Proceedings of the International MultiConference of Engineers and Computer Scientists 2016 Vol II,
IMECS 2016, March 16 - 18, 2016, Hong Kong

Data Analysis with Fuzzy Measure on Intuitionistic Fuzzy Sets

Sanghyuk Lee*, Ka Lok Man, Eng Gee Lim, Mark Leach

***Abstract*- Study of fuzzy entropy on intuitionistic fuzzy sets (IFSs) were proposed, and analyzed. Built in uncertainty in IFSs, it is named as hesitance. It is contained in fuzzy membership function in itself by definition. Hence, designing fuzzy entropy is not easy because there is no general entropy definition about IFSs. By considering existing fuzzy entropy definitions, fuzzy entropy on IFSs is designed and proved its usefulness. Similarity measure was also extended with the information of entropy, and the similarity measure is verified its usefulness by the proof.**

I. INTRODUCTION

Generally, data uncertainty is calculated with the help of entropy [1]. Then, the difference between fuzzy set and intuitionistic fuzzy sets is clear, because IFSs include hesitance – uncertainty - by definition. Hence, there are difference and difficulty if we design fuzzy entropy on IFSs. In previous results, fuzzy entropy was designed as explicitly [2-4]. By the dual concepts, similarity measure was also proposed to compute the degree of similarity between fuzzy sets [5-7]. With the obtained fuzzy entropy and similarity measure, result represents the complementary information for each other. Hence, the relation was clear that the similarity measure (entropy) was derived from entropy (similarity measure) in the previous literature [6]. Previous research on fuzzy entropy has been derived by many researchers [2-4]. It was shown that entropy design on fuzzy sets is enough than the research on IFSs, it means that entropy design on fuzzy sets is rather easy and convenient. They derived rather explicit formula, hence it is easy to apply calculation. Specially, Liu proposed axiomatic definitions of entropy, distance measures, and similarity measures and discussed the relationships among these three concepts.

By the way, similarity measure design is simplified than fuzzy entropy, it can be designed easily with the help of fuzzy numbers [5]. Even it has restriction, it has only application to triangular or trapezoidal membership functions [5]. For the general formulation of similarity measure even for IFSs, the distance measures are applicable to general fuzzy membership functions, including non-convex fuzzy membership functions [8].

Sanghyuk Lee, Eng Gee Lim and Mark Leach are with the Department of Electrical and Electronic Engineering, and Ka Lok Man is with the Department of Computer Science and Software Engineering, Xi'an Jiaotong-Liverpool University, 111 Ren'ai Road, Suzhou, Jiangsu, China 215123.

Sanghyuk Lee is also with the Biomedical Engineering Center, Chiang Mai University, Thailand

{Sanghyuk.Lee; ka.man; enggee.lim; Mark.Leach}@xjtlu.edu.cn

*Author to whom correspondence should be addressed; E-Mail: Sanghyuk.Lee@xjtlu.edu.cn

As an extension of fuzzy sets, IFSs and vague sets were introduced by Atanassov and Gau and Buehrer, and other authors [9-12]. Basically, IFSs include hesitancy, that is not belonging to membership and non-membership. Furthermore, comparison between IFSs and vague sets was pointed out by Bustince and Burillo, two sets are the same [13]. Data distribution with IFSs shows more realistic than fuzzy sets, and it is more practical and accurate. Hence, we do want to point out that design of fuzzy entropy and similarity measure on IFSs are very important to obtain more reliable results.

In this paper, fuzzy entropy on IFSs has been derived by the definition. Proposed fuzzy entropy was considered for the general fuzzy membership functions, it is applicable even it is non-convex structure. Unfortunately, still there is no unified fuzzy entropy definition on IFSs yet. Hence, we apply two fuzzy entropy definitions, and the fuzzy entropy was considered for both cases. By the dual concept, similarity measure was also derived through fuzzy entropy results.

II. PRELIMINARIES

IFS was proposed as general formulation of fuzzy sets with hesistancy by Atanassov [9].

