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The Hamiltonian Connected Property of Some Shaped Supergrid Graphs

Ruo-Wei Hung[1,*], Jong-Shin Chen[2], Jun-Lin Li[1], and Chin-Han Lin[1]

***Abstract*—A Hamiltonian path (cycle) of a graph is a simple path (cycle) which visits each vertex of the graph exactly once. The Hamiltonian path (cycle) problem is to determine whether a graph contains a Hamiltonian path (cycle). A graph is called Hamiltonian connected if there exists a Hamiltonian path between any two distinct vertices. Supergrid graphs were first introduced by us and include grid graphs and triangular grid graphs as their subgraphs. The Hamiltonian path (cycle) problem for grid graphs and triangular grid graphs was known to be NP-complete. Recently, we have proved that they are also NP-complete for supergrid graphs. These problems on supergrid graphs can be applied to control the stitching traces of computerized sewing machines. Very recently, we showed that rectangular supergrid graphs are Hamiltonian connected except two trivial forbidden conditions. In this paper, we will study the Hamiltonian connectivity of some shaped supergrid graphs, including triangular, parallelogram, and trapezoid. We prove that these shaped supergrid graphs are always Hamiltonian connected except few trivial forbidden conditions.**

***Index Terms*—Hamiltonian connectivity, supergrid graphs, triangular supergrid graphs, parallelogram supergrid graphs, trapezoid supergrid graphs, computer sewing machines.**

I. Introduction

A *Hamiltonian path* of a graph is a simple path in which each vertex of the graph appears exactly once. A *Hamiltonian cycle* in a graph is a simple cycle with the same property. The *Hamiltonian path* (resp., *cycle*) *problem* involves deciding whether or not a graph contains a Hamiltonian path (resp., cycle). A graph is called to be *Hamiltonian* if it contains a Hamiltonian cycle. A graph G is said to be *Hamiltonian connected* if for each pair of distinct vertices u and v of G, there is a Hamiltonian path between u and v in G. If (u, v) is an edge of a Hamiltonian connected graph, then there must exist a Hamiltonian cycle containing (u, v). Thus, a Hamiltonian connected graph has many Hamiltonian cycles, and, hence, the sufficient conditions of Hamiltonian connectivity are stronger than those of Hamiltonicity. It is well known that the Hamiltonian path and cycle problems are NP-complete for general graphs [5], [15]. The same holds true for bipartite graphs [16], split graphs [6], circle graphs [4], undirected path graphs [1], grid graphs [14], triangular grid graphs [7], and supergrid graphs [9]. In the literature, there are many studies for the Hamiltonian connectivity of interconnection networks. Li *et al.* [17] proved the Hamiltonian connectivity of the recursive dual-net. The popular hypercubes are Hamiltonian but are not Hamiltonian connected. However, many variants of hypercubes, including augmented hypercubes [8], twisted cubes [13], crossed cubes [12], and Möbius cubes [3], have been known to be Hamiltonian connected.

The *two-dimensional integer grid* G^∞ is an infinite graph whose vertex set consists of all points of the Euclidean plane with integer coordinates and in which two vertices are adjacent if and only if the (Euclidean) distance between them is equal to 1. The *two-dimensional triangular grid* T^∞ is an infinite graph obtained from G^∞ by adding all edges on the lines traced from up-left to down-right. A *grid graph* is a finite, vertex-induced subgraph of G^∞. For a node v in the plane with integer coordinates, let v_x and v_y be the x and y *coordinates* of node v, respectively, denoted by $v = (v_x, v_y)$. If v is a vertex in a grid graph, then its possible neighboring vertices include $(v_x, v_y + 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, and $(v_x, v_y - 1)$. For example, Fig. 1(a) shows a grid graph. A *triangular grid graph* is a finite, vertex-induced subgraph of T^∞. If v is a vertex in a triangular grid graph, then its possible neighboring vertices include $(v_x, v_y + 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, $(v_x, v_y - 1)$, $(v_x - 1, v_y + 1)$, and $(v_x + 1, v_y - 1)$. For example, Fig. 1(b) depicts a triangular grid graph. Thus, triangular grid graphs contain grid graphs as subgraphs. Note that triangular grid graphs defined above are isomorphic to the original triangular grid graphs studied in the literature [7] but these graphs are different when considered as geometric graphs. By the same construction of triangular grid graphs from grid graphs, we have proposed a new class of graphs, namely *supergrid graphs*, in [9]. The *two-dimensional supergrid* S^∞ is an infinite graph obtained from T^∞ by adding all edges on the lines traced from up-right to down-left. A *supergrid graph* is a finite, vertex-induced subgraph of S^∞. The possible adjacent vertices of a vertex $v = (v_x, v_y)$ in a supergrid graph include $(v_x, v_y + 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, $(v_x, v_y - 1)$, $(v_x - 1, v_y + 1)$, $(v_x + 1, v_y - 1)$, $(v_x + 1, v_y + 1)$, and $(v_x - 1, v_y - 1)$. Then, supergrid graphs contain grid graphs and triangular grid graphs as subgraphs. For example, Fig. 1(c) shows a supergrid graph. Notice that grid and triangular grid graphs are not subclasses of supergrid graphs, and the converse is also true: these classes of graphs have common elements (points) but in general they are distinct since the edge sets of these graphs are different. Obviously, all grid graphs are bipartite [14] but triangular grid graphs and supergrid graphs are not bipartite.

