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Stabilization of Linear Differential-algebraic Equations with Time-varying Delay

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***Abstract*—Stabilization involves finding feedback controllers which stabilize the closed-loop system in the finite-time sense. Stability and control have been developed in the literature using Lyapunov-like method. In this paper, we develop a general framework for stabilization of linear differential-algebraic equations with time-varying delay. Based on Lyapunov-like function method and new bound estimation technique, we provide sufficient conditions for global stabilization. The proposed conditions expressed in terms of linear matrix inequalities allow us to find state feedback controllers which stabilize the closed-loop system in the interval.**

***Index Terms*—stabilization, differential-algebraic equations, feedback controller, time-varying delay, linear matrix inequalities**

I. Introduction

Linear differential-algebraic equations are basic models in control theory (see, e.g., [1,2] and the references therein). When they are generalized to include state delays, the resulting models are described by a system of linear delay-differential-algebraic equations (LDDAEs). Since LDDAEs are matrix delay differential equations coupled with matrix difference equations, the study of such systems is much more complicated than that for standard state-space time-delay systems or singular systems.

Most of the results in the literature are focused on Lyapunov stability of LDDAEs [3]. Some early results on stabilization of linear time-delay systems can be found in [4]; more recently the concept of stability has been revisited in the light of recent results coming from linear matrix inequalities (LMIs) theory [5] which has enabled us to find less conservative conditions guaranteeing Lyapunov stability of linear differential-algebraic equations [6].

However, to date and to the best of our knowledge, the problem of stabilization for linear differential-algebraic equations with time-varing delay has not fully investigated. The problem is important and challenging in many practice applications, which motivates the main purpose of our research.

In this paper, we develop a general framework for stabilization of linear differential-algebraic equations with time-varying delay. This is the first instance where the linear differential-algebraic equations is considered with time-varying delay in the state. Under the practical constraints that not all of the state variables of the system are available for feedback control and the real-time knowledge of the time-varying delay is not available, our objective is to design an state feedback controller to finite-time stabilize the closed-loop system. The main contribution of the paper is to find a state feedback controller which guarantees finite-time stability of the resulting closed-loop system. By using new bound estimation techniques we select a simpler set of Lyapunov-like functionals to derive delay-dependent sufficient conditions for designing the state feedback control stabilizer. The conditions are obtained in terms of LMIs.

The outline of the paper is as follows. Section II presents definitions and some well-known technical propositions needed for the proof of the main result. Delay-dependent sufficient conditions for finite-time stabilization of linear differential-algebraic equations with time-varing delays with a numerical example are presented in Section III. Some conclusions are drawn in Section IV.

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II. Preliminaries

The following notations will be used throughout this paper. R^+ denotes the set of all nonnegative real numbers; R^n denotes the $n-$ dimensional space with the scalar product $x^\top y$; $R^{n\times r}$ denotes the space of all matrices of $(n\times r)-$ dimension. $A^\top$ denotes the transpose of A; a matrix A is symmetric if $A = A^\top$; I denotes the identity matrix; $0_{n\times m}$ denotes the zero matrix in $R^{n\times m}$; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{Re(\lambda) : \lambda \in \lambda(A)\}; \lambda_{min}(A) = \min\{Re(\lambda) : \lambda \in \lambda(A)\}; C^1([-\tau,0],R^n)$ denotes the set of all R^n- valued continuously differentiable functions on $[-\tau,0]; L_2([0,T],R^r)$ stands for the set of all square-integrable R^r-valued functions on $[0,T]$. The symmetric terms in a matrix are denoted by $*$. Matrix A is positive defnite $(A>0)$ if $(Ax,x)>0$ for all $x\neq 0$. The following norms will be used: $\|\cdot\|$ refers to the Euclidean vector norm; $\|\varphi\| = \max\{\sup_{t\in[-\tau,0]}\|\varphi(t)\|, \sup_{t\in[-\tau,0]}\|\dot{\varphi}(t)\|\}$ stands for the norm of a function $\varphi(\cdot)\in C^1([-\tau,0],R^n)$. The segment of the trajectory $x(t)$ is denoted by $x_t = \{x(t+s) : s\in[-\tau,0]\}$ with its norm $\|x_t\| = \sup_{s\in[-\tau,0]}\|x(t+s)\|$.

Consider a linear differential-algebraic equation with time-varying delay of the form

$$\begin{cases} E\dot{x}(t) &= Ax(t) + Dx(t-h(t)) + Bu(t) + B_1w(t),\\ t\geq 0, & \\ x(t) &= \psi(t), \quad \forall t\in[-h,0], \end{cases} \tag{1}$$

where $x(t)$ is state vector in R^n, $u(t)\in R^{m_1}$ is control vector, $w(t)$ is the disturbance vector; A, D are constant matrices in $R^{n\times n}$, $B\in R^{n\times m_1}, B_1\in R^{n\times m_2}$, $E\in R^{n\times n}$ is a singular matrix, rank $E = r < n$, $\psi(t)\in C^1([-h,0],R^n)\,; h(t)$ is a continuous function satisfying the condition:

$$0 < h(t) \leq h, \quad t\geq 0.$$

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The disturbance $w(t)$ is any continuous function satisfying the condition

$$\exists d > 0: \qquad w^\top(t)w(t) \le d, \quad t \in [0,T]. \tag{2}$$

Definition 1. (i) System (1) is regular if $det(sE - A)$ is not identical zero.
(ii) System (1) is impulse-free if $deg(det(sE - A)) = r = rankE$.

