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A Dynamic Geometry Approach to the Fagnano's Problem

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Abstract—**The Fagnano's problem is a famous optimization problem in plane geometry, whose solution is given by the inscribed orthic triangle of a given acute triangle. In this paper, we discuss how to solve this problem by the principle of reflection via a dynamic geometry approach. Compared with the method of solutions based on Calculus or Euclidean Geometry only, this approach can be easily adapted to study similar problems in other settings. We will also introduce a useful formula for finding the perimeter of the orthic triangle.**

Index Terms—**Fagnano problem, minimal perimeter, orthic triangle, periodic billiard path, dynamic geometry.**

I. INTRODUCTION

In 1775, the Italian mathematician called Giovanni Fagnano posed an interesting optimization problem as follows [1-3]:

"For a given acute triangle ΔABC, determine the inscribed triangle with minimal perimeter."

Fagnano used the technique of calculus to solve the problem by himself. He concluded that the inscribed triangle with minimal perimeter is the orthic triangle of ΔABC. By definition, the orthic triangle of ΔABC is the triangle formed by joining the endpoints of the altitudes drawn from the vertices of ΔABC (see Fig. 1).

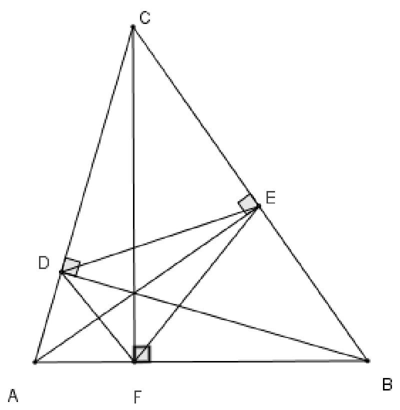

Fig. 1. ΔDEF is the orthic triangle of ΔABC

There are some important and interesting results concerned with the orthic triangle, which are stated below. These results can be proved by elementary plane geometry and they will not be discussed in this paper (see [2, 3, 5]). For convenience, we will use the same notations as above to describe the results.

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Theorem 1.1. The altitudes CF, AE and BD bisect the angles ∠DFE, ∠FED and ∠EDF of ΔDEF, respectively.

Theorem 1.2. The following angles in ΔABC are equal.
(i) ∠ACB = ∠AFD = ∠BFE
(ii) ∠ABC = ∠ADF = ∠CDE
(iii) ∠CAB = ∠BEF = ∠CED
(iv) ∠CAE = ∠CFD = ∠CFE = ∠CBD
(v) ∠BCF = ∠BDE = ∠BDF = ∠BAE
(vi) ∠ABD = ∠AEF = ∠AED = ∠ACF

Besides using calculus, there are other methods for solving the Fagnano's problem, which can be found in [2, 3, 6, 8, 9, 11, 13]. In the next section, we will introduce a simple approach to solve this problem by the principle of reflection, via the use of the dynamic geometry software GeoGebra as a tool. Then, we will describe a useful formula for finding the length of the perimeter of the inscribed orthic triangle ΔDEF. This approach can be easily adapted to study similar problems in other settings (e.g. ΔABC is right or obtuse triangle). We expect this approach will be found useful for reference by researchers working on the related areas, as well as lecturers or teachers who are interested to introduce the Fagnano's problem for illustration or discussion purposes in their own classes.

II. HOW TO SOLVE THE FAGNANO'S PROBLEM

Before we discuss how to solve the Fagnano's problem, let us introduce a very useful technique for solving the shortest path problem below [4].

"If A, B are two points lying on the same side of a line, find a point O on the line such that the path A-O-B is the shortest."

Fig. 2 shows that we can first use the given line as the axis of reflection to locate the image of reflection of B, say B_1, on the opposite side, and then determine the position of the required point O, which is the intersection between AB_1 and the given line. According to the principle of reflection, OB = OB_1, so AO + OB = AO + OB_1 = AB_1. Since AB_1 is a line segment, so the path obtained is the shortest possible. In fact, if one choose an arbitrary point P not at the position O, the path A-P-B will be longer than A-O-B since AP + PB = AP + PB_1 > AB_1, due to the well-known "triangular inequality"

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(see Fig. 3).