The Hamiltonian problems on supergrid graphs can be applied to control the stitching trace of a computerized sewing machine as stated in [9]. We have proved that the Hamiltonian cycle and path problems are NP-complete for

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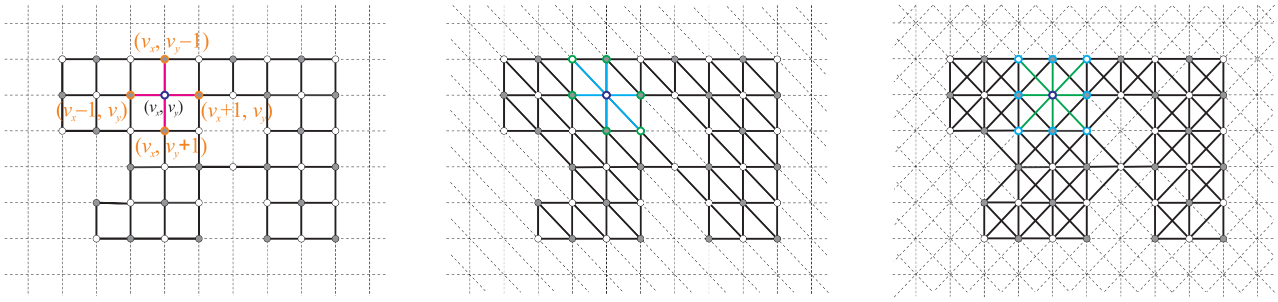
[1]Ruo-Wei Hung, Jun-Lin Li, and Chin-Han Lin are with the Department of Computer Science and Information Engineering, Chaoyang University of Technology, Wufeng, Taichung 41349, Taiwan.

[2]Jong-Shin Chen is with Department of Information and Communication Engineering, Chaoyang University of Technology, Wufeng, Taichung 41349, Taiwan.

[*]Corresponding author e-mail: rwhung@cyut.edu.tw.



(a) (b) (c)

Fig. 1. (a) A grid graph, (b) a triangular grid graph, and (c) a supergrid graph, where circles represent the vertices and solid lines indicate the edges in the graphs.

supergrid graphs [9]. Thus, an important line of investigation is to discover the complexities of the Hamiltonian related problems when the input is restricted to be in special subclasses of supergrid graphs. In [10], we showed that the Hamiltonian cycle problem for linear-convex supergrid graphs is linear solvable. Recently, we proved that rectangular supergrid graphs are always Hamiltonian connected except two trivial forbidden conditions [11]. In this paper, we will show that some shaped supergrid graphs, including triangular, parallelogram, and trapezoid, are always Hamiltonian connected except few trivial forbidden conditions.

II. Notations and Background Results

In this section, we will introduce some terminologies and symbols used in the paper. Some previously observations are also presented. For graph-theoretic terminology not defined here, the reader is referred to [2].

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let S be a subset of vertices in G, and let u, v be two distinct vertices in G. We write $G[S]$ for the subgraph of G *induced* by S, $G - S$ for the subgraph $G[V - S]$, i.e., the subgraph induced by $V - S$. If (u, v) is an edge in G, we say that u is *adjacent* to v. The notation $u \sim v$ (resp., $u \nsim v$) means that vertices u and v are adjacent (resp., non-adjacent). Two edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ are said to be *incident* if $u_1 \sim v_1$ and $u_2 \sim v_2$, denote this by $e_1 \approx e_2$. The *degree* of vertex v, denoted by $deg(v)$, is the number of vertices adjacent to vertex v. A path P of length $|P|$ in G, denoted by $v_1 \to v_2 \to \cdots \to v_{|P|-1} \to v_{|P|}$, is a sequence $(v_1, v_2, \cdots, v_{|P|-1}, v_{|P|})$ of distinct vertices such that $(v_i, v_{i+1}) \in E$ for $1 \leqslant i < |P|$. The first and last vertices visited by P are denoted by $start(P)$ and $end(P)$, respectively. We will use $v_i \in P$ to denote "P visits vertex v_i" and use $(v_i, v_{i+1}) \in P$ to denote "P visits edge (v_i, v_{i+1})". A path from v_1 to v_k is denoted by (v_1, v_k)-path. In addition, we use P to refer to the set of vertices visited by path P if it is understood without ambiguity. A path P is a cycle if $|V(P)| \geqslant 3$ and $end(P) \sim start(P)$. Two paths (or cycles) P_1 and P_2 of graph G are called vertex-disjoint if and only if $V(P_1) \cap V(P_2) = \emptyset$.

