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Some Results of the Bipolar Max-product Fuzzy Relational Equations

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***Abstract*—In the literature, some studies have shown that determining the consistency of bipolar fuzzy relational equations is NP-complete. Namely, the solution procedure for solving a system of bipolar fuzzy relational equations with max-product composition contains a high computational complexity. In this study, some sufficient conditions for consistency of bipolar fuzzy relational equations are proposed to reduce the difficulty of detecting the case of an empty solution set. Numerical examples illustrate that the proposed properties are simple to use and do not require to generate the set of possible feasible solution pairs.**

***Index Terms*—bipolar fuzzy relational equalities, max-product composition, NP-complete.**

I. Introduction

In the literature, a system of fuzzy relational equations usually formulates in a matrix form as follows:

$$x \circ A = b,$$

where $x = (x_i)_{1\times m}$, $A = [a_{ij}]_{m\times n}$ and $b = (b_j)_{1\times n}$ are all defined over [0, 1]. The operation "$\circ$" represents a well-defined algebraic composition for matrix multiplication.

Fuzzy relational equations have played an important role in the field of fuzzy set theory [5], [13], [16] since the first study proposed by Sanchez [17] in 1976. After then, fuzzy relational equations or inequalities with different kinds of compositions have been proposed over the years [11], [12], [19]. The commonly seen max-min, $\max_{i\in\mathcal{I}}(x_i \wedge a_{ij}) = b_j, \forall j \in \mathcal{J}$, and max-product compositions, $\max_{i\in\mathcal{I}}(a_{ij}x_i) = b_j, \forall j \in \mathcal{J}$, are special cases of the max-triangular-norm (max-t-norm) composition. Di Nola et al. [3] indicated that the solution set of fuzzy relational equations with max-continuous t-norm composition can be completely determined by a unique maximum solution and a finite number of minimal solutions. The maximum solution can easily be computed by an analytic formula while finding all of the minimal solutions become much more difficult because it is NP-hard [1], [2], [10]. However, many researchers continuously investigated relevant properties of minimal solution and proposed novel solution methods [9], [14], [15], [18], [20]. Furthermore, Lin et al. [8] presented that all systems of max-continuous u-norm fuzzy relational equations (e.g., max-product, max-continuous Archimedean t-norm and max-arithmetic mean) are essentially equivalent, because they all are equivalent to the covering problem.

In 2012, Li and Yang [7] left from the field of max-t-norm composition to introduce the fuzzy relational inequalities with addition-min composition. In 2013, Perfilieva [15] proposed a new sufficient condition and new solvability criteria for two types of fuzzy relation equations with sup-* and inf-$\rightarrow$ compositions. Peeva [14] proposed a universal and exact method, algorithms and software for solving fuzzy linear systems of equations with max-min, min-max, and max-product compositions etc. Matusiewicz et al. [11] showed that the structure of the solution set of fuzzy relational inequality with max-t-norm composition is similar to that of the fuzzy relational equation.

Recently, Freson et al. [4] considered a generation of the linear optimization problem subject to a system of bipolar fuzzy relational equations with max-min composition. They wanted to pursue the idea of taking into account antagonistic effects for this new optimization problem. For instance, consider a supplier who wants to optimize its public awareness and attributes a degree of appreciation to their products. Such a degree of appreciation can be denoted by a real number x_i in the unit interval [0, 1] whose complement $\tilde{x}_i = 1-x_i$ in [0, 1] stands for the degree of disappreciation. Generally, when the positive effect x_i increases, the negative effect $\tilde{x}_i = 1-x_i$ will fall. It is called the bipolar character. It is clear that the bipolar fuzzy relational equations contain the decision vector and its negation simultaneously. Motivated by Freson et al. [4], Li and Liu [6] considered the linear optimization problem with bipolar max-Łukasiewicz equation constraints and transformed this problem into a 0-1 integer linear programming problem.

A system of bipolar fuzzy relational equations with max-product composition formulates in the matrix form as follows:

$$x \circ A^+ \vee \tilde{x} \circ A^- = b \tag{1}$$

where $x = (x_i)_{1\times m}$, $\tilde{x} = (\tilde{x}_i)_{1\times m}$, $A^+ = [a^+_{ij}]_{m\times n}$, $A^- = [a^-_{ij}]_{m\times n}$ and $b = (b_j)_{1\times n}$ are all defined over [0, 1]. The notation "$\vee$" denotes max operation and the operation "$\circ$" represents the max-product composition. $\tilde{x}_i = 1-x_i$ denotes the bipolar character.