The singular delay system (1) may have an impulsive solution, however, the regularity and the absence of impulses of the pair (E, A) ensure the existence and uniqueness of an impulse free solution to this system, which is shown in [7].

Proposition 1. [7] *Let $\tau > 0, \sigma > 0, \gamma \in (0,1)$ and $v(t)$ be a continuous function satisfying*

$$0 \le v(t) \le \gamma \sup_{-\tau \le s \le 0} v(t+s) + \sigma, \quad t \ge 0,$$

then the following condition holds

$$v(t) \le \gamma \sup_{-\tau \le s \le 0} v(s) + \frac{\sigma}{1-\gamma} \quad t \ge 0. \tag{3}$$

Proposition 2. (Generalized Jensen inequality [7]) *Given a symmetric matrix $R > 0$ and a differentiable function $\phi : [a,b] \to \mathbb{R}^n$, we have*

$$\int_a^b \dot{\phi}^\top(u) R \dot{\phi}(u) du \ge \frac{1}{b-a}(\phi(b) - \phi(a))^\top R (\phi(b) - \phi(a)) + \frac{12}{b-a}\Omega^\top R \Omega,$$

where $\Omega = \dfrac{\phi(b) + \phi(a)}{2} - \dfrac{1}{b-a}\displaystyle\int_a^b \phi(u)du.$

III. Main result

Consider system (1) with $u(t) = Kx(t)$, since rank $E = r < n$, then there are two nonsingular matrices M, G such that $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = MEG.$
Let

$$M(A + BK)G = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix},$$

$$MDG = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

$$MB_1 = \begin{pmatrix} B_{11} \\ B_{12} \end{pmatrix}.$$

Under coordinate transformation $y = G^{-1}x := [y_1, y_2], y_1 \in R^r$,
$y_2 \in R^{n-r}$, the system (1) is reduced to the system

$$\begin{cases} \dot{y}_1(t) & = \bar{A}_{11}y_1(t) + \bar{A}_{12}y_2(t) + D_{11}y_1(t - h(t)) \\ & \quad + D_{12}y_2(t - h(t)) + B_{11}\omega(t) \\ 0 & = \bar{A}_{21}y_1(t) + \bar{A}_{22}y_2(t) + D_{21}y_1(t - h(t)) \\ & \quad + D_{22}y_2(t - h(t)) + B_{12}\omega(t) \\ y(t) & = G^{-1}\psi(t) := [\phi_1(t), \phi_2(t)], \quad t \in [-h, 0]. \end{cases} \tag{4}$$

Theorem 1. *The system (1) is robustly stabilizable if there exist symmetric positive definite matrices $N \in R^{m_1 \times m_1}, Q \in R^{n \times n}$, a nonsingular matrix $P \in R^{n \times n}$, matrices $W \in R^{n \times n}, U \in R^{m_1 \times n}$ and a positive number $\eta > 0$ such that the following LMIs:*

$$PE = E^\top P^\top \ge 0, \tag{5}$$

$$\Gamma = \begin{pmatrix} \Gamma_1 & X_1 & X_2 \\ * & \Gamma_2 & 0 \\ * & * & \Gamma_3 \end{pmatrix} < 0, \tag{6}$$

$$\|\bar{A}_{22}^{-1} D_{22}\| < 1. \tag{7}$$

Moreover, the state feedback control is defined by $u(t) = N^{-1}Ux(t)$, where

$$M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, \hat{M} = \begin{pmatrix} 0_{r \times n} \\ M_2 \end{pmatrix}, G^\top P M^{-1} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

$$M^{-\top} \mathbb{R} M^{-1} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \alpha_1 = \frac{\lambda_{\min}(P_{11})}{\lambda_{\max}(R_{11})},$$

$$\alpha_2 = \frac{\lambda_{\max}(P_{11})}{\lambda_{\min}(G^\top \mathbb{R} G)}, \quad b_1 = \frac{h^3}{2}\lambda_{\max}(E^\top Q E)\|\psi\|,$$

$$\gamma_1 = \|\bar{A}_{22}^{-1}\bar{A}_{21}\| + \|\bar{A}_{22}^{-1}D_{21}\|, \quad \gamma_2 = \|\bar{A}_{22}^{-1}D_{22}\|,$$

$$\gamma_3 = \|\bar{A}_{22}^{-1}B_{12}\|\sqrt{d}, \Gamma_{11} = PA + A^\top P^\top + BU + U^\top B^\top - \frac{1}{2+h^2}BNB^\top - 4E^\top QE, \Gamma_{12} = PD - 2E^\top QE - A^\top W^\top, \Gamma_{24} = h^2 D^\top Q^\top + W(I_n + \hat{M}), \Gamma_{22} = -8E^\top QE - WD - D^\top W^\top, \quad \Gamma_{33} = -4Q, \quad \Gamma_{44} = -h^2 Q,$$

$$\Gamma_{55} = -12Q, \Gamma_{66} = -12Q, \Gamma_2 = \text{diag}(-N, -\frac{1}{2+h^2}N),$$