Let S^∞ be the infinite graph whose vertex set consists of all points of the plane with integer coordinates and in which two vertices are adjacent if and only if the difference of their x or y coordinates is not larger than 1. A *supergrid graph* is a finite, vertex-induced subgraph of S^∞. For a vertex v in a supergrid graph, let v_x and v_y be respectively x and y coordinates of v. We color vertex v to be *white* if $v_x + v_y \equiv 0 \pmod 2$; otherwise, v is colored to be *black*. Then there are eight possible adjacent vertices of vertex v including four white vertices and four black vertices. Obviously, all grid graphs are bipartite [14] but supergrid graphs are not bipartite. The edge (u, v) in S^∞ is said to be *horizontal* (resp., *vertical*) if $u_y = v_y$ and $u_x \neq v_x$ (resp., $u_x = v_x$ and $u_y \neq v_y$), and is called *skewed* if it is neither a horizontal nor a vertical edge. In the figures, we assume that $(1, 1)$ are coordinates of the up-left vertex, i.e., the leftmost vertex of the first row, in a supergrid graph.

Rectangular supergrid graphs first appeared in [9], in which we have solved the Hamiltonian cycle problem. Let $R(m, n)$ be the supergrid graph whose vertex set $V(R(m, n)) = \{v = (v_x, v_y) \mid 1 \leqslant v_x \leqslant m$ and $1 \leqslant v_y \leqslant n\}$. A *rectangular supergrid graph* is a supergrid graph which is isomorphic to $R(m, n)$. Then m and n, the *dimensions*, specify a rectangular supergrid graph up to isomorphism. The size of $R(m, n)$ is defined to be mn, and $R(m, n)$ is called n-rectangle. Let $v = (v_x, v_y)$ be a vertex in $R(m, n)$. The vertex v is called the *up-left* (resp., *up-right*, *down-left*, *down-right*) *corner* of $R(m, n)$ if $w_x \geqslant v_x$ and $w_y \geqslant v_y$ (resp., $w_x \leqslant v_x$ and $w_y \geqslant v_y$, $w_x \geqslant v_x$ and $w_y \leqslant v_y$, $w_x \leqslant v_x$ and $w_y \leqslant v_y$) for any vertex $w = (w_x, w_y) \in R(m, n)$. There are four boundaries (borders) in a rectangular supergrid graph $R(m, n)$ with $m, n \geqslant 2$. The edge in the boundary of $R(m, n)$ is called *boundary edge*. For example, Fig. 2(a) shows a rectangular supergrid graph $R(10, 10)$ which is called 10-rectangle and contains $4 \times 9 = 36$ boundary edges. Fig. 2(a) also indicates the types of corners.

The triangular supergrid graphs are subgraphs of rectangular supergrid graphs and are defined as follows.

Definition 1. Let ℓ be a diagonal line of $R(n, n)$ with $n \geqslant 2$ from the up-left corner to the down-right corner. Let $\Delta(n, n)$ be the supergrid graph obtained from $R(n, n)$ by removing all vertices under ℓ. A *triangular supergrid graph* is a supergrid graph which is isomorphic to $\Delta(n, n)$.

For example, Fig. 2(b) shows a triangular supergrid graph $\Delta(10, 10)$. Each triangular supergrid graph contains three boundaries, namely *horizontal*, *vertical*, and *skewed*, and these boundaries form a triangle, as illustrated in Fig. 2(b). The triangular supergrid graph $\Delta(n, n)$ is called n-triangle, and the vertex v in $\Delta(n, n)$ is called *triangular corner* if $deg(v) = 2$ and it is the intersection of horizontal (or

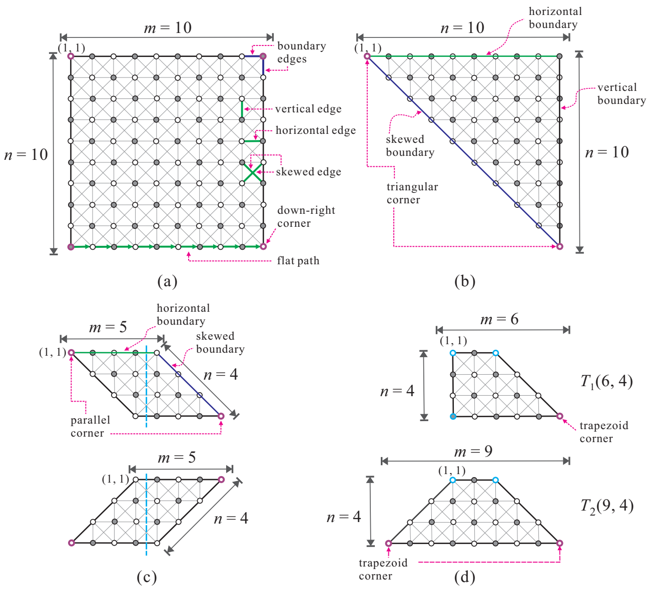


Fig. 2. (a) A rectangular supergrid graph $R(10, 10)$, (b) a triangular supergrid graph $\Delta(10, 10)$, (c) two types of parallelogram supergrid graph $P(5, 4)$, and (d) two types of trapezoid supergrid graphs $T_1(6, 4)$ and $T_2(9, 4)$, where solid arrow lines in (a) indicate a flat path on $R(10, 10)$ and dashed line in (c) indicates a vertical cut.

vertical) and skewed boundaries.