A. Intuitionistic fuzzy sets

Atanassov gave the definition of IFSs, in which the uncertainty of data, hesitancy is included more practically [9].

Definition 2.1 IFSs V in the universe of discourse $X = \{x_1, x_2, \cdots, x_n\}$ defined as follows:

$$V = \{(x, t_v(x), f_v(x)) \mid x \in X, t_v(x) \in [0.1], \\ f_v(x) \in [0.1], 0 \le t_v(x) + f_v(x) \le 1\}.$$

Where, $t_v(x)$ and $f_v(x)$ denote the membership function and the non-membership function of x in X, respectively. And $t_v(x)$ is the lowest bound of membership degree of x, $f_v(x)$ is the lowest bound of non-membership degree of x, respectively. From the definition, it is clear that membership degree in IFS V, in between $[t_v, 1 - f_V]$ is not belonging in anywhere. Uncertainty degree can be calculated by $1 - t_v - f_V$ for all x in X. Furthermore, if $t_v + f_V = 1$, then V is expressed as the same as fuzzy set definition. In order to evaluate the uncertainty or entropy on IFSs, analysis on relation among the hesitance information, membership and

ISBN: 978-988-14047-6-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

non-membership degree is needed to be considered.

By the meaning of fuzzy set, null set has meaning that there is no any other information about data themselves. By graphical representation, analysis between the membership degree and non-membership degree is illustrated in Fig. 1. It shows the projection with Szmidt and Kacprzyk's, it represents the coordination of $t_v(x)$ - $f_v(x)$ plane.

From the Fig. 1, under diagonal area is the point that hesistancy is included. Hence A-F-B, and lines under the fuzzy line satisfies $t_v + f_V + \text{hesitancy} = 1$ under the variation of hesistancy. By the inspection, it is clear that hesitance satisfies one when $t_v + f_V = 0$, that is origin.

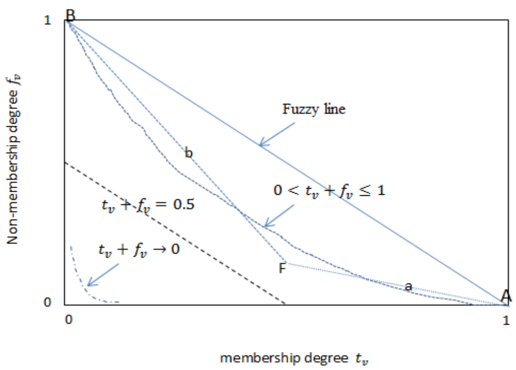

Fig. 1 F membership function and non-membership function.

Analysis indicates that the fuzzy line means that it has no hesitance. IFSs satisfies lines under the fuzzy line, hence the relation between membership degree and non-membership degree are defined by $0 \le t_v + f_V \le 1$. Especially the line of $t_v + f_V = 0.5$ is specified in dotted line. As the point approach to the origin (0, 0), degree of hesitance also approaches one. Graphical information provide rather geometrical information about membership, non-membership degree and hesistancy information.

B. Uncertainty on IFSs

Typical IFSs membership function is illustrated in Fig. 2. Research on uncertainty of IFSs, fuzzy entropy on IFSs has been considered by some researchers [13, 17]. However, similarity measure design on IFSs reported more than fuzzy entropy design on IFSs. Burillo and Bustince [13] and Szmidt and Kacprzyk [17] have proposed the fuzzy entropy on IFSs [13, 17]. The knowledge provides us methodology how to measure the degree of intuitionism on IFS, and non-probabilistic type of entropy measure with a geometric interpretation on IFSs.

Burillo and Bustince proposed an axiomatic definition of IFSs, which was considered by taking into account fuzzy set consideration. Gaussian type IFS membership function is illustrated in Fig. 2.

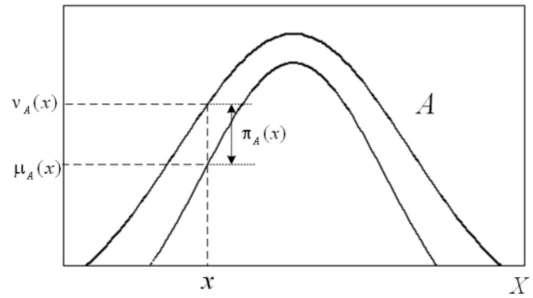

Fig. 2 Gaussian type IFS Membership Function.