If either A^+ or A^- is the zero matrix, the system (1) degenerates into unipolar max-product fuzzy relational equations as $x \circ A^+ = b$ or $\tilde{x} \circ A^- = b$, respectively. Essentially, solving the bipolar fuzzy relational equations with max-product composition is to find a set of solution vectors $x = (x_i)_{i\in\mathcal{I}}$ such that

$$\max_{i\in\mathcal{I}}\{a^+_{ij}x_i, a^-_{ij}\tilde{x}_i\} = b_j, j \in \mathcal{J}, \tag{2}$$

where index sets $\mathcal{I} = \{1, 2, \cdots, m\}$ and $\mathcal{J} = \{1, 2, \cdots, n\}$, respectively.

For investigating the solution set of bipolar max-min fuzzy relational equations to (1),

$$\max_{i\in\mathcal{I}}\{\min(a^+_{ij}, x_i), \min(a^-_{ij}, \tilde{x}_i)\} = b_j, j \in \mathcal{J}, \tag{3}$$

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Freson et al. [4] first analyzed each single equation by a piecewise linear function. Based on the analyzed results which obtained from all of equations, then they structured the solution set of (3) by taking proper intersections and unions. They also figured out that the solution set of a system of bipolar fuzzy relational equations can be determined by a finite set of maximal and minimal solution pairs. However, Li and Liu [6] presented that determining the consistency of a system of bipolar fuzzy relational equations is NP-complete. That is to say, applying the proposed solution procedure by Freson et al. [4] for solving bipolar max-product fuzzy relational equations of (2) contains the high computation complexity. To improve the difficulty of solving this problem, this study proposes some properties for the bipolar fuzzy relational equations with max-product composition. Numerical examples illustrate that the proposed properties can be easily used to detect the case of an empty solution set of (2).

II. Some results

For a system of fuzzy relational equations with continuous max-t-norm composition, a well-known property exists according to which its solution set, if non-empty, can be completely determined using a unique maximum solution and a finite number of minimal solutions. However, this structural property can not extend to the solution set of bipolar fuzzy relational equations because the (2) contains the decision vector and its negation simultaneously. To investigate the property of the solution set, denoted by $X(A^+, A^-, b)$, for the bipolar fuzzy relational equations with max-product composition in (2), some results are given as follows:

Lemma 1. If $a_{ij}^+ < b_j$ and $a_{ij}^- < b_j$, $\forall i \in \mathcal{I}$ holds for some $j \in \mathcal{J}$ in (2), the solution set $X(A^+, A^-, b)$ is empty.

Proof. Due to $0 \le x_i \le 1$ for each $i \in \mathcal{I}$, if $a_{ij}^+ < b_j$ and $a_{ij}^- < b_j$, $\forall i \in \mathcal{I}$ holds for some $j \in \mathcal{J}$ in (2), then this result leads to

$$a_{ij}^+ x_i < b_j \text{ and } a_{ij}^- \tilde{x}_i < b_j$$

and no solution for $x \in X(A^+, A^-, b)$ can satisfy the jth equation in (2). □

According to Lemma 1 we can conclude that if solution set $X(A^+, A^-, b)$ of (2) is nonempty, for each $j \in \mathcal{J}$, $a_{ij}^+ \ge b_j$ or $a_{ij}^- \ge b_j$, for $i \in \mathcal{I}$, that is $b_j \le \max_{i\in\mathcal{I}}\{a_{ij}^+, a_{ij}^-\}$ must hold true.

Henceforth, this study assumes $\frac{b_j}{a_{ij}^+} \to \infty$ and $\frac{b_j}{a_{ij}^-} \to \infty$ for all $i \in \mathcal{I}, j \in \mathcal{J}$ if $a_{ij}^+ = 0$ or $a_{ij}^- = 0$. We also define $\frac{a_{ij}^- a_{ij}^+}{a_{ij}^- + a_{ij}^+} = 0$ if $a_{ij}^+ = 0$ and $a_{ij}^- = 0$.

Lemma 2. If $x = (x_i)_{i\in\mathcal{I}} \in X(A^+, A^-, b) \ne \emptyset$ is a feasible solution for (2), $\max_{j\in\mathcal{J}}\{1 - \frac{b_j}{a_{ij}^-}, 0\} \le x_i \le \min_{j\in\mathcal{J}}\{\frac{b_j}{a_{ij}^+}, 1\}, \forall\, i \in \mathcal{I}$.