Parallelogram supergrid graphs are defined similar to rectangular supergrid graphs as follows.

Definition 2. Let $P(m, n)$ be the supergrid graph with $m \geqslant n$ whose vertex set $V(P(m, n)) = \{v = (v_x, v_y) \mid 1 \leqslant v_y \leqslant n$ and $v_y \leqslant v_x \leqslant v_y + m - 1\}$ or $\{v = (v_x, v_y) \mid 1 \leqslant v_y \leqslant n$ and $-v_y + 2 \leqslant v_x \leqslant m - (v_y - 1)\}$. A *parallelogram supergrid graph* is a supergrid graph which is isomorphic to $P(m, n)$.

In the above definition, there are two types of parallelogram supergrid graphs. We can see that they are isomorphic although they are different when considered as geometric graphs. In this paper, we can only consider the parallelogram supergrid graph $P(m, n)$ with $V(P(m, n)) = \{v = (v_x, v_y) \mid 1 \leqslant v_y \leqslant n$ and $v_y \leqslant v_x \leqslant v_y + m - 1\}$. Each parallelogram supergrid graph contains four boundaries, two *horizontal* boundaries and two *skewed* boundaries, and these boundaries form a parallelogram, as illustrated in Fig. 2(c). The size of $P(m, n)$ is defined to be mn, and $P(m, n)$ is called n-parallelogram. The vertex w of $P(m, n)$ is called *parallel corner* if $deg(w) = 2$. We can see that a parallelogram supergrid graph contains two parallel corners and it can be decomposed into disjoint triangular and rectangular supergrid subgraphs. For instance, Fig. 2(c) depicts a parallelogram supergrid graph $P(5, 4)$ which can be partitioned into two triangular supergrid graphs $\Delta(4, 4)$.

Next, we introduce trapezoid supergrid graphs. Let $R(m, n)$ be a rectangular supergrid graph with $m \geqslant n \geqslant 2$. A trapezoid graph $T_1(m, n)$ or $T_2(m, n)$ is obtained from $R(m, n)$ by removing one or two triangular supergrid graphs $\Delta(n-1, n-1)$. The definitions of $T_1(m, n)$ and $T_2(m, n)$ are as follows.

Definition 3. Let $R(m, n)$ be a rectangular supergrid graph with $m \geqslant n \geqslant 2$. A trapezoid supergrid graph $T_1(m, n)$ with $m \geqslant n + 1$ is obtained from $R(m, n)$ by removing a triangular supergrid graph $\Delta(n-1, n-1)$ from the corner of $R(m, n)$. A trapezoid supergrid graph $T_2(m, n)$ is constructed from $R(m, n)$ with $m \geqslant 2n$ by removing two triangular supergrid graphs $\Delta(n-1, n-1)$ from the up-left and up-right corners of $R(m, n)$. Fig. 2(d) illustrates these two types of trapezoid graphs.

In a trapezoid supergrid graph, a vertex v is called *trapezoid corner* if $deg(v) = 2$. We can see that $T_1(m, n)$ contains a trapezoid corner, $T_2(m, n)$ contains two trapezoid corners, $T_1(m, n)$ contains two horizontal, one vertical and one skewed boundaries, and $T_2(m, n)$ contains two horizontal and two skewed boundaries. By definition, each boundary of $T_1(m, n)$ and $T_2(m, n)$ contains at least two vertices. On the other hand, $T_1(m, n)$ and $T_2(m, n)$ are called n_{T_1}-trapezoid and n_{T_2}-trapezoid, respectively. For instance, Fig. 2(d) shows $T_1(6, 4)$ and $T_2(9, 4)$.

Let G be a rectangular, triangular, parallelogram, or trapezoid supergrid graph. A path on one boundary of G is called *flat* if it contains all boundary edges in the boundary. For example, the solid arrow lines in Fig. 2(a) indicate a flat path of $R(10, 10)$.

In proving our results, we need to partition a shaped supergrid graph into two disjoint parts. The decomposition is defined as follows.

Definition 4. Let $S(m, n)$ be a triangular, parallelogram, or trapezoid supergrid graph. A *cut* operation on $S(m, n)$ is a line partition through a set Z of edges so that the removal of Z from $S(m, n)$ results in two disjoint supergrid subgraphs S_1 and S_2. A cut is called *vertical* (resp., *horizontal*) if it is a vertical (resp., horizontal) line to separate $S(m, n)$ into S_1 and S_2 such that S_1 is to the left (resp., upper) of S_2, i.e., Z is a set of horizontal (resp., vertical) edges.

For instance, the bold dashed line in Fig. 2(c) shows a vertical cut on $P(5, 4)$ to partition it into two disjoint triangular supergrid subgraphs $\Delta(4, 4)$.