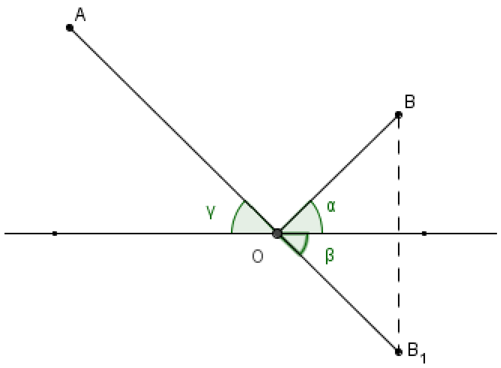

Fig. 2. Finding the shortest path by using reflection.

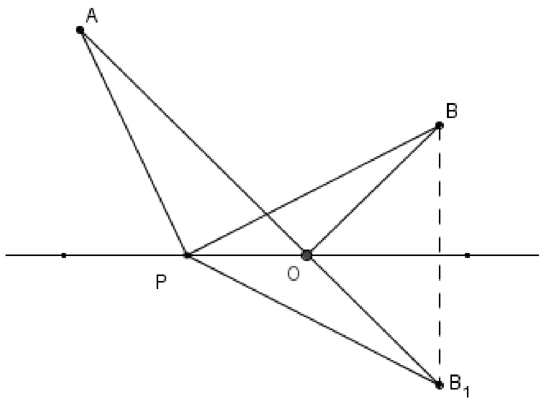

Fig. 3. The path A-P-B is longer than A-O-B.

By the principle of reflection, we have $\angle\alpha = \angle\beta$ (see Fig. 2). On the other hand, $\angle\beta = \angle\gamma$, as they are vertically opposite angles. Hence, $\angle\alpha = \angle\gamma$. We can summarize this important result as follows [9].

Theorem 2.1. The shortest path joining two given points on one side of a line, and meeting this line, is a broken line whose parts make equal angles with the given line.

We now illustrate how to solve the Fagnano's problem by using the dynamic geometry software GeoGebra as an aids. First, we draw an acute triangle ΔABC with altitudes AE, BD and CF, and a point P on AB as shown in Fig. 4.

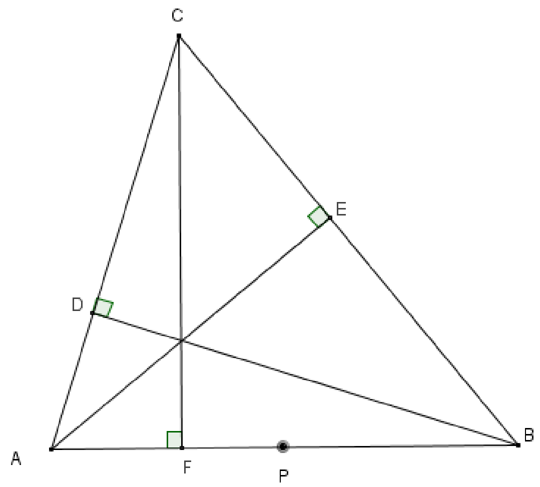

Fig. 4. An acute triangle ΔABC with 3 altitudes.

Next, using BC and AC as the axes of reflection, we locate the images of reflection of P, say Q and R, respectively, outside the triangle ΔABC. Then, the line segment QR is drawn to determine its intersections with AC and BC, say M and N respectively, as illustrated in Fig. 5.