Proof. For any solution $x = (x_i)_{i\in\mathcal{I}} \in X(A^+, A^-, b) \ne \emptyset$,

$$\max_{i\in\mathcal{I}}\{a_{ij}^+ x_i, a_{ij}^- \tilde{x}_i\} = b_j, j \in \mathcal{J},$$

This implies that $a_{ij}^+ x_i \le b_j$ and $a_{ij}^- \tilde{x}_i \le b_j$, for all $i \in \mathcal{I}$, for each $j \in \mathcal{J}$.

Consider the situation where $a_{ij}^+ x_i \le b_j$ to yield $x_i \le \frac{b_j}{a_{ij}^+}$, for each $j \in \mathcal{J}$.

The other situation is where $a_{ij}^- \tilde{x}_i = a_{ij}^-(1 - x_i) \le b_j$ to yield $x_i \ge 1 - \frac{b_j}{a_{ij}^-}$, for each $j \in \mathcal{J}$.

Combining the results of these two situations with each variable $x_i \in [0, 1], i \in \mathcal{I}$, can yield

$$\max_{j\in\mathcal{J}}\{1 - \frac{b_j}{a_{ij}^-}, 0\} \le x_i \le \min_{j\in\mathcal{J}}\{\frac{b_j}{a_{ij}^+}, 1\}, \forall\, i \in \mathcal{I}. \square$$

Lemma 2 shows that if $x = (x_i)_{i\in\mathcal{I}} \in X(A^+, A^-, b) \ne \emptyset$ is a feasible solution for (2), the value of each variable x_i is bound between $\max_{j\in\mathcal{J}}\{1 - \frac{b_j}{a_{ij}^-}, 0\}$ and $\min_{j\in\mathcal{J}}\{\frac{b_j}{a_{ij}^+}, 1\}, \forall\, i \in \mathcal{I}$. They can be called the lower and upper bounds of variable x_i, denoted using $\underline{x}_i$ and $\bar{x}_i$, respectively.

The lower bound $\underline{x}_i = \max_{j\in\mathcal{J}}\{1 - \frac{b_j}{a_{ij}^-}, 0\}$ and the upper bound $\bar{x}_i = \min_{j\in\mathcal{J}}\{\frac{b_j}{a_{ij}^+}, 1\}$ of variable $x_i,\ i \in \mathcal{I}$ can be easily computed, but they may not be solutions for (2).

Remark. Clearly, Lemma 2 can further deduce that if variable x_i exists the case $\underline{x}_i > \bar{x}_i$, for some $i \in \mathcal{I}$, then the solution set of (2) is empty.

Lemma 3. If $x = (x_i)_{i\in\mathcal{I}} \in X(A^+, A^-, b) \ne \emptyset$ is a feasible solution for (2), $\max_{i\in\mathcal{I}}\{\frac{a_{ij}^- a_{ij}^+}{a_{ij}^- + a_{ij}^+}\} \le b_j \le \max_{i\in\mathcal{I}}\{a_{ij}^+, a_{ij}^-\}$ for all $j \in \mathcal{J}$.

Proof. According to Lemma 2, if $x = (x_i)_{i\in\mathcal{I}} \in X(A^+, A^-, b) \ne \emptyset$ is a feasible solution for (2), $\underline{x}_i = \max_{j\in\mathcal{J}}\{1 - \frac{b_j}{a_{ij}^-}, 0\} \le x_i \le \bar{x}_i = \min_{j\in\mathcal{J}}\{\frac{b_j}{a_{ij}^+}, 1\}, \forall\, i \in \mathcal{I}$. This implies that for each $j \in \mathcal{J}$ the following inequalities hold true:

$$1 - \frac{b_j}{a_{ij}^-} \le \underline{x}_i \text{ and } \bar{x}_i \le \frac{b_j}{a_{ij}^+}, i \in \mathcal{I}, \text{ for all } j \in \mathcal{J}.$$

Since $\underline{x}_i \le \bar{x}_i$, there exists $1 - \frac{b_j}{a_{ij}^-} \le \frac{b_j}{a_{ij}^+}, i \in \mathcal{I}$ such that $\frac{a_{ij}^- a_{ij}^+}{a_{ij}^- + a_{ij}^+} \le b_j, i \in \mathcal{I}$. Hence, $\max_{i\in\mathcal{I}}\{\frac{a_{ij}^- a_{ij}^+}{a_{ij}^- + a_{ij}^+}\} \le b_j$, for all $j \in \mathcal{J}$.