In this paper, we will construct a canonical Hamiltonian path of a triangular, parallelogram, or trapezoid supergrid graph $S(m, n)$. Let s, t be two distinct vertices of $S(m, n)$. A Hamiltonian (s, t)-path of $S(m, n)$ is called *canonical* if it contains at least one boundary edge of each boundary in $S(m, n)$.

Let (G, s, t) denote the supergrid graph G with two given distinct vertices s and t. Without loss of generality, we will assume that $s_x \leqslant t_x$, i.e., s is to the left of t, in the rest of the paper. The notation $L(G, s, t)$ indicates the length of longest path between s and t in G, where the length of a path is defined to be the number of vertices visited by the path. We denote a Hamiltonian path between s and t in G by $HP(G, s, t)$. We say that $HP(G, s, t)$ exists if there is a Hamiltonian (s, t)-path of G. By the definition, $L(G, s, t) = |V(G)|$ if $HP(G, s, t)$ does exist. In [11], we have proved that $HP(R(m, n), s, t)$ always exists for $m, n \geqslant 3$. For $(R(m, n),\ s,\ t)$ with $m \geqslant n \geqslant 3$, a Hamiltonian (s, t)-path of $R(m, n)$ is called *canonical* if it contains at least one boundary edge of each side (boundary) in $R(m, n)$. We then proved the following lemma to show the Hamiltonian connectivity of rectangular supergrid graphs.

Lemma 1. *(See [11]) For* $(R(m, n),\ s,\ t)$ *with* $m \geqslant n \geqslant 3$,

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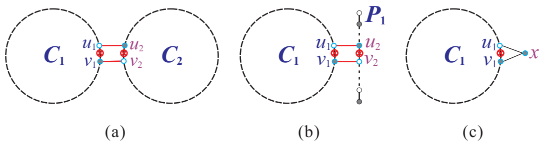

Fig. 3. A schematic diagram for (a) Proposition 2, (b) Proposition 3, and (c) Proposition 4, where bold dashed lines indicate the cycles (paths) and ⊗ represents the destruction of an edge while constructing a cycle or path.

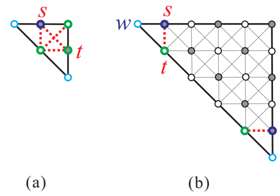

Fig. 4. Triangular supergrid graph in which there exists no Hamiltonian (s,t)-path for (a) condition (F1), and (b) condition (F2), where dotted lines indicate the forbidden edges (s,t).

$R(m,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(R(m,n),s,t)$ does exist.

For the 1-rectangle, $HP(R(m,1),s,t)$ does not exist if s or t is not a corner. On the other hand, $HP(R(m,2),s,t)$ does not exist if (s,t) is a vertical and nonboundary edge of $R(m,2)$. For $n=1$ or 2, $HP(R(m,n),s,t)$ does exist except the above two trivial forbidden conditions [11].

Next, we review some observations on the relations among cycle, path, and vertex. These propositions are presented in [11] and will be used in proving our results. Let C_1 and C_2 be two vertex-disjoint cycles of a graph G. If there exist two edges $e_1 \in C_1$ and $e_2 \in C_2$ such that $e_1 \approx e_2$, then C_1 and C_2 can be combined into a cycle of G. Thus we have the following proposition.

Proposition 2. *(See [11]) Let C_1 and C_2 be two vertex-disjoint cycles of a graph G. If there exist two edges $e_1 \in C_1$ and $e_2 \in C_2$ such that $e_1 \approx e_2$, then C_1 and C_2 can be combined into a cycle of G. (see Fig. 3(a))*

Let C_1 be a cycle and let P_1 be a path in a graph G such that $V(C_1) \cap V(P_1) = \emptyset$. If there exist two edges $e_1 \in C_1$ and $e_2 \in P_1$ such that $e_1 \approx e_2$, then C_1 and P_1 can be combined into a path P of G with $start(P) = start(P_1)$ and $end(P) = end(P_1)$. Fig. 3(b) depicts such a construction, and, hence, the following proposition holds true.

Proposition 3. *(See [11]) Let C_1 and P_1 be a cycle and a path, respectively, of a graph G such that $V(C_1) \cap V(P_1) = \emptyset$. If there exist two edges $e_1 \in C_1$ and $e_2 \in P_1$ such that $e_1 \approx e_2$, then C_1 and P_1 can be combined into a path of G. (see Fig. 3(b))*

The above observation can be extended to a vertex x, where $P_1 = x$, as shown in Fig. 3(c), and we then have the following proposition.