$$\Gamma_3 = \text{diag}(-N, -N, -I_{3m_2}),$$

$$\Gamma_1 = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & 0 & h^2 A^\top Q^\top & 0 & 6E^\top Q \\ * & \Gamma_{22} & -2E^\top Q & \Gamma_{24} & 6E^\top Q & 6E^\top Q \\ * & * & \Gamma_{33} & 0 & 6Q & 0 \\ * & * & * & \Gamma_{44} & 0 & 0 \\ * & * & * & * & \Gamma_{55} & 0 \\ * & * & * & * & * & \Gamma_{66} \end{pmatrix},$$

$$X_1 = \begin{pmatrix} PB & U^\top - \frac{1}{2+h^2}BN \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0 & 0 & PB_1 & 0 & 0 \\ WB & 0 & 0 & WB_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & hQB & 0 & 0 & hQB_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof: We first prove the system (1) with the feedback control is regular and impulse-free. We obtain from (5) that

$$G^\top PEG = G^\top PM^{-1}MEG = \begin{pmatrix} P_{11} & 0_{r \times (n-r)} \\ P_{21} & 0_{(n-r)\times(n-r)} \end{pmatrix}$$
$$= G^\top E^\top P^\top G = G^\top E^\top M^\top M^{-1^\top} P^\top G$$
$$= \begin{pmatrix} P_{11}^\top & P_{21}^\top \\ 0_{(n-r)\times r} & 0_{(n-r)\times(n-r)} \end{pmatrix} \ge 0.$$

Therefore, $P_{21} = 0, P_{11} = P_{11}^\top \geq 0$. Since P is nonsingular, then $G^\top PM^{-1} = \begin{pmatrix} P_{11} & P_{12} \\ 0_{(n-r)\times r} & P_{22} \end{pmatrix}$ is nonsingular, we have $det(P_{11}) \neq 0$, hence $P_{11} > 0$.
Since $\Gamma < 0$, we obtain $\begin{pmatrix} \Gamma_1 & X_1 \\ * & \mathrm{diag}(-N, -\frac{1}{2+h^2}N) \end{pmatrix} < 0.$
Applying Proposition 1, we have

$$\Gamma_1 + X_1\Big(\mathrm{diag}(N, \frac{1}{2+h^2}N)\Big)^{-1} X_1^\top < 0. \tag{8}$$

Letting $U = NK$, we have:

$$X_1\,\mathrm{diag}(N, \frac{1}{2+h^2}N)^{-1}X_1^\top = \begin{pmatrix} Y & 0_{n\times 5n} \\ 0_{5n\times n} & 0_{5n\times 5n} \end{pmatrix} \tag{9}$$

where $Y := PBN^{-1}B^\top P^\top + (2+h^2)K^\top NK - BNK - K^\top N^\top B^\top + \frac{1}{2+h^2}BNB^\top$. From (8) and (9), we obtain

$$\Gamma_{11} + PBN^{-1}B^\top P^\top + (2+h^2)K^\top NK - BNK \\ -K^\top N^\top B^\top + \frac{1}{2+h^2}BNB^\top < 0.$$

Therefore

$$PA + A^\top P^\top + PBN^{-1}B^\top P^\top + K^\top NK - 4E^\top QE < 0.$$

Using Cauchy matrix inequality for the inequality

$$K^\top B^\top P^\top + PBK = 2PBK \leq PBN^{-1}B^\top P^\top + K^\top NK,$$

we obtain

$$\begin{aligned} 0 &> PA + A^\top P^\top + PBN^{-1}B^\top P^\top + K^\top NK - 4E^\top QE \\ &> PA + A^\top P^\top + K^\top B^\top P^\top + PBK - 4E^\top QE \\ &= P\bar{A} + \bar{A}^\top P^\top - 4E^\top QE. \end{aligned}$$

Since G is nonsingular, $G^\top \Gamma_{11} G < 0$, we have

$$\begin{aligned} G^\top XG =& G^\top(P\bar{A} + \bar{A}^\top P^\top - 4E^\top QE)G \\ =& G^\top PM^{-1}M\bar{A}G + G^\top \bar{A}^\top M^\top M^{-\top}P^\top G \\ & - 4G^\top E^\top M^\top M^{-\top}QM^{-1}MEG \\ =& \begin{pmatrix} W_{11} & P_{11}\bar{A}_{12} + \bar{A}_{12}\bar{A}_{22} + \bar{A}_{21}^\top P_{22}^\top \\ * & \bar{A}_{22}^\top P_{22}^\top + P_{22}\bar{A}_{22} \end{pmatrix} < 0. \end{aligned}$$

where $W_{11} := P_{11}\bar{A}_{11} + P_{12}\bar{A}_{21} + \bar{A}_{11}^\top P_{11}^\top + \bar{A}_{21}^\top P_{12}^\top - 4Q_{11}$. Therefore, $det(\bar{A}_{22}) \neq 0$.
Let

$$\bar{M} = \begin{pmatrix} I_r & -\bar{A}_{12}\bar{A}_{22}^{-1} \\ 0 & I_{n-r} \end{pmatrix} M, \quad \bar{G} = G\begin{pmatrix} I_r & 0 \\ -\bar{A}_{22}^{-1}\bar{A}_{21} & \bar{A}_{22}^{-1} \end{pmatrix}.$$

