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The Problem of Counting Semi Pandiagonal Magic Squares

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Abstract— **In this paper we introduce and study special types of magic squares of order six. We present the property preserving transformations. We list the enumerations of some sets of the squares. The codes for the enumeration is based on parallel computing.**

Index Terms—**magic squares, four corner property, semi pandiagonal magic squares, parallel computing**

I. Introduction

IN this paper we consider the old famous problem of magic squares. A semi magic square is a square matrix, where the sum of all entries in each column or row yields the same number. Some authors call it magic square. This number is called the magic constant. We call a semi magic square a magic square if both main diagonals sum up to the magic constant. A natural magic square of order n is a matrix of size of size $n \times n$ such that its entries consist of all integers from one to n^2. The magic constant in this case is

$$\frac{n(n^2+1)}{2}.$$

A pandiagonal magic square is a magic square such that the sum of all entries in all broken diagonals equals the magic constant. A symmetric magic square is a natural magic square of order n such that the sum of both elements of each pair of dual (opposite entries) equals n^2+1. For example,

TABLE I
A NATURAL SYMMETRIC MAGIC SQUARE

15	14	1	18	17
19	16	3	21	6
2	22	13	4	24
20	5	23	10	7
9	8	25	12	11

If the two main diagonals sum to the magic constant then the square is called a magic square. An off-diagonal is a combination of two parallel diagonal lines to the same main diagonal. The two parallel diagonal lines must occur on opposite sides of the main diagonal and they can only be combined if the combination has the same number of entries as the main diagonal. Two examples of an off-diagonal line are 13, 7, 4, 10 and 1, 7, 16, 10 as shown in table II. There are 3 offdiagonals corresponding to each main diagonal. A pandiagonal square is a magic square where all off-diagonals sum to the magic constant.

TABLE II
A NATURAL PANDIAGONAL MAGIC SQUARE

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

The number of natural magic squares of order five is known. It is well-known that there are pandiagonal magic squares and symmetric squares of order five. The number of natural magic squares of order six is till now unknown. We give here the number of a subset of such squares. It is well-known that there are neither pandiagonal magic squares nor symmetric squares of order six. We define here classes of magic squares of order six, which satisfy some of the conditions for both types. It is well known that there are pandiagonal and symmetric magic squares of order seven.

II. Types of Magic Squares

A. Four corner magic squares 6 by 6

A four corner magic square of order 6 is magic square with magic constant 3s such that

$$a_{ij} + a_{i(j+3)} + a_{(i+3)j} + a_{(i+3)(j+3)} = 2s \quad (1)$$

holds for each i = 1,2,3 and j = 1,2,3 and

$$a_{33} + a_{44} + a_{34} + a_{43} = 2s \quad (2)$$

The entries of a four corner magic square of order 6 satisfy

$$\begin{aligned} a_{14} + a_{25} + a_{36} + a_{41} + a_{52} + a_{63} &= 2s, \\ a_{13} + a_{22} + a_{31} + a_{61} + a_{55} + a_{64} &= 2s. \end{aligned} \quad (3)$$

These two conditions represent the sum of the entries of two broken diagonals.

B. FC magic squares with symmetric center

A four corner magic square of order 6 can be written as

TABLE III
A FC MAGIC SQUARE

x	f	g	t	G	M
z	h	n	j	q	N
w	E	e	a	m	D
A	k	b	$s-a-b-e$	H	R
I	p	d	o	Q	T
B	F	W	J	L	$p+q-x$

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where

$$A = s + e - t - x,\ B = j+o+t - w - e;$$
$$D = d+g+n+x- a - p - q,$$
$$E = 3s - e - a - m - w - D,$$
$$F = 3s- f -h-k- p-E,$$
$$G = j +o+ p+q+s-e -g- f -w-x,$$
$$H = e+g+s+w+x - j- k - o - p - q,$$
$$I=2s - o - j - z,\ J = 2s+e- j - o - a - t,$$
$$M = 3s- f -g-t -x-G,$$
$$N = 3s- j-n-q-h-z,$$
$$L = f +h+k-m+ p-s,$$
$$Q=2s-h-p-q,\ R = s+a+e-k-A-H,$$
$$T = h + j+q +z -d - s,$$
$$W = a+2s - d - g - n - e.$$

We see that it has seventeen independent variables. This formula was computed by maple. In case that

$$b= s - a\ ,\ s - a - b - e = s - e$$

we call the square a FC magic square with symmetric center. In [1] we find for first time the concept of FC squares with some simple enumerations.