Proposition 4. *(See [10]) Let C_1 be a cycle (path) of a graph G and let x be a vertex in $G - V(C_1)$. If there exists an edge (u_1, v_1) in C_1 such that $u_1 \sim x$ and $v_1 \sim x$, then C_1 and x can be combined into a cycle (path) of G. (see Fig. 3(c))*

III. The Hamiltonian Connectivity of Triangular and Parallelogram Supergrid Graphs

In this section, we will verify the Hamiltonian connectivity (except few trivial conditions) of triangular and parallelogram supergrid graphs. For a triangular supergrid graph $\Delta(n,n)$, we first observe two conditions for $HP(\Delta(n,n),s,t)$ does not exist. These two forbidden conditions are described as follows:

(F1) $\Delta(n,n)$ is a 3-triangle, and (s,t) is a nonboundary edge of $\Delta(n,n)$ (see Fig. 4(a)).

(F2) $\Delta(n,n)$ satisfies $n \geqslant 3$, and (s,t) is an edge of $\Delta(n,n)$ such that both s and t are adjacent to a triangular corner w of $\Delta(n,n)$ (see Fig. 4(b)).

The conditions of (F1) and (F2) are called *forbidden* for $HP(\Delta(n,n),s,t)$. Note that $|V(\Delta(n,n))| = \frac{n(n+1)}{2}$. The following lemma computes the longest (s,t)-path with length $L(\Delta(n,n),s,t)$ when $(\Delta(n,n),s,t)$ satisfies condition (F1) or (F2). Due to the space limitation, the proof of the lemma is omitted.

Lemma 5. *Let $\Delta(n,n)$ be a triangular supergrid graph with $n \geqslant 3$, and let s,t be two distinct vertices of $\Delta(n,n)$. If $(\Delta(n,n),s,t)$ satisfies condition (F1) or (F2), then $L(\Delta(n,n),s,t) = \frac{n(n+1)}{2} - 1$.*

We have computed the longest (s,t)-path of $\Delta(n,n)$ when $(\Delta(n,n),s,t)$ satisfies forbidden condition (F1) or (F2). When $(\Delta(n,n),s,t)$ does not satisfy conditions (F1) and (F2), we will construct a canonical Hamiltonian (s,t)-path of $\Delta(n,n)$ in the following lemma. Due to the space limitation, we omit the proof of the following lemma.

Lemma 6. *Let $\Delta(n,n)$ be a triangular supergrid graph with $n \geqslant 3$, and let s,t be two distinct vertices of $\Delta(n,n)$. If $(\Delta(n,n),s,t)$ does not satisfy conditions (F1) and (F2), then $\Delta(n,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(\Delta(n,n),s,t)$ does exist.*

Next, we will verify the Hamiltonian connectivity of parallelogram supergrid graphs. In a parallelogram supergrid graph $P(m,n)$, we can only consider that $V(P(m,n)) = \{v = (v_x,v_y) \mid 1 \leqslant v_y \leqslant n$ and $v_y \leqslant v_x \leqslant v_y + m - 1\}$. Note that there are two horizontal and two skewed boundaries in $P(m,n)$. We first observe three forbidden conditions for $HP(P(m,n),s,t)$. Then, we prove that $HP(P(m,n),s,t)$ does exist except the forbidden conditions. We first consider 1-parallelogram $(P(m,1),s,t)$. The following condition implies $HP(P(m,1),s,t)$ does not exist.

(F3) $P(m,n)$ is a 1-parallelogram, and s or t is not a corner vertex (see Fig. 5(a)).

Since the possible path between s and t in $P(m,1)$ is unique, the longest (s,t)-path in $(P(m,1),s,t)$ is unique and its length equals $t_x - s_x + 1$. Note that $s_x < t_x$, i.e., s is to the left of t. Then, $HP(P(m,1),s,t)$ does exist if $(P(m,1),s,t)$ does not satisfy condition (F3).

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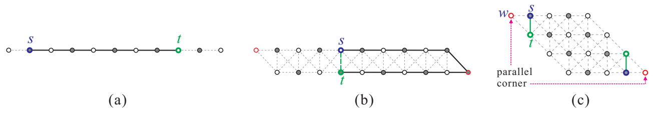

Fig. 5. Parallelogram supergrid graph in which there exists no Hamiltonian (s,t)-path for (a) condition (F3), (b) condition (F4), and (c) condition (F5), where solid lines indicate the longest (s,t)-path.

Next, we consider $(P(m,2),s,t)$ with $m \geqslant 2$. By inspection, the following condition implies $P(m,2)$ contains no Hamiltonian (s,t)-path.

(F4) $P(m,n)$ is a 2-parallelogram with $m \geqslant 2$, and (s,t) is a vertical edge of $P(m,n)$ (see Fig. 5(b)).

Consider that $(R(m,2),s,t)$ satisfies condition (F4). In this case, $s_x = t_x$. Note that the left parallel corner is coordinated as $(1,1)$. Without loss of generality, assume that $s_y \leqslant t_y$. We can easily see that the longest (s,t)-path $L(P(m,2),s,t)$ is either $2s_x - 1$ or $2(m - s_x + 1) + 1$. Then, $L(P(m,2),s,t) = \max\{2s_x - 1, 2m - 2s_x + 3\}$. When $(P(m,2),s,t)$ does not satisfy condition (F4), it is not difficult to verify that $HP(P(m,2),s,t)$ does exist. Thus, we have the following lemma.