It is easy to verify that

$$\bar{E} = \bar{M}E\bar{G} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \ \hat{A} = \bar{M}(A+BK)\bar{G} \\ = \begin{pmatrix} \hat{A}_{11} & 0 \\ 0 & I_{n-r} \end{pmatrix}.$$

$$\bar{M}(sE - (A+BK))\bar{G} = s\bar{M}E\bar{G} - \bar{M}(A+BK)\bar{G} \\ = \begin{pmatrix} sI_r - \hat{A}_{11} & 0 \\ 0 & I_{n-r} \end{pmatrix}.$$

Therefore

$$det(sE-(A+BK)) = det(\bar{M}^{-1}\bar{M}(sE-(A+BK))\bar{G}\bar{G}^{-1}) \\ = det(\bar{M}^{-1})det(sI_r - \hat{A}_{11})det(\bar{G}^{-1}).$$

Moreover, note that $det(sI_r - \hat{A}_{11}) = \sum_{i=0}^r a_i s^i,\ a_r = 1$, and $det(\bar{M}) \neq 0, \det(\bar{G}) \neq 0$ because of the nonsingularity of $\bar{M}$ and $\bar{G}$, then the polynomial $det(sE - (A+BK))$ is not identically zero and

$$deg\,(det(sE - (A+BK))) = r = rank(E),$$

which implies that the system is regular and impulse-free. We now prove the finite-time stabilization of system (1). Consider the following non-negative quadratic functional $V(t, x_t) = \sum_{i=1}^2 V_i(t, x_t)$, where

$$\begin{aligned} V_1(t, x_t) &= e^{\eta t}x^\top(t)PEx(t), \\ V_2(t, x_t) &= e^{\eta t}h\int_{-h}^0\int_{t+s}^t \dot{x}^\top(\tau)E^\top QE\dot{x}(\tau)d\tau ds. \end{aligned}$$

Note that

$$\begin{aligned} G^\top E^\top \mathbb{R}EG =& G^\top E^\top M^\top M^{-\top}\mathbb{R}M^{-1}MEG \\ =& \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \\ & \begin{pmatrix} R_{11} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{10}$$

$$\begin{aligned} G^\top PEG =& G^\top PM^{-1}MEG \\ =& \begin{pmatrix} P_{11} & P_{21} \\ 0 & P_{22} \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P_{11} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{11}$$

Since $M^{-\top}\mathbb{R}M^{-1} > 0$, by Silvester's criterion the matrix R_{11} is positive definite, we then have

$$\langle G^\top E^\top \mathbb{R}EGx, x\rangle \geq \lambda_{\min}(R_{11})\sum_{i=1}^r ||x_i||^2, \quad \forall x \in R^n.$$

$$\langle G^\top \mathbb{R}Gx, x\rangle \geq \lambda_{\min}(G^\top \mathbb{R}G)\sum_{i=1}^n ||x_i||^2, \quad \forall x \in R^n.$$

$$\langle G^\top PEGx, x\rangle \leq \lambda_{\max}(P_{11})\sum_{i=1}^r ||x_i||^2, \quad \forall x \in R^n.$$

Moreover, we have

$$\Big\langle [\frac{\lambda_{\max}(P_{11})}{\lambda_{\min}(G^\top \mathbb{R}G)}G\mathbb{R}G - G^\top PEG]x, x\Big\rangle \geq 0, \quad \forall x \in R^n,$$

which implies

$$G^\top(PE - \frac{\lambda_{\max}(P_{11})}{\lambda_{\min}(G^\top \mathbb{R}G))}\mathbb{R})G \leq 0.$$

Since G is a nonsingular matrix, it follows $PE \leq \frac{\lambda_{\max}(P_{11})}{\lambda_{\min}(G^\top \mathbb{R}G)}\mathbb{R}$ and then

$$x^\top(0)PEx(0) \leq \frac{\lambda_{\max}(P_{11})}{\lambda_{\min}(G^\top \mathbb{R}G)}\sup_{t\in[-h,0]}\psi^\top(t)\mathbb{R}\psi(t). \tag{12}$$

On the other hand, we have

$$h\int_{-h}^0\int_s^0 \dot{x}^\top(\tau)E^\top QE\dot{x}(\tau)d\tau ds \leq \frac{h^3}{2}\lambda_{\max}(E^\top QE)||\psi||^2. \tag{13}$$

Combining the conditions (12)-(13) gives

$$V(0, x_0) \leq \alpha_2 c_1 + b_1. \tag{14}$$

We now show that

$$\alpha_1 x^\top(t)E^\top \mathbb{R}Ex(t) \leq V(t, x_t), \quad \forall t \in [0, T]. \tag{15}$$

