An Improved Lower Bound for Bipartite Ramsey Numbers br(2,7) and br(2,8)

Decha Samana and Peerasak Intarapaiboon

***Abstract*—For complete bipartite graphs $K_{s,s}, K_{t,t}$, the bipartite Ramsey number $br(s,t)$ is the least positive integer b such that if the edges of $K_{b,b}$ are colored with red and blue, then there always exists a red $K_{s,s}$ or a blue $K_{t,t}$. We obtain new lower bounds of $br(2,7)$ and $br(2,8)$.**

***Index Terms*—bipartite Ramsey numbers, lower bounds, graphs.**

I. Introduction

For any graphs G_1, G_2, the *bipartite Ramsey number* $br(G_1; G_2)$ is the smallest integer b such that any subgraph G of the complete bipartite graph $K_{b,b}$, either G contains a copy of G_1 or its complement relative to $K_{b,b}$ contains a copy of G_2. For any graphs G_1, G_2, the *bipartite Ramsey number* $br(G_1; G_2)$ is the smallest integer b such that any subgraph G of the complete bipartite graph $K_{b,b}$, either G contains a copy of G_1 or its complement relative to $K_{b,b}$ contains a copy of G_2. The determination of exact values of bipartite Ramsey numbers to be very difficult.

In this paper, we consider only finite undirected simple graphs. For a graph G with vertex set $V(G)$ and edge set $E(G)$. Let $A(G)$ be a adjacency matrix of graph G and $K_{m,n}$ be a complete bipartite graph with order $m+n$ and size mn whose vertices can be partitioned to V_1 and V_2, $|V_1| = m$ and $|V_2| = n$, respectively. For convenience, let $V(K_{m,n}) = V_1 \cup V_2$ where $V_1 = \{u_i | 1 \leq i \leq m\}$ and $V_2 = \{v_j | 1 \leq j \leq n\}$ and $E(K_{m,n}) = \{u_i v_j | 1 \leq i \leq m, 1 \leq j \leq n\}$. The neighborhood of a vertex $v \in V(G)$ are denoted by $N(v) = \{u \in V(G) | uv \in E(G)\}$.

For complete bipartite graphs $K_{s,s}, K_{t,t}$, the *bipartite Ramsey number* $br(s,t)$ is the least positive integer b such that if the edges of $K_{b,b}$ are colored with red and blue, then there always exists a red $K_{s,s}$ or a blue $K_{t,t}$. In Table I, shown the exact values and bound of some bipartite Ramsey numbers $br(s,t)$.

Table I Exact values and bound of some bipartite Ramsey numbers $br(s,t)$

s \ t	2	3	4	5	6	7	8	9	10
2	5	9	14	17	18 25	21	26	29	32
3		17	29	41	56				
4			48	52	101				
5				115	168				

Manuscript received January 22, 2017; revised February 11, 2017.

D. Samana is with the Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang,Chalongkrung Road, Ladkrabang, Bangkok, Thailand 10520 and Centre of Excellence in Mathematics, CHE, Sri Ayutthaya Road, Bangkok 10400, Thailand e-mail: dechasamana@hotmail.com.

P. Intarapaiboon is with the Department of Mathematics and Statistics, Faculty of Science and Technology,Thammasat University, Rangsit Center, Pathumthani 12121, Thailand.

Furthermore, in [2],[3], [4], [5], [6], [7], [8], and [9], they shown exact values and general bounds of some bipartite Ramsey numbers $br(G_1, G_2)$ as follow that

$$br(K_{1,n}; K_{1,n}) = 2n - 1,$$
$$br(K_{2,n}; K_{2,n}) \leq 4n - 3,$$
$$br(K_{3,n}; K_{3,n}) \leq 8n - 5,$$
$$\text{and} \quad br(K_{2,2}; C_{2m}) = m + 1 \text{ for } m \geq 4.$$

For $q^2 - q + 1 \leq n \leq q^2$, $br(K_{2,2}; K_{1,n}) = n + q$ where q is a prime power.

$$br(m,n) \leq \binom{m+n}{m} - 1$$
$$\text{and} \quad br(n,n) < 2^{n-1}(n-1) \text{ for } n \geq 21.$$

One of our aims in the present paper is to obtain some new lower bounds of bipartite Ramsey numbers $br(2,7)$ and $br(2,8)$.

II. Lower Bounds of $br(2,7)$ and $br(2,8)$

In [1], they found some lower bounds bipartite Ramsey numbers $br(2,7) \geq 21, br(2,8) \geq 26, br(2,9) \geq 29$ and $br(2,10) \geq 32$ as shown in Table I. This article, we shall concentrate on lower bounds of bipartite Ramsey numbers $br(2,7)$ and $br(2,8)$ in Theorem 2.1 and 2.2, respectively.

Theorem II.1. $br(2,7) \geq 25$.