Lemma 7. *Let $P(m,2)$ be a 2-parallelogram with $m \geqslant 2$ and let s,t be its two distinct vertices with $s_x \leqslant t_x$ and $s_y \leqslant t_y$. Then, $L(P(m,2),s,t) = \max\{2s_x - 1, 2m - 2s_x + 3\}$ if $(P(m,2),s,t)$ satisfies condition (F4); and $L(P(m,2),s,t) = 2m$, i.e., $HP(P(m,2),s,t)$ does exist, otherwise.*

The third forbidden condition for $HP(P(m,n),s,t)$ is as follows:

(F5) $P(m,n)$ satisfies $m \geqslant n \geqslant 2$, and (s,t) is an edge of $P(m,n)$ such that $s \sim w$ and $t \sim w$ for any parallel corner w of $P(m,n)$, where $s \neq w$, $t \neq w$, and $deg(w) = 2$ (see Fig. 5(c)).

When $(P(m,n),s,t)$ satisfies condition (F5), we can compute the longest (s,t)-path by removing the vertex w from the Hamiltonian cycle of $P(m,n)$. Note that the Hamiltonian cycle of $P(m,n)$ can be constructed in [9]. Thus, we have the following lemma.

Lemma 8. *Let $P(m,n)$ be a parallelogram supergrid graph with $m \geqslant n \geqslant 2$, and let s,t be its two distinct vertices. If $(P(m,n),s,t)$ satisfies condition (F5), then $L(P(m,n),s,t) = mn - 1$, and the longest (s,t)-path contains at least one boundary edge of each boundary in $P(m,n)$ when $n \geqslant 3$.*

In the following, we consider that $(P(m,n),s,t)$ does not satisfy conditions (F3)–(F5). Then, we will construct a canonical Hamiltonian (s,t)-path of $P(m,n)$. We first consider 3-parallelogram $P(m,3)$ as follows. Due to the space limitation, we omit its proof.

Lemma 9. *Let $P(m,n)$ be a 3-parallelogram with $n = 3$ and $m \geqslant 3$, and let s,t be two distinct vertices of $P(m,n)$ with $s_x \leqslant t_x$. If $(P(m,n),s,t)$ does not satisfy condition (F5), then $P(m,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(P(m,3),s,t)$ does exist.*

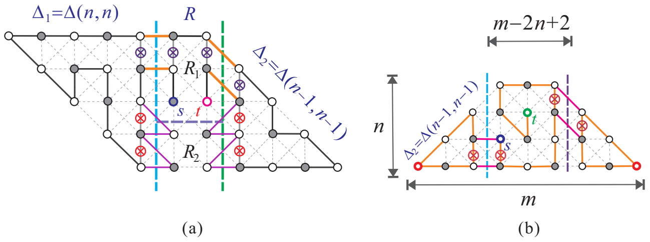

Fig. 6. The Hamiltonian (s,t)-path for (a) a parallelogram supergrid graph $P(7,5)$ and (b) a trapezoid supergrid graph $T_2(9,4)$, where the solid lines indicate the Hamiltonian (s,t)-path and $\otimes$ represents the destruction of an edge while constructing such a Hamiltonian path.

We next verify the Hamiltonian connectivity of parallelogram supergrid graph $P(m,n)$ with $m \geqslant n \geqslant 4$ as follows. Due to the space limitation, we omit the proof of the lemma.

Lemma 10. *Let $P(m,n)$ be a parallelogram supergrid graph with $m \geqslant n \geqslant 4$, and let s,t be two distinct vertices of $P(m,n)$ with $s_x \leqslant t_x$. If $(P(m,n),s,t)$ does not satisfy condition (F5), then $P(m,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(P(m,n),s,t)$ does exist.*

For instance, Fig. 6(a) depicts the Hamiltonian (s,t)-path for a parallelogram supergrid graph $P(7,5)$, where $P(7,5)$ is decomposed into two triangular supergrid subgraphs $\Delta(5,5), \Delta(4,4)$ and one rectangular supergrid subgraph $R(2,5)$.

It immediately follows from Lemmas 9 and 10 that we conclude the following theorem.

Theorem 1. *Let $P(m,n)$ be a parallelogram supergrid graph with $m \geqslant n \geqslant 1$, and let s,t be two distinct vertices of $P(m,n)$. If $(P(m,n),s,t)$ does not satisfy conditions (F3)–(F5), then $P(m,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(P(m,n),s,t)$ does exist.*

IV. The Hamiltonian Connectivity of Trapezoid Supergrid Graphs

In this section, we will verify the Hamiltonian connectivity (except two trivial conditions) of trapezoid supergrid graphs. There are two types of trapezoid supergrid graphs $T_1(m,n)$ and $T_2(m,n)$. Let $T(m,n)$ be a trapezoid supergrid graph, where $T(m,n) = T_1(m,n)$ or $T_2(m,n)$. We first observe the conditions so that $HP(T(m,n),s,t)$ does not exist. For a 2_{T_1}-trapezoid or 2_{T_2}-trapezoid, the following condition implies that $HP(T(m,2),s,t)$ does not exist.