By the principle of reflection, we have ∠CMN = ∠BMP and ∠CNM = ∠ANP. However, in general, ∠APN may not equal to ∠BPM. Now, by moving P along AB and using the angle measurement tool built-in GeoGebra, we can observe that ∠APN = ∠BPM, ∠CMN = ∠BMP and ∠CNM = ∠ANP occur when P, M, N coincide with the points F, E and D, respectively (see Fig. 6). According to Theorem 2.1, the shortest path from N to M via P is N-P-M. Similarly, the shortest path from P to N via M is P-M-N, and the shortest path from M to P via N is M-P-N. This implies that the inscribed triangle of ΔABC with minimal perimeter is the orthic triangle ΔDFE, which is the solution of the Fagnano's problem. In fact, it is a unique solution since there is only one inscribed orthic triangle in ΔABC.

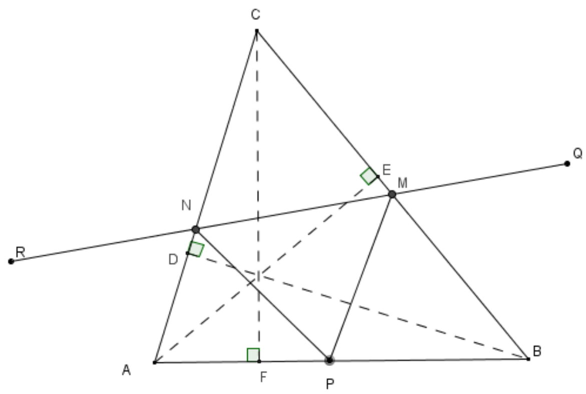

Fig. 5. Draw a triangle ΔPMN inside ΔABC.

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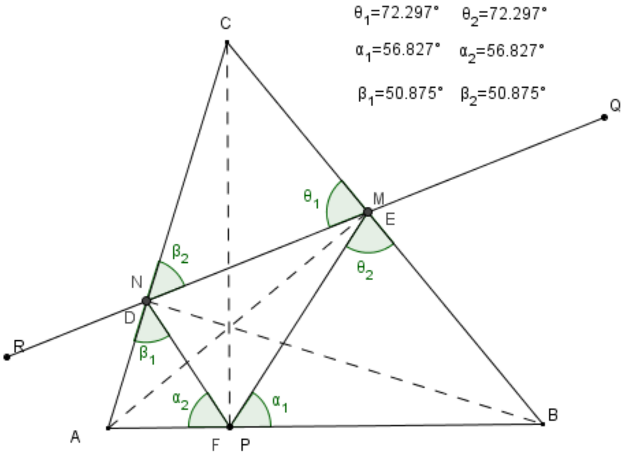

Fig. 6. Solution of the Fagnano's problem

We can summarize the conclusion by the following theorem.

Theorem 2.2. Of all triangles inscribed in the given acute triangle ΔABC, the orthic triangle ΔDEF has the minimal perimeter.

III. FINDING THE PERIMETER OF THE ORTHIC TRIANGLE

Let BC = a, AC = b, AB = c, and $\angle CAB = \theta_1$, $\angle ACB = \theta_2$, $\angle ABC = \theta_3$. We now derive a formula for finding the perimeter of the inscribed orthic triangle ΔDEF.

We can refer to the diagram below (Fig. 7).

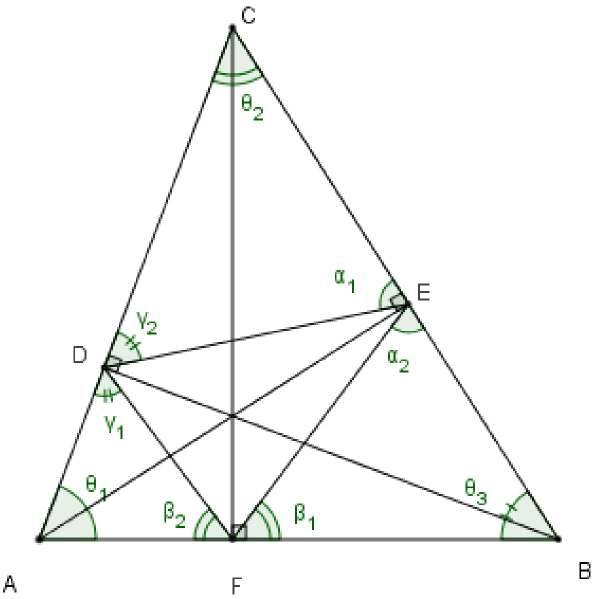

Fig. 7. Several equal angles in ΔABC

By Theorem 1.2, $\theta_1 = \alpha_1 = \alpha_2$, $\theta_2 = \beta_1 = \beta_2$, $\theta_3 = \gamma_1 = \gamma_2$. Since ΔADF~ΔABC, so DF/BC = AF/AC. Hence,