Proof: To show that $br(2,7) > 24$, consider a red blue coloring of $K_{24,24}$ as follows. Let $V_1(K_{24,24}) = \{u_1, u_2, ..., u_{24}\}$ and $V_2(K_{24,24}) = \{v_1, v_2, ..., v_{24}\}$ denote the partition sets of $K_{24,24}$. The 2-coloring of the edges of $K_{24,24}$ using the colors red(R) and blue(B)shown in Table II contains no a red $K_{2,2}$ and a blue $K_{7,7}$. Thus $br(2,7) \geq 25$. ■

Theorem II.2. $br(2,8) \geq 27$.

Proof: Let H_1 be a bipartite graph with 52 vertices and H_2 be complement relative of H_1 to $K_{26,26}$. We will show that $br(2,8) > 26$ by representing the graph H_1 and H_2 in adjacency matrices.

We construct a graph H_1 with $|V_1(H_1)| = |V_2(H_1)| = 26$ and $E(H_1) = \{u_i v_j | j \equiv k \text{ mod } 26, \forall k \in \{i, i+1, i+5, i+15, i+24\} \text{ and } 1 \leq i, j \leq 26\}$. Then

$$A(H_1) = \begin{bmatrix} 0 & M_1 \\ M_1^T & 0 \end{bmatrix}$$
$$\text{and} \quad A(H_2) = \begin{bmatrix} 0 & M_2 \\ M_2^T & 0 \end{bmatrix}$$

where M_1 and M_2 are the matrix in Fig 1.

By the preceding remark, no two rows of any M_1 have a common pair of 1's and so, no monochromatic $K_{2,2}$ occurs in $K_{26,26}$. Each any 8 vertices of $V_1(H_1)$ have $|\bigcap N(u_i)| < 8$ for all $1 \leq i \leq 26$ and so, no monochromatic $K_{8,8}$ occurs in $K_{26,26}$. Thus $br(2,8) \geq 27$. ■

Table II A graph showing bipartite Ramsey numbers $br(2,7) > 24$

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}	u_{21}	u_{22}	u_{23}	u_{24}
v_1	R	B	B	B	R	B	B	B	B	B	B	B	B	R	B	B	B	B	B	B	B	R	B	R
v_2	R	R	B	B	B	R	B	B	B	B	B	B	B	B	R	B	B	B	B	B	B	B	R	B
v_3	B	R	R	B	B	B	R	B	B	B	B	B	B	B	B	R	B	B	B	B	B	B	B	R
v_4	R	B	R	R	B	B	B	R	B	B	B	B	B	B	B	B	R	B	B	B	B	B	B	B
v_5	B	R	B	R	R	B	B	B	R	B	B	B	B	B	B	B	B	R	B	B	B	B	B	B
v_6	B	B	R	B	R	R	B	B	B	R	B	B	B	B	B	B	B	B	R	B	B	B	B	B
v_7	B	B	B	R	B	R	R	B	B	B	R	B	B	B	B	B	B	B	B	R	B	B	B	B
v_8	B	B	B	B	R	B	R	R	B	B	B	R	B	B	B	B	B	B	B	B	R	B	B	B
v_9	B	B	B	B	B	R	B	R	R	B	B	B	R	B	B	B	B	B	B	B	B	R	B	B
v_{10}	B	B	B	B	B	B	R	B	R	R	B	B	B	R	B	B	B	B	B	B	B	B	R	B
v_{11}	B	B	B	B	B	B	B	R	B	R	R	B	B	B	R	B	B	B	B	B	B	B	B	R
v_{12}	R	B	B	B	B	B	B	B	R	B	R	R	B	B	B	R	B	B	B	B	B	B	B	B
v_{13}	B	R	B	B	B	B	B	B	B	R	B	R	R	B	B	B	R	B	B	B	B	B	B	B
v_{14}	B	B	R	B	B	B	B	B	B	B	R	B	R	R	B	B	B	R	B	B	B	B	B	B
v_{15}	B	B	B	R	B	B	B	B	B	B	B	R	B	R	R	B	B	B	R	B	B	B	B	B
v_{16}	B	B	B	B	R	B	B	B	B	B	B	B	R	B	R	R	B	B	B	R	B	B	B	B
v_{17}	B	B	B	B	B	R	B	B	B	B	B	B	B	R	B	R	R	B	B	B	R	B	B	B
v_{18}	B	B	B	B	B	B	R	B	B	B	B	B	B	B	R	B	R	R	B	B	B	R	B	B
v_{19}	B	B	B	B	B	B	B	R	B	B	B	B	B	B	B	R	B	R	R	B	B	B	R	B
v_{20}	B	B	B	B	B	B	B	B	R	B	B	B	B	B	B	B	R	B	R	R	B	B	B	R
v_{21}	R	B	B	B	B	B	B	B	B	R	B	B	B	B	B	B	B	R	B	R	R	B	B	B
v_{22}	B	R	B	B	B	B	B	B	B	B	R	B	B	B	B	B	B	B	R	B	R	R	B	B
v_{23}	B	B	R	B	B	B	B	B	B	B	B	R	B	B	B	B	B	B	B	R	B	R	R	B
v_{24}	B	B	B	R	B	B	B	B	B	B	B	B	R	B	B	B	B	B	B	B	R	B	R	R