(F6) $T(m,n)$ is a 2_{T_1}-trapezoid or 2_{T_2}-trapezoid, and (s,t) is a vertical and nonboundary edge of $T(m,n)$ (see Fig. 7(a)).

For a trapezoid corner w of $T(m,n)$, we can easily see that $HP(T(m,n),s,t)$ does not exist when $s,t \neq w$, $s \sim w$, and $t \sim w$.

(F7) $T(m,n)$ is a trapezoid supergrid graph for $n \geqslant 2$, w is a trapezoid corner of $T(m,n)$, $s,t \neq w$, $s \sim w$, and $t \sim w$ (see Fig. 7(b)).

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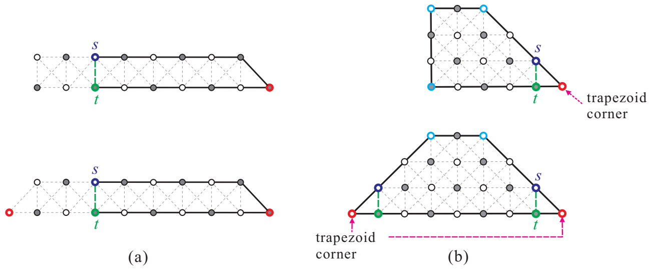

Fig. 7. Trapezoid supergrid graph in which there exists no Hamiltonian (s,t)-path for (a) condition (F6), and (b) condition (F7), where solid lines indicate the longest (s,t)-path.

By similar arguments in proving Lemma 5, we can prove the following lemma.

Lemma 11. *Let $T(m,n)$ be a trapezoid supergrid graph with $n \geqslant 2$, and let s,t be two distinct vertices of $T(m,n)$. Then, the following statemsts hold true:*
(1) if $(T(m,n),s,t)$ satisfies condition (F6), then $L(T_1(m,n),s,t) = \max\{2(m-s_x+1)-1, 2s_x\}$ and $L(T_2(m,n),s,t) = \max\{2(m-s_x+1)-1, 2s_x+1\}$.
(2) if $(T(m,n),s,t)$ satisfies condition (F7), then $L(T(m,n),s,t) = |V(T(m,n))|-1$.

In the following, we will assume that $(T(m,n),s,t)$ does not satisfy conditions (F6) and (F7). Then, we will construct a Hamiltonian (s,t)-path of $T(m,n)$. We first prove $T_1(m,n)$ to be canonical Hamiltonian connected as follows. Due to the space limitation, its proof is omitted.

Lemma 12. *Let $T_1(m,n)$ be a trapezoid supergrid graph with $m-1 \geqslant n \geqslant 2$, and let s,t be two distinct vertices of $T_1(m,n)$. If $(T_1(m,n),s,t)$ does not satisfy conditions (F6)–(F7), then $T_1(m,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(T_1(m,n),s,t)$ does exist.*

Next, we consider the other type of trapezoid supergrid graph $T_2(m,n)$ as follows. Due to the space limitation, we omit the proof of the lemma.

Lemma 13. *Let $T_2(m,n)$ be a trapezoid supergrid graph with $\frac{m}{2} \geqslant n \geqslant 2$, and let s,t be two distinct vertices of $T_2(m,n)$. If $(T_2(m,n),s,t)$ does not satisfy conditions (F6)–(F7), then $T_2(m,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(T_2(m,n),s,t)$ does exist.*

For instance, Fig. 6(b) depicts the Hamiltonian (s,t)-path for a trapezoid supergrid graph $T_2(9,4)$, where $T_2(9,4)$ is decomposed into two triangular supergrid subgraphs $\Delta(3,3)$ and one rectangular supergrid subgraph $R(3,4)$. It immediately follows from Lemmas 12–13 that the following theorem holds true.

Theorem 2. *Let $T(m,n)$ be a trapezoid supergrid graph with $n \geqslant 2$, and let s,t be two distinct vertices of $T(m,n)$, where $T(m,n) = T_1(m,n)$ or $T_2(m,n)$. If $(T(m,n),s,t)$ does not satisfy conditions (F6)–(F7), then $T(m,n)$ contains a canonical Hamiltonian (s,t)-path, and, hence, $HP(T(m,n),s,t)$ does exist.*

V. Concluding Remarks

In this paper, we provide constructive proofs to show that some shaped supergrid graphs, including triangular, parallelogram, and trapezoid, are Hamiltonian connected except few trivial conditions. These constructive proofs give linear time algorithms to construct the longest paths or Hamiltonian paths between two distinct vertices of shaped supergrid graphs. A supergrid graph is called alphabet if its boundaries form an alphabet. There are 26 types of alphabet supergrid graphs. We can see from the structures of alphabet supergrid graphs that they can be decomposed into triangular, parallelogram, or trapezoid supergrid subgraphs. In the future, we would like to apply our results to study the Hamiltonian connectivity of alphabet supergrid graphs.

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