C. Semi pandiagonal magic squares

We can generalize the concept of four corner magic square to the semi pandiagonal magic square. It has the following structure:

TABLE IV
A SEMI PANDIAGONAL MAGIC SQUARE

a	D	c	d	f	H
h	Q	k	l	m	E
A	r	u	v	J	I
q	p	z	$2s-u-v-z$	y	L
n	o	i	x	e	T
B	F	M	N	G	Y

where

$$A = d-c+l+m+o+ p+q-s-2u-v+x+y-z,$$
$$B = 3s-a-h-q-n-A,$$
$$D = 4s-2d- f -h-l-n- p-2q-2a+2u+2v-x-y+2z,$$
$$E = o-k-l-h+s+e,$$
$$F = 2a+2d+ f +h+l+m+n+2q-r-3s-2u-2v+x+y-2z+e,$$
$$G = k- f +l-m+ p+r+i-s+x-e,$$
$$H = 3s-a-c-d- f -D,$$
$$I = c-d+k-m-o-q+i+u+z,$$
$$J = 4s-l- p-r-i-k-x-y,$$
$$L = s-q- p+u+v-y,\ M = 3s-k-i-c-u-z,$$
$$N = s-l-d+u-x+z,\ Q=2s-e-m-o$$
$$T=3s-e-i-o-n-x,$$
$$Y = m-a+o-s+v+z.$$

It is easy to check that every FC square is a semi pandiagonal magic square. Hence, we are dealing with a generalization of the FC squares. The number of independent variables is now twenty.

It is worth mentioning that the two dependent variables in the frame of center square (E and H) depends only on the variables in the outer frame. This is helpful by programming in order to reduce run time for counting such squares.

III. Property preserving transformations

There are seven classical transformations, which take a magic square into another magic square. They are the combinations of the rotations with angles $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and transpose operation. Now, a four corner magic squares with symmetric center can be transformed as follows into another one of the same kind: we make these interchanges simultaneously: interchange a_{12} (res. a_{62}) with a_{15} (res. a_{65}); interchange a_{21} (res. a_{26}) with a_{51} (res. a_{56}); interchange a_{22} (res. a_{55}) with a_{25} (res. a_{52}); interchange a_{23} (res. a_{24}) with a_{53} (res. a_{54}); interchange a_{32} (res. a_{42}) with a_{35} (res. a_{45}). As a practical example on this transformation, the previous semi pandiagonal magic square can be transformed into this one

TABLE V
THE TRANSFORMED SQUARE

a	f	c	d	D	H
n	e	i	x	o	T
A	J	u	v	r	I
q	y	z	$2s-u-v-z$	p	L
h	m	k	l	Q	E
B	G	M	N	F	Y

without losing its properties.

We can use this transformation to reduce the run time for computing the number of natural magic squares. In order to eliminate the effect of the previous transformations in developing a code we compute all natural four corner magic squares for which the following conditions hold:

$$a_{52} < a_{25}, a_{34} < a_{33} < a_{44} \quad (4)$$

When we calculate the number of all natural squares, we multiply then the number with sixteen.

IV. Enumertaion of semi pandiagonal squares

We want to talk about the problem of counting the natural squares of the previous types. In the papers [2], [3], ..., [7] there is an enumertion of four corner magic squares, which was carried out over several stages. The problem of enumerating the semi pandiagonal squares can be performed very similarly. But, the run time will be too long. We are working on this problem and we present here some results. Also, we shall mention that we can use parallelizable codes as in the case of four corner magic squares.

As a special case we consider the following square

TABLE VI
A SPECIAL SEMI PANDIAGONAL MAGIC SQUARE

a	D	c	d	f	$s-B$
h	$s-e$	k	l	m	E
A	r	u	$s-z$	J	I
q	p	z	$s-u$	y	L
n	$s-m$	i	x	e	T
B	F	M	N	G	$s-a$

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We will call it semi pandiagonal magic square with centrally symmetric diagonals. For this special type we have another property preserving transformation. The last square can be transformed into this one

