]. However it is not clear whether the complexity decreases if one requires $C \in \mathcal{A}$ instead. An interesting question arises whether there are fixed-parameter algorithms [6] recognizing members in UNSAT_k with respect to parameter k. A natural question is whether there are formulas for which it is sufficient only to check for the existence of a monotone part for an eliminating set. In other words, it would be comfortagble to characterize those satisfiable formulas which can be tested only regarding a monotone part. Observe that in case that the monotone parts are of different sizes we can always achieve by flipping all variables that the positive monotone part is smaller. Perfectly symmetric formulas admit unit classes. It might be interesting to investigate how the classes with exactly, two, three, respectively k elements can be characterized for fixed positive integer k. The recognition of members of $\mathcal{P}$ even in linear time, might be attacked by designing appropriate data structures. Finally, the question whether there exist strictly diagonal hypergraphs is open.

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