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Indeed, from (11) it follows that

$$\langle GE^\top \mathbb{R}EGx, x\rangle \leq \lambda_{\max}(R_{11})\sum_{i=1}^{r}||x_i||^2,$$

$$\langle G^\top PEGx, x\rangle \geq \lambda_{\min}(P_{11})\sum_{i=1}^{r}||x_i||^2, \quad \forall x \in R^n,$$

hence

$$\langle[\alpha_1 GE^\top \mathbb{R}EG - G^\top PEG]x, x\rangle \leq 0, \quad \forall x \in R^n,$$

which gives

$$G^\top(PE - \alpha_1 E^\top \mathbb{R}E)G \geq 0.$$

Since G is a nonsingular matrix, we obtain $PE \geq \alpha_1 E^\top \mathbb{R}E$, and hence

$$V(t, x_t) \geq x^\top(t)PEx(t) \geq \alpha_1 x^\top(t)E^\top \mathbb{R}Ex(t),$$

as desired. Taking the derivative of $V(.)$ along the solution of system (1) we have

$$\begin{aligned}\dot{V}_1(.) &= e^{\eta t}\Big[x^\top(t)(PA + A^\top P^\top + PBK \\ &+ K^\top B^\top P^\top)x(t) + 2x^\top(t)PDx(t-h(t)) \\ &+ 2x^\top(t)PB_1 w(t)\Big] + \eta V_1(.) \\ &\leq e^{\eta t}\Big[x^\top(t)(PA + A^\top P^\top + PBN^{-1}B^\top P^\top \\ &+ K^\top NK)x(t) \ + 2x^\top(t)PDx(t-h(t)) \\ &+ 2x^\top(t)PB_1 w(t)\Big] + \eta V_1(.) \quad (16)\end{aligned}$$

$$\begin{aligned}\dot{V}_2(.) &= e^{\eta t}\Big[h^2\dot{x}^\top(t)E^\top QE\dot{x}(t) \\ &- h\int_{t-h}^{t}\dot{x}^\top(s)E^\top QE\dot{x}(s)ds\Big] + \eta V_2(.) \\ &= e^{\eta t}\Big[h^2\dot{x}^\top(t)E^\top QE\dot{x}(t) \\ &- h\int_{t-h}^{t-h(t)}\dot{x}^\top(s)E^\top QE\dot{x}(s)ds \\ &- h\int_{t-h(t)}^{t}\dot{x}^\top(s)E^\top QE\dot{x}(s)ds\Big] + \eta V_2(.). \quad (17)\end{aligned}$$

To estimate $V_2(.)$ we apply the Proposition 2 for the following inequalities

$$\begin{aligned}&- h\int_{t-h}^{t-h(t)}\dot{x}^\top(s)E^\top QE\dot{x}(s)ds \leq \\ &- 4x(t-h(t))^\top E^\top QEx(t-h(t)) \\ &- 4x(t-h)^\top E^\top QEx(t-h) \\ &- 4x(t-h(t))^\top E^\top QEx(t-h) \\ &+ \frac{12}{h-h(t)}x(t-h(t))^\top E^\top QE\int_{t-h}^{t-h(t)}x(s)ds \\ &+ \frac{12}{h-h(t)}x(t-h)^\top E^\top QE\int_{t-h}^{t-h(t)}x(s)ds \\ &- \frac{12}{(h-h(t))^2}\int_{t-h}^{t-h(t)}x(s)^\top dsE^\top QE\int_{t-h}^{t-h(t)}x(s)ds\end{aligned}$$

$$\begin{aligned}&- h\int_{t-h(t)}^{t}\dot{x}^\top(s)E^\top QE\dot{x}(s)ds \leq \\ &- 4x(t-h(t))^\top E^\top QEx(t-h(t)) - 4x(t)^\top E^\top QEx(t) \\ &- 4x(t-h(t))^\top E^\top QEx(t) \\ &+ \frac{12}{h(t)}x(t-h(t))^\top E^\top QE\int_{t-h(t)}^{t}x(s)ds \\ &+ \frac{12}{h(t)}x(t)^\top E^\top QE\int_{t-h(t)}^{t}x(s)ds \\ &- \frac{12}{h(t)^2}\int_{t-h(t)}^{t}x(s)^\top dsE^\top QE\int_{t-h(t)}^{t}x(s)ds. \quad (18)\end{aligned}$$

Therefore, from (12)-(15) we obtain

$$\begin{aligned}\dot{V}_2(.) \leq &e^{\eta t}\Big[h^2\dot{x}^\top(t)E^\top QE\dot{x}(t) \\ &- 4x(t-h(t))^\top E^\top QEx(t-h(t)) \\ &- 4x(t-h)^\top E^\top QEx(t-h) \\ &- 4x(t-h(t))^\top E^\top QEx(t-h) \\ &+ \frac{12}{h-h(t)}x(t-h(t))^\top E^\top QE\int_{t-h}^{t-h(t)}x(s)ds \\ &+ \frac{12}{h-h(t)}x(t-h)^\top E^\top QE\int_{t-h}^{t-h(t)}x(s)ds \\ &- \frac{12}{(h-h(t))^2}\int_{t-h}^{t-h(t)}x(s)^\top dsE^\top QE \quad (19) \\ &\int_{t-h}^{t-h(t)}x(s)ds - 4x(t-h(t))^\top E^\top QEx(t-h(t)) \\ &- 4x(t)^\top E^\top QEx(t) \\ &- 4x(t-h(t))^\top E^\top QEx(t) \\ &+ \frac{12}{h(t)}x(t-h(t))^\top E^\top QE\int_{t-h(t)}^{t}x(s)ds \\ &+ \frac{12}{h(t)}x(t)^\top E^\top QE\int_{t-h(t)}^{t}x(s)ds \\ &- \frac{12}{h(t)^2}\int_{t-h(t)}^{t}x(s)^\top dsE^\top QE\int_{t-h(t)}^{t}x(s)ds\Big] \\ &+ \eta V_2(.). \quad (20)\end{aligned}$$