Definition 2.2 [13] A real function $I : \text{IFS}(X) \to R^+$ is called an entropy on IFS(X) if I has the following properties:

(IP1) $I(A) = 0$, if and only if A is a fuzzy set,

(IP2) $I(A) = Cardinal(X) = N$ if and only if $\mu_A(x) = \nu_A(x) = 0$ for all $x \in X$,

(IP3) $I(A) = I(A^C)$ for all $A \in \text{IFSs}(X)$,

(IP4) if $A \prec B$, then $I(A) \ge I(B)$.

Where $A \prec B$ denotes that $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \le \nu_B(x)$ for all $x \in X$, which means that IFS B has less hesitancy than IFS A. μ_A, ν_A, and π_A are degree of membership, non-membership, and hesitancy of x to A, that is, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Szmidt and Kacprzyk also described the fuzzy entropy on IFSs by the ratio of intuitionistic fuzzy cardinalities [17]. Their definition was interesting, for the point datum "F", entropy of IFS point represents as

$$E(F) = \frac{a}{b}. \tag{1}$$

Where a and b are distance from F to the nearer and farther point among "A" and "B". Actually, hesitancy is contained in distance a and b. However, hesitance is not illustrated in Fig.1.

C. Entropy and Similarity Measure Definition on IFSs

Fuzzy entropy was proposed by De Luca and Termini at first time [4], and the axiomatic definition was referred to Shannon's probability entropy.

Definition 2.3 [4] A real function $e : \text{FS}(X) \to R^+$ is called an entropy on fuzzy set FS(X) if e has the following properties:

(E1) $e(\widetilde{A}) = 0$, if $\widetilde{A}$ is a crisp set.

(E2) $e(\widetilde{A})$ assumes a unique maximum if $\mu_{\widetilde{A}} = 1/2$.

(E3) $e(\widetilde{A}) \le e(\widetilde{B})$ if $\widetilde{A}$ is crisper than $\widetilde{B}$, that is, if $\mu_{\widetilde{A}} \le \mu_{\widetilde{B}}$ for $\mu_{\widetilde{B}} \le 1/2$ and $\mu_{\widetilde{A}} \ge \mu_{\widetilde{B}}$ for $\mu_{\widetilde{B}} \ge 1/2$.

ISBN: 978-988-14047-6-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

(E4) $e(\widetilde{A}) = e(\widetilde{A}^C)$ where $\widetilde{A}^C$ is the complement of $\widetilde{A}$.

Similarity measure has the dual meaning of fuzzy entropy on fuzzy set, and it was suggested by Liu [2]. Liu suggested similarity measure definition, and it was based on the distance measure of membership value. And the similarity measure on IFSs has been introduced by Dengfeng and Chuntian, it has similar formulation with definition of Liu's [14].

Definition 2.4 For a mapping S: $\mathrm{IFS}(X) \times \mathrm{IFS}(X) \to [0,1]$, IFSs(X) denotes the set of all IFSs in $X = \{x_1, x_2, \cdots, x_n\}$. $S(A,B)$ is said to be the degree of similarity between $A \in \mathrm{IFS}(X)$ and $B \in \mathrm{IFS}(X)$, if $S(A,B)$ satisfies the properties of conditions:

(P1) $S(A,B) \in [0,1]$,
(P2) $S(A,B) = 1 \Leftrightarrow A = B$,
(P3) $S(A,B) = S(B,A)$,
(P4) $S(A,C) \le S(A,B)$ and $S(A,C) \le S(B,C)$ if $A \subset B \subset C$, $C \in \mathrm{IFS}(X)$,
(P5) $S(A,B) = 0 \Leftrightarrow A = \Phi$ and $B = \overline{A}$, or, $A = \overline{B}$ and $B = \Phi$,

where Φ means that information of IFS is very clear, $\mu_A = 0$ and $\nu_A = 1$. $\overline{A}$ denotes the IFS complement of A.