In addition, $b_j \le \max_{i\in\mathcal{I}}\{a_{ij}^+, a_{ij}^-\}$ according to Lemma 1. Hence,

$$\max_{i\in\mathcal{I}}\{\frac{a_{ij}^- a_{ij}^+}{a_{ij}^- + a_{ij}^+}\} \le b_j \le \max_{i\in\mathcal{I}}\{a_{ij}^+, a_{ij}^-\} \text{ for all } j \in \mathcal{J}. \square$$

Lemma 3 shows that if the value of b_j is not in the range of $\max_{i\in\mathcal{I}}\{\frac{a_{ij}^- a_{ij}^+}{a_{ij}^- + a_{ij}^+}\}$ to $\max_{i\in\mathcal{I}}\{a_{ij}^+, a_{ij}^-\}$ for some $j \in \mathcal{J}$, the system of (2) is inconsistent. Hence, Lemma 3 can be used to check whether the solution set of (2) is empty or not.

Example 1. Consider the following matrix form of bipolar fuzzy relational equations with max-product composition. We use this example to detect the case of an empty solution set by verifying Lemma 3.

$$x \circ A^+ \vee \tilde{x} \circ A^- = b$$

where $x = (x_1, x_2, \cdots, x_6), \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_6), \tilde{x}_i = 1 - x_i, i \in \mathcal{I} = \{1, 2, \cdots, 6\}$,

$$A^+ = \begin{bmatrix} 0.45 & 0.11 & 0.18 & 0.05 & 0.16 & 0.23 \\ 0.21 & 0.12 & 0.11 & 0.08 & 0.17 & 0.16 \\ 0.32 & 0.32 & 0.15 & 0.18 & 0.37 & 0.20 \\ 0.05 & 0.19 & 0.30 & 0.25 & 0.24 & 0.35 \\ 0.96 & 0.32 & 0.25 & 0.21 & 0.37 & 0.36 \\ 0.27 & 0.21 & 0.22 & 0.12 & 0.26 & 0.27 \end{bmatrix},$$

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$$A^- = \begin{bmatrix} 0.18 & 0.09 & 0.12 & 0.14 & 0.10 & 0.21 \\ 0.37 & 0.29 & 0.10 & 0.01 & 0.09 & 0.49 \\ 0.16 & 0.77 & 0.07 & 0.12 & 0.02 & 0.44 \\ 0.24 & 0.11 & 0.20 & 0.04 & 0.13 & 0.47 \\ 0.09 & 0.19 & 0.09 & 0.27 & 0.13 & 0.30 \\ 0.01 & 0.17 & 0.08 & 0.17 & 0.06 & 0.22 \end{bmatrix},$$

$$b = (\ 0.32,\ \ 0.25,\ \ 0.10,\ \ 0.16,\ \ 0.24,\ \ 0.18).$$

Followong Lemma 3, we use the range of terms, $\max_{i\in\mathcal{I}}\{\frac{a^-_{ij}a^+_{ij}}{a^-_{ij}+a^+_{ij}}\}$ and $\max_{i\in\mathcal{I}}\{a^+_{ij}, a^-_{ij}\}$ for all $j \in \mathcal{J} = \{1, 2, \cdots, 6\}$, to check whether Example 1 is empty or not.

Compute

$$\max_{i\in\mathcal{I}}\{\frac{a^-_{i1}a^+_{i1}}{a^-_{i1}+a^+_{i1}}\}$$
$$= \max\{0.129, 0.134, 0.107, 0.041, 0.082, 0.010\} = 0.134$$

and

$$\max_{i\in\mathcal{I}}\{a^+_{i1}, a^-_{i1}\}$$
$$= \max\{0.45, 0.37, 0.32, 0.24, 0.96, 0.27\} = 0.96$$

to yield $0.134 \leq b_1 = 0.32 \leq 0.96$.