TABLE VII
THE TRANSFORMED SQUARE

$s-a$	F	M	N	G	B
E	$s-e$	k	l	m	h
I	r	u	$s-z$	J	A
L	p	z	$s-u$	y	q
T	$s-m$	i	x	e	n
$s-B$	D	c	d	f	a

without losing its properties. We note that the two property preserving transformations do not change the 2 by 2 center. Hence, if we fix a 2 by 2 center, then it suffices to consider the centrally symmetric semi pandiagonal magic squares, for which the following relation holds

$$a < s - a, m < s - m \tag{5}$$

We computed the semi pandiagonal magic squares with centrally symmetric diagonals such that

$$u = 1, z = 35, 1 \le a \le 18, 1 \le m \le 18 \tag{6}$$

We have the following enumeration:

TABLE VIII
NUMBER OF SEMI PANDIAGONAL MAGIC SQUARE (1)

m	number	m	Number
3	2345934	11	2654448
4	2426940	12	2766964
5	2569262	13	2834900
6	2646828	14	2775044
7	1732994	15	2857630
8	1830440	16	2854512
9	2038438	17	2784088
10	2198412	18	2743892

The total number of squares is 40 060 726. As a subset we computed the natural centrally symmetric squares such that

$$u = 1, z = 35, 1 \le a \le 18, 1 \le m \le 18$$

$$\det\begin{bmatrix} s-e & k & l & m \\ r & u & s-z & J \\ p & z & s-u & y \\ s-m & i & x & e \end{bmatrix} = 0 \tag{7}$$

The condition on value of the determinant is invariant regarding both property preserving transformations. We have the following enumeration:

TABLE IX
NUMBER OF SEMI PANDIAGONAL MAGIC SQUARE (2)

m	number	m	Number	m	Number
3	38508	8	37768	14	39434
4	35514	9	35740	15	36864
5	30618	10	40020	16	34992
6	34966	11	38340	17	34690
7	34368	12	38780	18	41674
*	*	13	38528	*	*

The total number of squares is 590 804. We computed also the natural squares such that

$$u = 18, z = 36, 1 \le a \le 18, 1 \le m \le 18$$

$$\det\begin{bmatrix} s-e & k & l & m \\ r & u & s-z & J \\ p & z & s-u & y \\ s-m & i & x & e \end{bmatrix} = 0$$

We obtained the following enumeration:

TABLE X
NUMBER OF SEMI PANDIAGONAL MAGIC SQUARE (3)

m	number	m	Number
2	65985	10	142108
3	62273	11	102625
4	68435	12	121622
5	69004	13	136832
6	85859	14	137848
7	105378	15	149437
8	92754	16	156867
9	109185	17	150147

The total number of squares is 1 756 359. These two possibilities for u and z are out of 153 possibilities, which shall be calculated. All other possibilities can be calculated by symmetry aspects.

V. Symbolic Computations

It is sometimes of interest to determine the determinant of the magic square as a square matrix. In the case of the semi pandiagonal magic squares there are cases when the determinant is zero. In general the determinant is not zero for any semi pandiagonal magic square. If we have all entries of the frame of outer 4 by 4 center (*e, i, k, l, m, r, p, o, x, y, Q* and *J*) as the value $\frac{s}{2}$, Then, we can prove using symbolic (e. g. maple) calculation software that the determinant is zero. In general the determinant of the following square

TABLE XI
SYMBOLIC SQUARE

Y	P	X	D	K	H
B	$σ$	$σ$	$σ$	$σ$	$s-B$
A	$σ$	U	V	$σ$	I
Q	$σ$	Z	$2s-U-V-Z$	$σ$	L
C	$σ$	$σ$	$σ$	$σ$	$s-C$
B	$s-P$	M	N	$s-K$	R

with $\sigma = \frac{s}{2}$ is zero. Here all capital letters are variables taking any real values. In [11] Rosser and Walker show that a pandiagonal 4×4 magic square with magic constant $2s$ has in general the following structure

TABLE XII
PANDIAGONAL MAGIC SQUARE

A	B	C	ω
E	θ	ς	ρ
s – C	s – ω	s – A	s – B
s – ς	s – ρ	s – E	s – θ

Here the Greek letters depend on the letters A, B, C and E. Also, it is well known that the determinant of this matrix is zero. In [12] we find other structures, whose determinant is zero. It is worth mentioning here that there are no natural pandiagonal magic squares 6 by 6. In [9] we find other structures similar to this structure with some enumerations.

VI. TRUMP'S WORK

Ian Bethune shared a code for counting the squares, which is available online at https://bitbucket.org/ibethune/sp_squares/.

