The Solution of Initial-value Wave-like Models via Perturbation Iteration Transform Method

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***Abstract*— This work is based on the application of the new Perturbation Iteration Transform Method (PITM), which is a combined form of the Perturbation Iteration Algorithm (PIA) and the Laplace Transform (LT) method on some wave-like models with constant and variable coefficients. The method provides the solution in closed form, is efficient and it involves less computational work.**

***Index Terms*— Laplace transform method, perturbation iteration algorithm, wave-like equations**

I. INTRODUCTION

Many real life problems can be mathematically modeled into Differential Equations (DEs), which comprise the Ordinary DEs (ODEs) and the Partial DEs (PDEs). A variety of methods to obtain both exact and approximate solutions of various forms of DEs have been proposed in literature. They include: Homotopy Perturbation Method (HPM), Differential Transform Method (DTM), Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), modified HPM and so on [1-7].

The wave-like equation is a second-order PDE whose purpose is the description of waves. This equation has great relevance in applied Mathematics, Engineering and Physics.

A lot of methods for the solution of wave-like equations have been proposed by different authors. Among them, we mention the DTM used in [8] to solve both wave-like and heat-like equations. Akinlabi and Edeki in [9] also solved some wave-like equations using the modified DTM. In [10], Keskin and Oturanc applied the reduced DTM in the solution of nonlinear wave equations.

The aim of this study is to approximate the wave-like equations using the Perturbation Iteration Transform Method (PITM), which was proposed in [11]. The PITM is the union of the Perturbation Iteration Algorithm (PIA) and the Laplace Transform (LT) method. The PIA has been studied extensively studied by several authors [12-16]. Some authors have combined the LTM with variety of other methods and this has proven to be very effective. Examples of such approach include but not limited to: [11], [17].

The remaining parts of this study are arranged thus: in section II, we review the PIA. Section III involves the description of the PITM. In section IV, the PITM is applied to some wave-like equations to show its efficiency. Section V involves the discussion of results with the aid of graphs. The concluding remark is given in section VI.

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II. PERTURBATION ITERATION ALGORITHM [11], [13]

In this section, we illustrate how the PIA works. If we derive a perturbation algorithm by taking the Taylor series of the correction terms of the first derivatives and also taking a correction term in the perturbed expansion. The perturbation algorithm will be referred to as: PIA (1, 1).

Now, considering a 2nd -order DE:

$$G\left(u,u',u'',\dot{u},\ddot{u},\varepsilon\right)=0 \tag{2.1}$$

where $u=u\left(x,t\right)$, $\ddot{u}\left(x,t\right)=\frac{\partial^2 u\left(x,t\right)}{\partial t^2}$, $\dot{u}\left(x,t\right)=\frac{\partial u\left(x,t\right)}{\partial t}$, $u''\left(x,t\right)=\frac{\partial^2 u\left(x,t\right)}{\partial x^2}$, $u'\left(x,t\right)=\frac{\partial u\left(x,t\right)}{\partial x}$ and ε is the introduced perturbed parameter.

Using only a correction term in the perturbed expansion yield:

$$u_{n+1}=u_n+\varepsilon\left(u_c\right)_n. \tag{2.2}$$

Putting (2.2) into (2.1) and taking the Taylor series expansion with first derivatives gives.

$$\begin{cases} G\left(u',u'',\dot{u},\ddot{u},u,0\right)+G_u\left(u',u'',\dot{u},\ddot{u},u,0\right)\varepsilon\left(u_c\right)_n+ \\ G_{u'}\left(u',u'',\dot{u},\ddot{u},u,0\right)\varepsilon\left(u'_c\right)_n+G_{u''}\left(u',u'',\dot{u},\ddot{u},u,0\right)\varepsilon\left(u''_c\right)_n+ \\ G_{\dot{u}}\left(u',u'',\dot{u},\ddot{u},u,0\right)\varepsilon\left(\dot{u}_c\right)_n+G_{\ddot{u}}\left(u',u'',\dot{u},\ddot{u},u,0\right)\varepsilon\left(\ddot{u}_c\right)_n+ \\ G_{\varepsilon}\left(u',u'',\dot{u},\ddot{u},u,0\right)\varepsilon=0 \end{cases} \tag{2.3}$$