Using the same calculation, we can get the other terms as follows:

$$0.226 \leq b_2 = 0.25 \leq 0.77,\quad 0.120 \nleq b_3 = 0.10 \leq 0.30,$$
$$0.118 \leq b_4 = 0.16 \leq 0.27,\quad 0.096 \leq b_5 = 0.24 \leq 0.37,$$
$$\text{and } 0.201 \nleq b_6 = 0.18 \leq 0.49.$$

Clearly, Lemma 3 is not satisfied by the above results. Hence, Example 1 is inconsistent. □

Moreover, Lemma 2 can also be used to check the consistency of (2). Let us compute the lower bound and upper bound of variable x_i for Example 1 to yield

$$\bar{x} = (\bar{x}_i)_{i\in\mathcal{I}} = (0.556, 0.909, 0.649, 0.333, 0.333, 0.455)$$
and
$$\underline{x} = (\underline{x}_i)_{i\in\mathcal{I}} = (0.167, 0.633, 0.675, 0.617, 0.407, 0.182).$$

Because above results show $\bar{x}_3 = 0.649 < \underline{x}_3 = 0.675$, $\bar{x}_4 = 0.333 < \underline{x}_4 = 0.617$ and $\bar{x}_5 = 0.333 < \underline{x}_5 = 0.407$, such that Example 1 is inconsistent by Lemma 2.

Definition 1. For any variable $x_i, i \in \mathcal{I}$ in (2), x_i is called a *binding variable* for the jth bipolar fuzzy relational equation if $a^+_{ij}x_i = b_j$ or $a^-_{ij}\tilde{x}_i = b_j$ holds true for some $j \in \mathcal{J}$. The set $J(x_i) := \{j \in \mathcal{J} | a^+_{ij}x_i = b_j, \text{ or } a^-_{ij}\tilde{x}_i = b_j, \forall j \in \mathcal{J}\}$ denotes the binding set of the binding variable x_i.

Note that a feasible solution for bipolar fuzzy relational equations with max-product composition in (2) is to find a set of vector $x = (x_i)_{i\in\mathcal{I}}$ that satisfies all equations. By Definition 1, to find a solution for (2) can be considered the selection of binding variables from the binding sets $J(x_i)$ and $J(\tilde{x}_i)$ to satisfy all equations.

Theorem 1. Let $x = (x_i)_{i\in\mathcal{I}}$ be a solution for (2) and, $\underline{x} = (\underline{x}_i)_{i\in\mathcal{I}}$ and $\bar{x} = (\bar{x}_i)_{i\in\mathcal{I}}$ represent vectors of the lower and upper bounds, respectively. If x_i is binding in the jth equation, $\bar{x}_i$ or $\underline{x}_i$ is also binding there. Moreover, if $\bar{x}_i$ and $\underline{x}_i$ are non-binding variables, x_i is also non-binding any solution x.

Theorem 1 shows that for any solution $x = (x_i)_{i\in\mathcal{I}} \in X(A^+, A^-, b)$, if x_i is a binding variable, $\bar{x}_i$ or $\underline{x}_i$ is also binding there, that is $J(x_i) \subseteq J(\bar{x}_i) \bigcup J(\underline{x}_i)$.

Definition 2. Let $\underline{x}_i = \max_{j\in\mathcal{J}}\{1 - \frac{b_j}{a^-_{ij}}, 0\}$ and $\bar{x}_i = \min_{j\in\mathcal{J}}\{\frac{b_j}{a^+_{ij}}, 1\}$ be the corresponding lower bound and upper bound of variable $x_i,\ i \in \mathcal{I}$ for (2). Two index sets define as follows:

$$I_j := \{i \in \mathcal{I} | \underline{x}_i a^-_{ij} = b_j, i \in \mathcal{I}\},$$

and

$$\bar{I}_j := \{i \in \mathcal{I} | \bar{x}_i a^+_{ij} = b_j, i \in \mathcal{I}\}, \forall j \in \mathcal{J}.$$

For Definition 2, index sets I_j and $\bar{I}_j$ denote that the possible variables of x may be selected as a binding variable in the jth equation.

Lemma 4. If index sets with $I_j = \bar{I}_j = \emptyset$ for some $j \in \mathcal{J}$ exists, then the solution set $X(A^+, A^-, b)$ of (2) is empty.

Proof. For any solution $x = (x_i)_{i\in\mathcal{I}} \in X(A^+, A^-, b)$, all equations must be satisfied. Furthermore, Theorem 1 shows that if x_i is a binding variable, $\bar{x}_i$ or $\underline{x}_i$ is also binding there. Hence, index sets with $I_j = \bar{I}_j = \emptyset$ show that no any possible variables of x can be selected as a binding variable in the jth equation. Hence the solution set $X(A^+, A^-, b)$ of (2) is empty. □

Example 2. Consider the following matrix form of bipolar fuzzy relational equations with max-product composition. We use this example to detect the case of an empty solution set by verifying Lemma 4.