Let $\hat{M} = \begin{pmatrix}0_{r\times n} \\ M_2\end{pmatrix}$, we have

$$2e^{\eta t}x^\top(t-h(t))W\hat{M}E\dot{x}(t) = 0. \quad (21)$$

Multiplying both sides of (1) by

$$2h^2e^{\eta t}\dot{x}(t)^\top E^\top Q, 2e^{\eta t}x^\top(t-h(t))W$$

from the right, we obtain:

$$\begin{aligned}&2h^2e^{\eta t}\dot{x}(t)^\top E^\top Q\Big(-E\dot{x}(t) + (A+BK)x(t) \\ &+ Dx(t-h(t)) + B_1 w(t)\Big) = 0, \\ &2e^{\eta t}x^\top(t-h(t))W\Big(E\dot{x}(t) - (A+BK)x(t) \\ &- Dx(t-h(t)) - B_1 w(t)\Big) = 0,\end{aligned}$$

Combining the conditions (11), (15)-(19) gives:

$$\begin{aligned}
&\dot{V}(.) - \eta V(.) \leq \\
&e^{\eta t}\Big[x^\top(t)(PA + A^\top P^\top + PBN^{-1}B^\top P^\top \\
&+ K^\top NK)x(t) + 2x^\top(t)PDx(t-h(t)) \\
&+ 2x^\top(t)PB_1w(t) + h^2\dot{x}^\top(t)E^\top QE\dot{x}(t) \\
&- 4x(t-h(t))^\top E^\top \\
&QEx(t-h(t)) - 4x(t-h)^\top E^\top QEx(t-h) \\
&- 4x(t-h(t))^\top E^\top QEx(t-h) \\
&+ \frac{12}{h-h(t)}x(t-h(t))^\top E^\top QE\int_{t-h}^{t-h(t)}x(s)ds \\
&+ \frac{12}{h-h(t)}x(t-h)^\top E^\top QE\int_{t-h}^{t-h(t)}x(s)ds \\
&- \frac{12}{(h-h(t))^2}\int_{t-h}^{t-h(t)}x(s)^\top dsE^\top QE \qquad (22)\\
&\int_{t-h}^{t-h(t)}x(s)ds - 4x(t-h(t))^\top E^\top QEx(t-h(t)) \\
&- 4x(t)^\top E^\top QEx(t) - 4x(t-h(t))^\top E^\top QEx(t) \\
&+ \frac{12}{h(t)}x(t-h(t))^\top E^\top QE\int_{t-h(t)}^{t}x(s)ds \\
&+ \frac{12}{h(t)}x(t)^\top E^\top QE\int_{t-h(t)}^{t}x(s)ds \\
&- \frac{12}{h(t)^2}\int_{t-h(t)}^{t}x(s)^\top dsE^\top QE\int_{t-h(t)}^{t}x(s)ds \\
&+ h^2\Big(-\dot{x}(t)^\top(2E^\top QE - E^\top QBN^{-1}B^\top Q^\top E) \\
&\dot{x}(t) + 2\dot{x}(t)^\top E^\top QAx(t) \\
&+ x^\top(t)K^\top NKx(t) + 2\dot{x}(t)^\top E^\top QDx(t-h(t)) \\
&+ 2\dot{x}(t)^\top E^\top QB_1w(t)\Big) \\
&+ 2x^\top(t-h(t)WE\dot{x}(t) - 2x^\top(t-h(t)WAx(t) \\
&- 2x^\top(t-h(t)WDx(t-h(t)) \\
&+ x^\top(t-h(t))WBN^{-1}B^\top W^\top x(t-h(t)) \\
&+ x^\top(t)K^\top NKx(t) - 2x^\top(t-h(t))WB_1w(t) \\
&+ 2x^\top(t-h(t))W\hat{M}E\dot{x}(t)\Big] \qquad (23)
\end{aligned}$$

Therefore, we obtain from (20) that

$$\dot{V}(.) - \eta V(.) \leq e^{\eta t}\xi^\top(t)\Phi\xi(t) + e^{\eta t}(2+h^2)w^\top(t)w(t), \quad \forall t \in [0,T], \qquad (24)$$

where

$$\xi(t) = [x(t), x(t-h(t)), Ex(t-h), E\dot{x}(t), \frac{1}{h-h(t)}\int_{t-h}^{t-h(t)}Ex(s)ds, \frac{1}{h(t)}\int_{t-h(t)}^{t}Ex(s)ds]^\top,$$

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} & 0 & h^2A^\top Q^\top & 0 & 6E^\top Q \\ * & \Phi_{22} & -2E^\top Q & \Phi_{24} & 6E^\top Q & 6E^\top Q \\ * & * & \Phi_{33} & 0 & 6Q & 0 \\ * & * & * & \Phi_{44} & 0 & 0 \\ * & * & * & * & \Gamma_{55} & 0 \\ * & * & * & * & * & \Gamma_{66} \end{pmatrix}$$

$$\begin{aligned}
\Phi_{11} &= PA + A^\top P^\top + PBN^{-1}B^\top P^\top \\
&+ (2+h^2)K^\top NK + PB_1B_1^\top P^\top - 4E^\top QE \\
\Phi_{12} &= PD - 2E^\top QE - A^\top W^\top, \Phi_{24} = h^2D^\top Q^\top \\
&+ W + W\hat{M} \\
\Phi_{22} &= -8E^\top QE - WD - D^\top W^\top + WBN^{-1}B^\top W^\top \\
&+ WB_1B_1^\top W^\top \\
\Phi_{33} &= -4Q, \Phi_{44} = -h^2Q + h^2QBN^{-1}B^\top Q^\top \\
&+ h^2QB_1B_1^\top Q^\top.
\end{aligned}$$