Based on the definition of fuzzy entropy and similarity measure, numerous entropy and similarity measure have been introduced [6-8]. The usefulness also verified through previous results. Comparison between IFSs similarity measure was done by Y. Li, D.L. Olson, and Z. Qin [158]. They have compared conventional similarity measures. Furthermore, other fuzzy entropy on IFSs has been also defined by Hung and Yang [16]. In their definition, IP2 and IP4 are different from that of their properties. Difference of two definitions has their own characteristics. Next, entropy and similarity measure between IFSs has been derived based on Definition 2.4. Results are somewhat similar with that of fuzzy set.

III. Fuzzy Entropy and Similarity Measure Design on IFSs

Fuzzy entropy design on IFSs has been reported by fewer researchers than similarity measure design. Burillo and Bustince, and Szmidt and Kacprzyk proposed the entropy on IFS, and non-probabilistic type entropy measure with a geometric interpretation on IFSs.

A. Entropy Characteristics on FSs and IFSs

From the Definition 2.2, fuzzy entropy on IFSs is the meaning of hesitance area, Burillo and Bustince proposed entropy on IFSs as follows

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \tag{2}$$

and the proof was shown in [16]. The ratio of intuitionistic fuzzy cardinalities was also provided as an entropy on IFSs by Szmidt and Kacprzyk. They proposed entropy on IFSs as follows

$$E(F) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\max Count(F_i \cap F_i^c)}{\max Count(F_i \cup F_i^c)}\right). \tag{3}$$

Where $F_i \cap F_i^c = <\min(\mu_{F_i}, \mu_{F_i}^c), \max(\nu_{F_i}, \nu_{F_i}^c)>$

and $F_i \cup F_i^c = <\max(\mu_{F_i}, \mu_{F_i}^c), \min(\nu_{F_i}, \nu_{F_i}^c)>$.

Here, (2) and (3) have quite different meaning, that is, the difference could be found Example 1 of Szmidt and Kacprzyk [17].

Another fuzzy entropy on IFSs has been defined by Hung and Yang [16]. In their definition, IP2 and IP4 of Definition 2.2 are different from those of their properties. Difference of two definitions has their own characteristics, that is, in Definition 2.2, IP1 express that all fuzzy set has entropy zero property. However, it is not matched to the actual consideration, because of the property (E1) in Definition 2.3.

Now we introduce entropy on fuzzy set briefly with our previous result. Fig. 3 shows Gaussian fuzzy membership function and crisp set A_{near}, especially "near" is 0.5.

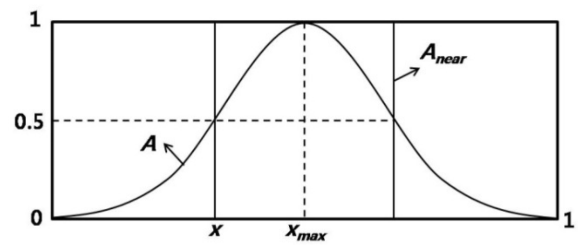

Fig. 3 Membership functions of fuzzy set A and crisp set $A_{near} = A_{0.5}$.

Then, it could be sufficient to calculate the entropy representation for "AB" line of Fig. 1, that is the case of $t_v + f_V = 1$.

B. Entropy on IFSs

Now we propose fuzzy entropy on IFSs with the definition of Szmidt and Kacprzyk that was generalized to IFSs case [17]. Now fuzzy entropy on IFSs satisfying definition is obtained as follows.

Theorem 3.1 Following equation satisfies a fuzzy entropy on IFSs.

$$E_L(A, A_{near}) = \frac{1}{N}\sum_{i=1}^{N}\{\pi_A(x_i) + d(\mu_A(x_i), \mu_{A_{near}})\} \tag{4}$$

Where subscript "near" means close crisp set to set A, in this theorem near = 0.5 is considered, then $\mu_{A_{near}} = 1$ when $\mu_A \ge 0.5$ and $\mu_{A_{near}} = 0$ when $\mu_A \le 0.5$.