where $u=u\left(x,t\right)$, $G_{\dot{u}}=\frac{\partial G}{\partial \dot{u}}$, $G_{\ddot{u}}=\frac{\partial G}{\partial \ddot{u}}$, $G_{u'}=\frac{\partial G}{\partial u'}$, $G_{u''}=\frac{\partial G}{\partial u''}$ $G_u=\frac{\partial G}{\partial u}$, $G_{\varepsilon}=\frac{\partial G}{\partial \varepsilon}$ and ε the perturbation parameter to be evaluated at $\varepsilon=0$.

Reorganizing (2.3), we have

$$\left(\ddot{u}_c\right)_n+\frac{G_{u''}}{G_{\ddot{u}}}\left(u''_c\right)_n+\frac{G_u}{G_{\ddot{u}}}\left(u_c\right)_n=-\frac{G_{\varepsilon}+\frac{G}{\varepsilon}}{G_{\ddot{u}}}-\frac{G_{\dot{u}}}{G_{\ddot{u}}}\left(\dot{u}_c\right)_n. \tag{2.4}$$

This is a variable coefficient linear 2nd-order DE.

The term, $\left(u_c\right)_0$ is calculated from (2.4) starting with an initial guess, u_0 and then substituted into (2.2) to evaluate u_1. We continue the iterative process using (2.2) and (2.4) until we get a satisfactory result.

III. Perturbation Iteration Transform Method [11]

To demonstrate the basic idea of the PITM, we consider a PDE with boundary conditions of the form:

$$Au(x,t)+Bu(x,t)=c(x,t) \quad (3.1)$$

subject to the conditions:

$$u(x,0)=g(x) \text{ and } u_t(x,0)=h(x) \quad (3.2)$$

where $A=\frac{\partial^2}{\partial t^2}$, $B=\frac{\partial^2}{\partial x^2}$ are the second order linear differential operators with $c(x,t)$ as the source term.

Taking the LT of both sides of (3.1), we get

$$L[Au(x,t)]+L[Bu(x,t)]=L[c(x,t)] \quad (3.3)$$

which on using the differential property of LT, yield

$$L[Au(x,t)]=\frac{g(x)}{s}+\frac{h(x)}{s^2}+\frac{1}{s^2}L[c(x,t)]-\frac{1}{s^2}L[Bu(x,t)]. \quad (3.4)$$

Applying the inverse LT to both sides of (3.4) yield

$$u(x,t)=D(x,t)-L^{-1}\left[\frac{1}{s^2}L[Bu(x,t)]\right] \quad (3.5)$$

where $D(x,t)$ is the term obtained from the imposed initial conditions and the source term.

Now, by using the PITM (3.5) becomes:

$$u(x,t)-G(x,t)+u_c(x,t)\varepsilon+L^{-1}\left[\frac{1}{s^2}L[Bu(x,t)]\right]\varepsilon=0 \ . \quad (3.6)$$

Thus,

$$u_c(x,t)=\frac{G(x,t)-u(x,t)}{\varepsilon}-L^{-1}\left[\frac{1}{s^2}L[Bu(x,t)]\right] \ . \quad (3.7)$$

This is the combined form of the LTM and the PIA. The term, $(u_c)_0$ is then obtained from (3.7) and substituted into (2.2) for u_1. The iterative process is repeated for u_2, u_3, $\cdots$

The solution is thus obtained by:

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots. \quad (3.8)$$

IV. Numerical Examples

In this section, the method discussed above is applied to the following wave-like equations with both constant and variable coefficients.