Using the Proposition 1, the condition $\Phi < 0$ holds if and only if

$$\Omega = \begin{pmatrix} \Omega_1 & \Omega_2 \\ * & \Omega_3 \end{pmatrix} < 0$$

where

$$\Omega_1 = \begin{pmatrix} \Omega_{11} & \Omega_{12} & 0 & h^2A^\top Q^\top & 0 & 6E^\top Q \\ * & \Gamma_{22} & -2E^\top Q & \Omega_{24} & 6E^\top Q & 6E^\top Q \\ * & * & \Gamma_{33} & 0 & 6Q & 0 \\ * & * & * & \Gamma_{44} & 0 & 0 \\ * & * & * & * & \Gamma_{55} & 0 \\ * & * & * & * & * & \Gamma_{66} \end{pmatrix},$$

$$\Omega_2 = \begin{pmatrix} PB & K^\top & 0 & 0 & PB_1 & 0 & 0 \\ 0 & 0 & WB & 0 & 0 & WB_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & hQB & 0 & 0 & hQB_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned}
\Omega_3 &= \text{diag}\Big(-N, -\frac{1}{(2+h^2)}N^{-1}, -N, -N, -I_{3m_2}\Big) \\
\Omega_{11} &= PA + A^\top P^\top - 4E^\top QE, \Omega_{12} = PD - 2E^\top QE \\
&- A^\top W^\top \\
\Omega_{24} &= h^2D^\top Q^\top + W + W\hat{M}
\end{aligned}$$

Define the matrix of full column

$$C = \begin{pmatrix} I_n & 0 & BN & 0 \\ 0 & I_{5n+m_1} & 0 & 0 \\ 0 & 0 & N & 0 \\ 0 & 0 & 0 & I_{2m_1+3m_2} \end{pmatrix},$$

we have

$$\Lambda = C\Omega C^\top < 0,$$

where $\Lambda = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ * & \Lambda_3 \end{pmatrix}$, and

$$\Lambda_1 = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & 0 & h^2A^\top Q^\top & 0 & 6E^\top Q \\ * & \Gamma_{22} & -2E^\top Q & \Lambda_{24} & 6E^\top Q & 6E^\top Q \\ * & * & \Gamma_{33} & 0 & 6Q & 0 \\ * & * & * & \Gamma_{44} & 0 & 0 \\ * & * & * & * & \Gamma_{55} & 0 \\ * & * & * & * & * & \Gamma_{66} \end{pmatrix}$$

$$\begin{aligned}
\Lambda_{11} &= PA + A^\top P^\top + BNK + K^\top N^\top B^\top - \frac{1}{2+h^2}BNB^\top \\
&- 4E^\top QE, \\
\Lambda_{12} &= PD - 2E^\top QE - A^\top W^\top, \Lambda_{24} = h^2D^\top Q^\top \\
&+ W(I_n + \hat{M}), \\
\Lambda_3 &= \text{diag}\Big(-N, -\frac{1}{(2+h^2)}N, -N, -N, -I_{3m_2}\Big),
\end{aligned}$$

Let $U = NK$, then the condition $\Lambda < 0$ holds iff $\Gamma < 0$. Thus, if the LMI (6) holds, then $\Lambda < 0$. Therefore, we finally obtain from (21) that

$$\dot{V}(t, x_t) - \eta V(t, x_t) < e^{\eta t}(2 + h^2)w^\top(t)w(t), \forall t \in [0, T]. \tag{25}$$

Multiplying both sides of (22) with $e^{-\eta t}$ and integrating both side from 0 to t we obtain

$$e^{-\eta t}V(t, x_t) - V(0, x_0) < (2 + h^2)\int_0^t w^\top(s)w(s)ds,$$
$$\forall t \in [0, T],$$

and hence using (10) we have

$$\begin{aligned} V(t, x_t) &< e^{\eta t}[V(0, x_0) + (2 + h^2)Td] \\ &\le e^{\eta T}[\alpha_2 c_1 + b_1 + (2 + h^2)Td], \quad \forall t \in [0, T]. \end{aligned} \tag{26}$$

Thus, from (11) and (22) it follows that

$$\begin{aligned} x^\top(t)E^\top \mathbb{R}Ex(t) =& y^\top(t)G^\top E^\top \mathbb{R}EGy(t) \\ &= y^\top(t)\begin{pmatrix} R_{11} & 0 \\ 0 & 0 \end{pmatrix} y(t) \\ =& y_1^\top(t)R_{11}y_1(t) \\ <& \frac{V(t, x_t)}{\alpha_1} \le e^{\eta T}\alpha_3, \quad t \in [0, T]. \end{aligned}$$

which implies

$$\|y_1(t)\| \le \sqrt{\frac{\alpha_3}{\lambda_{\min}(R_{11})}} e^{0.5\eta T} \le \beta e^{0.5\eta T}, \quad \forall t \in [0, T]. \tag{27}$$