Proof: (4) stands for the sum of conventional fuzzy entropy

ISBN: 978-988-14047-6-3
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

concept and hesistancy area on IFSs. For the conventional area in (4), $d(\mu_A(x_i), \mu_{A_{near}})$ represent easy of proof. Hence, it is the same result of [16]. For the non-specific distribution, (4) is clear because A_{near} satisfies A itself. Therefore, (IP1) is satisfied. For (IP2), $d(\mu_A(x_i), \mu_{A_{near}}) = 0$ is proved easily for the case of $\mu_A(x) = 0$ and $\nu_A(x) = 0$. And it is also quite natural, $I(A) = Cardinal(X) = N$, so (IP3) is clear from the definition. Finally, $A \prec B$ guarantee two properties

$$\frac{1}{N}\sum_{i=1}^{N}\pi_A(x_i) \geq \frac{1}{N}\sum_{i=1}^{N}\pi_A(x_i)$$
and $d(\mu_A(x_i), \mu_{A_{near}}) \geq d(\mu_A(x_i), \mu_{A_{near}})$.

Hence, Theorem 3.1 considers for more flexible fuzzy membership function including Gaussian fuzzy membership function. Hence, (4) is useful as a fuzzy entropy on IFSs. Fuzzy entropy has to be carried out between IFS and corresponding numeric data or set. Whereas, the similarity measure can be designed through comparison between IFSs.

C. Similarity Measure Design on IFSs with Distance Measure

It is well known fact that the similarity measure has counter-intuitive case of fuzzy entropy measure [15]. However, condition P2 is very strict to overcome, because similarity measure based on the difference calculation between two IFSs. For example, Hong and Kim showed similarity measure as follows;

$$S_H(A,B) = 1 - \frac{\sum_{i=1}^{n} (|t_A(x_i) - t_B(x_i)| + |f_A(x_i) - f_B(x_i)|)}{2n}$$

Equation has counter-intuitive case for P2, and the result can be shown in the related references. Other conventional similarity measure showed almost same results. Here novel similarity measure between IFSs is considered as follows.

Theorem 3.2 Following equation satisfies a similarity measure on IFS(X).

$$S_L(A,B) = 1 - E_L(A,B)$$

Where $E_L(A,B)$ is the same formulation of (1), and A_{near} is replaced into B.

Proof: From (P1) to (P3), it is clear from (4) itself. And $d(\mu_A(x_i), \mu_C(x_i))$ and $d(\mu_A(x_i), \mu_C(x_i))$ are greater than $d(\mu_A(x_i), \mu_B(x_i))$ and $d(\mu_B(x_i), \mu_C(x_i))$, respectively. Hence, (P4) is satisfied. Finally, $A = \Phi$ and $B = \overline{A}$, or, $A = \overline{B}$ and $B = \Phi$, IFSs A and B are complementary each other., therefore (P5) is also satisfied.

IV. DISCUSSION AND CONCLUSIONS

Since the fuzzy set was introduced by Zadeh, many new approaches and theories treating imprecision and uncertainty have been proposed, such as the interval-valued fuzzy sets. Among these theories, an extension of the classic fuzzy set is intuitionistic fuzzy set theory, which was introduced by Atanassov [9,10]. Since then, many researchers have investigated this topic and obtained some meaningful conclusions such as implication of intuitionistic fuzzy sets, generalization of intuitionistic fuzzy rough approximation operators, and multi-criteria decision-making methods based on intuitionistic fuzzy sets. Fuzzy entropy and similarity measure represent dual meaning each other, that is, dissimilarity and similarity between considering two sets. In the previous result, their relation was showed and discussed about each characteristics [6]. However, fuzzy entropy contains some debate from definition itself. In Definition 2.2, fuzzy entropy was defined by the hesitance area, there it was not necessary corresponding numerical data. Basically, fuzzy set contains the uncertainty in itself, which is evaluated by the entropy measure. Hence, additional term between $\mu_A(x_i)$ and $\mu_{A_{near}}$ is considered, which make possible to consider abnormal intuitionistic fuzzy membership function. Furthermore, relation between fuzzy entropy and similarity measure shows similar result compared to our previous result. With the help of fuzzy entropy and similarity measure of fuzzy set, analysis on IFSs also carried out. Usefulness of proposed measures is proved by evaluating the definitions. Relation between fuzzy entropy and similarity measure are also discussed. By discussing the relation, it was verified that one measure is derived from another measure, hence two measures satisfy dual concept for data analysis.