Walter Trump joined me recently in the search for solving the problem of counting the natural squares (see [9] and [10]). He developed a code for counting the squares. According to Trump's terminology: a magic series (of order 6) is a vector

$$\begin{aligned} &(k_1, k_2, ..., k_6) \text{ with} \\ &k_1 < k_2 < ... < k_6, \\ &k_1 + k_2 + ... + k_6 = 3s, \\ &\{k_1, k_2, ..., k_6\} \subset \{1,2,3,....,36\} \end{aligned} \quad (8)$$

A magic line is a permutation of a magic series. A general pair of magic diagonals consists of two disjoint magic lines, which represent the main two diagonals of a 6x6 matrix. The complement k^* of a number $k \in \{1, 2, 3, ... 36\}$ is the number $k^* = s - k$.

If two diagonals can be written as (k_1, k_2, k_3, k_3*, k_2*, k_1*) and (k_4, k_5, k_6, k_6*, k_5*, k_4*) we speak of a centrally symmetric pair of diagonals.

If two diagonals can be written as (k_1, k_2, k_3, k_4, k_5, k_6) and (k_1*, k_2*, k_3*, k_4*, k_5*, k_6*) we speak of an axially symmetric pair of diagonals.

In 2015 Trump enumerated all magic squares of order 6 with symmetric pairs of diagonals. But the number of semi-panmagic squares was not considered. The calculated number of magic squares as matrices is

8 x 96 x (1 459 201 633 806 + 2 355 312 270 384) =
2 929 546 678 417 920

TABLE XIII
MAGIC SQUARE WITH A CENRALLY SYMMMETRIC PAIR OF DIAGONALS

4	26	27	23	24	7
20	5	21	22	8	35
19	25	6	9	34	18
36	12	28	31	3	1
2	29	16	15	32	17
30	14	13	11	10	33

TABLE XIV
MAGIC SQUARE WITH AN AXIALLY SYMMMETRIC PAIR OF DIAGONALS

1	24	30	27	26	3
22	2	28	32	4	23
19	20	12	8	21	31
15	17	25	29	11	14
18	35	9	10	33	6
36	13	7	5	16	34

VII. CONCLUSION

The problem of counting the natural squares of the semi pandiagonal magic squares is yet unsolved. I hope that we will solve this problem in the near future. Our strategy is to split the problem into smaller problems by considering special cases of the squares. Also, it is possible to add constraints like the value of the determinant. The usage of high performance computers will play an essential role in this task. The counting of semi pandiagonal magic squares is also another step on the way to calculate the number of magic squares 6 by 6 (for estimates see [10]).

REFERENCES

[1] S. Al-Ashhab, "Magic Squares 5x5," the international journal of applied science and computations, 2008, Vol. 15, No.1, pp. 53-64.

[2] S. Al-Ashhab, "Even-order Magic Squares with Special Properties," Open Problems in Mathematics and Computer Science, 2012, Vol. 5, No. 2.

[3] S. Al-Ashhab, "Special Magic Squares of Order Six," Research Open Journal of Information Science and Application, 2013, Vol. 1, No. 1, pp 01-19.

[4] S. Al-Ashhab, "Special Magic Squares of Order Six and Eight," International Journal of Digital Information and Wireless Communications (IJDIWC) 2012, 1(4): pp. 769-781.

[5] S. Al-Ashhab, "Negative four corner magic squares of Order Six with a Between 1 and 5," Qatar Foundation Annual Research Conference (ARC 2014 conference).

[6] S. Al-Ashhab, "The Number of Four Corner Magic Squares of Order Six," British Journal of Applied Science and Technology 7(2): 2015, Article no BJAST. 2015. 132. pp. 141-155.

[7] S. Al-Ashhab, "The Complete Number of Four Corner Magic Squares 6 by 6," International Journal of Conceptions on Computing and Information Technology, Vol. 3, Issue 2, 2015.

[8] W. J. Bellew, "Counting the Number of Compound and Nasik Magic Squares," Mathematics Today, 1997, pp. 111-118.

[9] P. Loly, I. Cameron, W. Trump, D. Schindel, "Magic square spectra," Linear algebra and its applications.

[10] www.trump.de/ **magic-squares/**

[11] B. Rosser, J. Walker, "On the Transformation Group for Diabolic Magic Squares of Order Four," Amer. Math. Soc., Bult XLIV, 1938, pp. 416-420.

[12] A. Elshokry, S. Al-Ashhab, "The Nullspace of Nested Magic Squares," International journal of Research and reviews in applied sciences, 2011, Vol. 8, No. 3, pp. 311- 327.

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