Let us denote

$$p(t) = -\bar{A}_{22}^{-1}\bar{A}_{21}y_1(t) - \bar{A}_{22}^{-1}D_{21}y_1(t - h(t)).$$

To estimate $\|p(t)\|$, we consider two cases. First, if $t - h(t) \ge 0$, then by (23) we have

$$\|y_1(t - h(t))\| \le \beta e^{0.5\eta T}.$$

Secondly, if $t - h(t) < 0$, then

$$\|y_1(t - h(t))\| = \|\phi_1(t)\| \le \|G^{-1}\|\|\psi\| \le \beta e^{0.5\eta T}.$$

Thus, we have obtain that

$$\begin{aligned} \|p(t)\| \le& \|\bar{A}_{22}^{-1}\bar{A}_{21}\|\|y_1(t)\| + \|\bar{A}_{22}^{-1}D_{21}\|\|y_1(t - h(t))\| \\ \le& (\|\bar{A}_{22}^{-1}\bar{A}_{21}\| + \|\bar{A}_{22}^{-1}D_{21}\|)\beta e^{0.5\eta T}. \end{aligned}$$

Moreover, from the second equation of (4) we have

$$y_2(t) = p(t) - \bar{A}_{22}^{-1}D_{22}y_2(t - h(t)) - \bar{A}_{22}^{-1}B_{12}\omega(t).$$

Therefore,

$$\begin{aligned} \|y_2(t)\| \le \|p(t)\| + \|\bar{A}_{22}^{-1}D_{22}\|\|y_2(t - h(t))\| \\ + \|\bar{A}_{22}^{-1}B_{12}\omega(t)\|, \forall t \ge 0. \end{aligned}$$

Noting that $h(t) \le h$, we have

$$\|y_2(t - h(t))\| \le \sup_{-h \le s \le 0} \|y_2(t + s)\|,$$

and putting $f(t) = \|y_2(t)\|$, we have

$$f(t) \le \gamma_1 \beta e^{0.5\eta T} + \gamma_2 \sup_{-h \le s \le 0} f(t + s) + \gamma_3,$$

with $\gamma_2 = \|\bar{A}_{22}^{-1}D_{22}\| < 1, \gamma_3 = \|\bar{A}_{22}^{-1}B_{12}\|\sqrt{d}$. Applying the Proposition 2, we have

$$\begin{aligned} \|y_2(t)\|^2 \le& \frac{\gamma_1\gamma_4 + \gamma_3}{1 - \gamma_2} \\ &+ \gamma_2 \sup_{-h \le s \le 0} f(s) \le \frac{\gamma_1\gamma_4 + \gamma_3}{1 - \gamma_2} \\ &+ \gamma_2 \sup_{-h \le s \le 0} \|y_2(s)\| \\ &\le \frac{\gamma_1\gamma_4 + \gamma_3}{1 - \gamma_2} \\ &+ \gamma_2 \sup_{-h \le s \le 0} \|y(s)\| \le \frac{\gamma_1\gamma_4 + \gamma_3}{1 - \gamma_2} \\ &+ \gamma_2 \sup_{-h \le s \le 0} \|G^{-1}\psi(s)\| \\ &\le \frac{\gamma_1\gamma_4 + \gamma_3}{1 - \gamma_2} + \gamma_2\|G^{-1}\|\sqrt{\frac{c_1}{\lambda_{\min}(R)}}, \qquad (28) \\ &\forall t \in [0, T]. \qquad (29) \end{aligned}$$

Finally, taking (24), (25) into account, we obtain

$$\begin{aligned} x^\top(t)\mathbb{R}x(t) =& y^\top(t)G^\top \mathbb{R}Gy(t) \\ \le& \lambda_{\max}(G^\top \mathbb{R}G)(\|y_1(t)\|^2 + \|y_2(t)\|^2) \\ \le& \lambda_{\max}(G^\top \mathbb{R}G)\Big(\gamma_4^2 + (\frac{\gamma_1\gamma_4 + \gamma_3}{1 - \gamma_2} \\ &+ \gamma_2\|G^{-1}\|\sqrt{\frac{c_1}{\lambda_{\min}(\mathbb{R})}})^2\Big) \\ \le& c_2, \forall t \in [0, T], \end{aligned}$$

because of the LMI (3) is, by the Schur complement lemma, equivalent to the inequality

$$\lambda_{\max}(G^\top \mathbb{R}G)\Big(\gamma_4^2 + (\frac{\gamma_1\gamma_4 + \gamma_3}{1 - \gamma_2} + \gamma_2\|G^{-1}\|\sqrt{\frac{c_1}{\lambda_{\min}(\mathbb{R})}})^2\Big) \le c_2.$$

This completes the proof of the theorem. ∎

IV. Conclusion

In this paper, we have studied the stabilization of linear DAEs with interval time-varying delays. The designed state feedback controller guarantees the closed-loop system to be stable. By constructing a set of Lyapunov-Krasovskii functionals, sufficient conditions for the existence of such controllers are established in terms of LMIs.

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