ACKNOWLEDGMENT

This work was supported by the Centre for Smart Grid and Information Convergence (CeSGIC) at Xi'an Jiaotong-Liverpool University.

REFERENCES

[1] Zadeh LA. "Fuzzy sets and systems", Proc Symp on Systems Theory, Polytechnic Institute of Brooklyn, New York; pp 29–37, 1965
[2] Liu Xuecheng, "Entropy, distance measure and similarity measure of fuzzy sets and their relations", Fuzzy Sets and Systems, Vol. 52, 305–318, 1992.
[3] D. Bhandari and N.R. Pal, "Some new information measure of fuzzy sets", Inform. Sci. Vol. 67, 209–228, 1993.
[4] DeLuca and S. Termini, "A Definition of nonprobabilistic entropy in the setting of fuzzy entropy", J. General Systems, vol. 5, pp. 301-312, 1972.
[5] S.J. Chen and S.M. Chen, "Fuzzy risk analysis based on similarity measures of generalized fuzzy numbers", IEEE Trans. on Fuzzy Systems, Vol. 11, no. 1, pp. 45–56, 2003.
[6] S.H. Lee, W. Pedrycz, and Gyoyong Sohn, "Design of Similarity and Dissimilarity Measures for Fuzzy Sets on the Basis of Distance Measure", International Journal of Fuzzy Systems, vol. 11, pp. 67-72, 2009.
[7] S.H. Lee, K.H.Ryu, G.Y. Sohn, "Study on Entropy and Similarity Measure for Fuzzy Set", IEICE Trans. Inf. & Syst., vol. E92-D, pp. 1783-1786, Sep. 2009.

[8] S.H. Lee, S. J. Kim, N. Y. Jang, "Design of Fuzzy Entropy for Non Convex Membership Function", CCIS, vol. 15, pp. 55–60, doi:10.1007/978-3-540-85930-7, 2008.
[9] Atanassov K. "Intuitionistic fuzzy sets", Fuzzy Set and Syst . vol. 20, pp. 87–96, 1986.
[10] Atanassov K. Intuitionistic fuzzy sets: Theory and applications, Heidelberg: Physica-Verlag; 1999.
[11] Atanassov K. New operations defined over the intuitionistic fuzzy sets. Fuzzy Set and Sys., 61:137–142, 1994.
[12] W.L. Gau, Buehrer, D.J, "Vague sets", IEEE Trans. Syst. Man Cybernet. Vol 23, No. 2, pp. 610-614, 1993.
[13] P. Burillo, H. Bustince, "Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets", Fuzzy Sets and Systems, Vol. 78, pp. 305-316, 1996.
[14] Dengfeng, L., Chuntian, C., "New similarity measure of intiotionistic fuzzy sets and application to pattern recognitions", Pattern Recognotion Lett., vol. 23, pp. 221-225, 2002.
[15] Yanhong Li, David L. Olson, Zheng Qin, "Similarity measures between intuitionistic fuzzy (vague) set: A comparative analysis", Pattern Recognition Lett., vol 28, pp. 278-285, 2007.
[16] W. L. Hung, M. S. Yang, "Fuzzy entropy on inttuitionistic fuzzy sets", International Journal of Intelligent Systems, vol 21, pp. 443-451, 2006.
[17] E. Szmidt, J. Kacprzyk, "Entropy for intuitionistic fuzzy sets", Fuzzy Sets and Systems, vol 118, pp. 467-477,2001.