First Case:

Consider the variable coefficient wave-like model:

$$u_{tt}(x,t)=\frac{x^2}{2}u_{xx}(x,t) \quad (4.1)$$

subject to:

$$u(x,0)=1 \text{ and } u_t(x,0)=x^2 \ . \quad (4.2)$$

Solution to First Case:

Taking the LT of both sides of (4.1) yields

$$L[u(x,t)]=\frac{1}{s}+\frac{x^2}{s^2}+\frac{1}{s^2}L\left[\frac{x^2}{2}u_{xx}(x,t)\right]. \quad (4.3)$$

Applying the Inverse LT to both sides of (4.3) gives

$$u(x,t)=1+x^2t+L^{-1}\left[\frac{1}{s^2}L\left[\frac{x^2}{2}u_{xx}(x,t)\right]\right] \ . \quad (4.4)$$

Now, by the PITM, (3.4) becomes:

$$u(x,t)-1-x^2t+u_c(x,t)\varepsilon-L^{-1}\left[\frac{1}{s^2}L\left[\frac{x^2}{2}u_{xx}(x,t)\right]\right]\varepsilon=0. \quad (4.5)$$

Thus,

$$u_c(x,t)=\frac{-u(x,t)+1+x^2t}{\varepsilon}+L^{-1}\left[\frac{1}{s^2}L\left[\frac{x^2}{2}u_{xx}(x,t)\right]\right]. \quad (4.6)$$

This implies that:

$$u_0(x,t)=1+x^2t,$$

$$u_1(x,t)=\frac{x^2t^3}{3!},$$

$$u_2(x,t)=\frac{x^2t^5}{5!},$$

$$u_3(x,t)=\frac{x^2t^7}{7!},$$

$$\vdots$$

Hence, the closed-form solution is:

$$\begin{aligned} u(x,t)&=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots \\ &=1+x^2t+\frac{x^2t^3}{3!}+\frac{x^2t^5}{5!}+\frac{x^2t^7}{7!}+\cdots \\ &=1+x^2\left\{t+\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^7}{7!}+\cdots\right\} \\ &=1+x^2\sum_{\eta=0}^{\infty}\frac{t^{2\eta+1}}{(2\eta+1)!} \\ &=1+x^2\operatorname{Sinh}t. \end{aligned} \quad (4.7)$$

Second Case:

Consider the constant coefficient wave-like model:

$$u_{tt}(x,t)=u_{xx}(x,t)-3u(x,t) \quad (4.8)$$

subject to:

$$u(x,0)=0 \text{ and } u_t(x,0)=2\sin x. \quad (4.9)$$

Solution to Second Case:

Taking the LT of both sides of (4.8) yields

$$L[u(x,t)]=\frac{0}{s}+\frac{2\sin x}{s^2}+\frac{1}{s^2}L[u_{xx}(x,t)-3u(x,t)]. \quad (4.10)$$

Applying the Inverse LT to both sides of (4.10) gives

$$u(x,t)=2t\sin x+L^{-1}\left[\frac{1}{s^2}L[u_{xx}(x,t)-3u(x,t)]\right]. \quad (4.11)$$

Now, by the PITM (4.11) becomes:

$$u(x,t)-2t\sin x+u_c(x,t)\varepsilon -L^{-1}\left[\frac{1}{s^2}L[u_{xx}(x,t)-3u(x,t)]\right]\varepsilon=0. \quad (4.12)$$

Thus,

$$u_c(x,t)=\frac{-u(x,t)+2t\sin x}{\varepsilon}+L^{-1}\left[\frac{1}{s^2}L[u_{xx}(x,t)-3u(x,t)]\right]. \quad (4.13)$$

This implies that:

$$u_0(x,t)=2t\sin x,$$

$$u_1(x,t)=-\frac{(2t)^3\sin x}{3!},$$

$$u_2(x,t)=\frac{(2t)^5\sin x}{5!},$$

$$u_3(x,t)=-\frac{(2t)^7\sin x}{7!},$$

$$\vdots$$

Hence, the closed-form solution is:

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots$$

$$
\begin{aligned}
&= 2t\sin x - \frac{(2t)^3 \sin x}{3!} + \frac{(2t)^5 \sin x}{5!} - \frac{(2t)^7 \sin x}{7!} + \cdots \\
&= \sin x\left[2t - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \frac{(2t)^7}{7!} + \cdots\right] \\
&= \left(\sum_{j=0}^{\infty}(-1)^j \frac{(x)^{2j+1}}{(2j+1)!}\right)\left(\sum_{\eta=0}^{\infty}(-1)^\eta \frac{(2t)^{2\eta+1}}{(2\eta+1)!}\right) \\
&= \sin 2t \sin x. \qquad (4.14)
\end{aligned}
$$

V. Discussion of Results

In this section, graphs for the approximate and exact solutions to the problems discussed above are presented. The approximate solutions contain terms up to the seventh power.

Fig. 1a and Fig. 1b are solution graphs for first *case*:

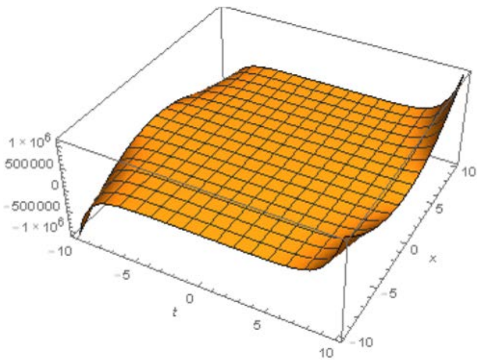

Fig. 1a: Exact solution of first *case*

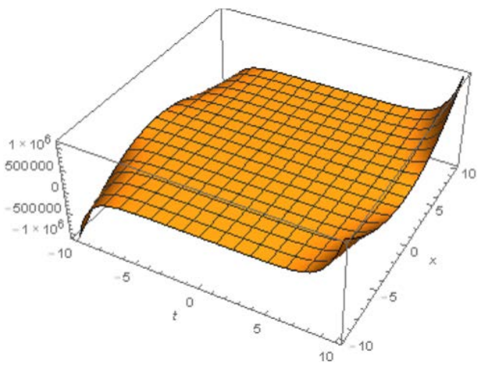

Fig. 1b: Approximate solution of first *case*

Fig. 2a and Fig. 2b are solution graphs for second *case*:

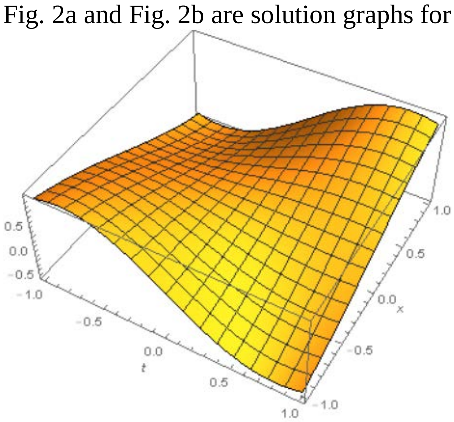

Fig. 2a: Exact solution of second *case*

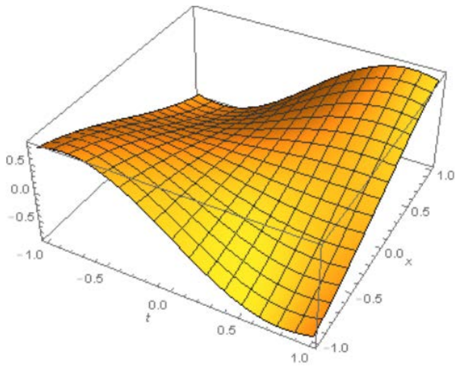

Fig. 2b: Approximate solution of second *case*

VI. Conclusion

In this study, the new Perturbation Iteration Transform Method (PITM) is applied to some wave-like models with constant and variable coefficients for closed-form solutions. The results obtained when compared with their exact solutions, showed that the proposed method is efficient and simple. We therefore, propose this method for solving both linear and non-linear PDEs.

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