$$x \circ A^+ \vee \tilde{x} \circ A^- = b$$

where $x = (x_1, x_2, \cdots, x_6), \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_6), \tilde{x}_i = 1 - x_i, i \in \mathcal{I} = \{1, 2, \cdots, 6\}$,

$$A^+ = \begin{bmatrix} 0.45 & 0.11 & 0.18 & 0.05 & 0.16 & 0.23 \\ 0.21 & 0.12 & 0.11 & 0.08 & 0.25 & 0.16 \\ 0.32 & 0.32 & 0.15 & 0.18 & 0.37 & 0.20 \\ 0.05 & 0.19 & 0.30 & 0.25 & 0.24 & 0.35 \\ 0.48 & 0.32 & 0.31 & 0.21 & 0.37 & 0.36 \\ 0.27 & 0.21 & 0.22 & 0.12 & 0.26 & 0.27 \end{bmatrix},$$

$$A^- = \begin{bmatrix} 0.18 & 0.09 & 0.12 & 0.24 & 0.10 & 0.21 \\ 0.37 & 0.29 & 0.10 & 0.01 & 0.09 & 0.49 \\ 0.16 & 0.64 & 0.07 & 0.12 & 0.02 & 0.44 \\ 0.24 & 0.11 & 0.20 & 0.04 & 0.13 & 0.47 \\ 0.09 & 0.19 & 0.50 & 0.27 & 0.13 & 0.30 \\ 0.01 & 0.17 & 0.08 & 0.17 & 0.06 & 0.22 \end{bmatrix},$$

$$b = (\ 0.32,\ \ 0.25,\ \ 0.36,\ \ 0.16,\ \ 0.24,\ \ 0.28).$$

Following Lemma 3, we consider the range of terms, $\max_{i\in\mathcal{I}}\{\frac{a^-_{ij}a^+_{ij}}{a^-_{ij}+a^+_{ij}}\}$ and $\max_{i\in\mathcal{I}}\{a^+_{ij}, a^-_{ij}\}$ for all $j \in \mathcal{J} = \{1, 2, \cdots, 6\}$, to check the consistency of Example 2.

We can get the terms as follows:

$$0.134 \leq b_1 = 0.32 \leq 0.48,\quad 0.213 \leq b_2 = 0.25 \leq 0.64,$$
$$0.191 \leq b_3 = 0.36 \leq 0.50,\quad 0.118 \leq b_4 = 0.16 \leq 0.27,$$
$$0.096 \leq b_5 = 0.24 \leq 0.37,\quad 0.201 \leq b_6 = 0.28 \leq 0.49.$$

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Clearly, Lemma 3 is satisfied by above results. However, we cannot verify the consistency of Example 2.

Computing the corresponding lower bound and upper bound of variable x_i for Example 2 can obtain

$$\bar{x} = (\bar{x}_i)_{i\in\mathcal{I}} = (0.711, 0.960, 0.649, 0.640, 0.649, 0.923)$$

and

$$\underline{x} = (\underline{x}_i)_{i\in\mathcal{I}} = (0.333, 0.429, 0.609, 0.404, 0.407, 0.059).$$

Above results show that each of the lower bound $\underline{x}_i$ is less than the upper bound $\bar{x}_i, \forall i \in \mathcal{I} = \{1, 2, \cdots, 6\}$. Namely, the above results do not violate Lemma 2. However, we also cannot verify the consistency of Example 2.

For Example 2, the corresponding index sets I_j and $\bar{I}_j, j \in \mathcal{J} = \{1, 2, \cdots, 6\}$ can be yielded by Definition 2 as follows:

$$I_1 = \emptyset,\ \bar{I}_1 = \{1\};\ I_2 = \{3\},\ \bar{I}_2 = \emptyset;$$
$$I_3 = \emptyset,\ \bar{I}_3 = \emptyset;\ I_4 = \{1, 5, 6\},\ \bar{I}_4 = \{4\};$$
$$I_5 = \emptyset,\ \bar{I}_5 = \{2, 3, 5, 6\};\ I_6 = \{2, 4\},\ \bar{I}_6 = \emptyset.$$

Since $I_3 = \bar{I}_3 = \emptyset$, it leads that the solution set of Example 2 is empty by Lemma 4. □

III. Conclusion

In this study, we propose some properties to detect the case of an empty solution set of bipolar fuzzy relational equations with max-product composition. Numerical examples illustrate that for detecting the case of an empty solution set, the proposed properties is simple to use and does not require to generate the set of possible feasible solution pairs.

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