$$M_1 = \begin{bmatrix}
1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1\\
1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0\\
0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
1&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&1&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&1&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&1&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&1&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1\\
1&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&1\\
1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&1&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&1&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&1
\end{bmatrix}$$

$$M_2 = \begin{bmatrix}
0&1&1&1&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&0&1&0\\
0&0&1&1&1&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&0&1\\
1&0&0&1&1&1&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&0\\
0&1&0&0&1&1&1&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1\\
1&0&1&0&0&1&1&1&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&0&1&0&0&1&1&1&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1\\
1&1&1&0&1&0&0&1&1&1&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1\\
1&1&1&1&0&1&0&0&1&1&1&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1\\
1&1&1&1&1&0&1&0&0&1&1&1&0&1&1&1&1&1&1&1&1&1&0&1&1&1\\
1&1&1&1&1&1&0&1&0&0&1&1&1&0&1&1&1&1&1&1&1&1&1&0&1&1\\
1&1&1&1&1&1&1&0&1&0&0&1&1&1&0&1&1&1&1&1&1&1&1&1&0&1\\
1&1&1&1&1&1&1&1&0&1&0&0&1&1&1&0&1&1&1&1&1&1&1&1&1&0\\
0&1&1&1&1&1&1&1&1&0&1&0&0&1&1&1&0&1&1&1&1&1&1&1&1&1\\
1&0&1&1&1&1&1&1&1&1&0&1&0&0&1&1&1&0&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1&1&0&1&0&0&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&0&1&1&1&1&1&1&1&1&0&1&0&0&1&1&1&0&1&1&1&1&1&1\\
1&1&1&1&0&1&1&1&1&1&1&1&1&0&1&0&0&1&1&1&0&1&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&1&1&0&1&0&0&1&1&1&0&1&1&1&1\\
1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&0&1&0&0&1&1&1&0&1&1&1\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&0&1&0&0&1&1&1&0&1&1\\
1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&0&1&0&0&1&1&1&0&1\\
1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&0&1&0&0&1&1&1&0\\
0&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&0&1&0&0&1&1&1\\
1&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&0&1&0&0&1&1\\
1&1&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&0&1&0&0&1\\
1&1&1&0&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&0&1&0&0
\end{bmatrix}$$

Fig 1 Matrices M_1 and M_2 for lower bound $br(2,8) > 26$

ISBN: 978-988-14047-7-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

ACKNOWLEDGMENT

The authors would like to thank King Mongkut's Institute of Technology Ladkrabang Research Fund, King Mongkut's Institute of Technology Ladkrabang, Thailand.

REFERENCES

[1] N. Adsawatithisakul, W. Summart, and D. Samana, Lower Bounds of Some Bipartite Ramsey Numbers $br(K_{2,2}; K_{n,n})$, *KMITL International Conference on Mathematics: Number Theory, Graph Theory and Applications*(2016).

[2] L.W. Beineke and A.J. Schwenk, On a bipartite form of the Ramsey problem, *Proc.*5^{th} *British Combin. Conf. 1975, Congressus Numer. XV* (1975), 17-22.

[3] W.A. Carnielli and E.L. M. Camelo, $K_{2,2} - K_{1,n}$ and $K_{2,n} - K_{2,n}$ bipartite Ramsey numebers, *Discrete Mathematics* **223** (2000), 83-92.

[4] W. Goddard, M. A. Henning and O. R. Oellermann, Bipartite Ramsey Numbers and Zarankiewicz Numbers, *Discrete Mathematics* **29** (2000), 85-95.

[5] J.H. Hattingh and M.A. Henning, Bipartite Ramsey theory, *Utilitas Math.* **53** (1998), 217-230.

[6] J.H. Hattingh and M.A. Henning, Star-path bipartite Ramsey number, *Discrete Mathematics* **185** (1998), 255-258.

[7] R.W. Irving, A bipartite Ramsey problem and the Zarankiewicz number, *Glasgow Math. J.* **19** (1978), 13-26.

[8] Z. Rui and S. Yongqi, The Bipartite Ramsey Numbers $b(C_{2m}; K_{2,2})$, *The Electronic Journal of Combinatrorics* **18** (2011),♯P51.

[9] V. Longani, Some Bipartite Ramsey Numbers, *Southeast Asian Bulletin of Mathematics.* V26, N4, (2002), 583-592.