$$DF = a \times \frac{b\sin(\pi/2-\theta_1)}{b} = a\cos\theta_1$$

Also, ΔBFE~ΔBCA, so FE/AC = BE/AB. Hence,

$$EF = b \times \frac{c\sin(\pi/2-\theta_3)}{c} = b\cos\theta_3$$

Similarly, ΔCED~ΔCAB, so DE/AB = CE/AC. Hence,

$$DE = c \times \frac{b\sin(\pi/2-\theta_2)}{b} = c\cos\theta_2$$

Thus, we have the following result.

Theorem 3.1. The perimeter of the orthic triangle ΔDEF inscribed in ΔABC can be computed by the formula:

$$\delta_{\Delta DEF} = a\cos\theta_1 + b\cos\theta_3 + c\cos\theta_2$$

where $\delta_{\Delta DEF}$ denotes the perimeter of ΔDEF.

Example. Find the perimeters of the sequence of orthic triangles in ΔABC if the latter is an equilateral triangle with unit side length.

Solution. Let Δ_i denotes the i-th orthic triangle in ΔABC. By Theorem 3.1, we have

$$\delta_{\Delta_1} = 3\cos\frac{\pi}{3} = \frac{3}{2}$$

$$\delta_{\Delta_2} = 3\times\frac{1}{2}\cos\frac{\pi}{3} = \frac{3}{4}$$

$$\delta_{\Delta_3} = 3\times\frac{1}{4}\cos\frac{\pi}{3} = \frac{3}{8}$$

In general, we have: $\delta_{\Delta_i} = 3\times\frac{1}{2^{i-1}}\cos\frac{\pi}{3} = \frac{3}{2^i}$.

IV. CONCLUDING REMARKS

In this paper, we have discussed how to adopt a dynamic geometry approach to solve the Fagnano's problem. We have also derived a useful formula for finding the perimeter of the inscribed orthic triangle of an acute triangle. Some recent researches on the study of periodic billiard trajectories in polygons are actually originated from the Fagnano's minimal perimeter problem, whose solution (i.e. the inscribed orthic triangle) is now often called a closed 3-periodic billiard trajectory or orbit. If the given triangle is a right triangle instead of an acute triangle, then we can use similar dynamic geometry approach to explore and conclude that every right triangle has a periodic billiard trajectory, although the latter is not closed. For instance, Fig. 8 illustrates an example of an open periodic billiard trajectory in a right triangle. In fact, there are some open problems related to periodic billiard trajectories in triangles or n-sided (n > 3) polygons. Readers may refer to [7, 11, 12] for getting some ideas on the recent results and open problems in this interesting research area.

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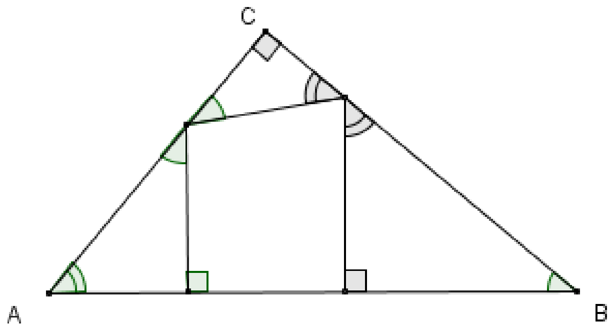

Fig. 8. A periodic billiard trajectory in a right triangle

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