General Disposition Strategy for Self-Blocking Queueing System

Yu-Li Tsai, *Member IAENG*, Daichi Yanagisawa, and Katsuhiro Nishinari

Abstract—**In this paper, we study the self-blocking queueing system consisting of three service stations with infinite capacity. Poisson arrivals and exponential service times are assumed. We apply matrix-geometric method to evaluate steady-state probabilities of the quasi-birth-death process. Performance measures include mean number in the system, mean waiting time in the system, mean throughput of each service station and blocking probability of the service stations in front of the terminal service station. The exact formulae of stability conditions are derived. We propose general disposition strategies for the queueing system with the arbitrary number of service stations.**

***Keywords*— Performance Analysis, Matrix-geometric method, Stability Conditions, Disposition Strategy**

I. INTRODUCTION

Self-blocking queueing systems are common in modern service systems. The self-blocking system with three service stations is represented in **Fig. 1** In this system, if all of the service stations are available, every customer has to directly enter the terminal service station (e.g. the station-3 in our case) to receive the service. The definition of a complete service is that when a customer finishes the service at any of the service stations. A customer who completes the service can leave the system in the condition that there is no customer receiving the service in the next service station. This system can be applied to model the performance of taxi stand within the train stations, computer networks and other similar queueing systems. In addition, we successfully derive the stability conditions of the system with both the same and different service rates. Theoretical results suggest that keeping higher service rate for the terminal service station and the stations near the terminal station are better strategies to increase the total operational efficiency of the system results.

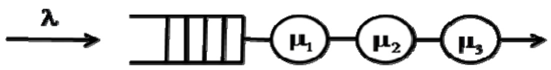

Fig 1. Self-blocking queueing system with three service stations.

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Queueing networks with no intermediate waiting queue between service stations and blocking phenomena was first studied by Hunt [1]. Neuts [2] systematically investigated mathematical analysis and related applications of matrix-geometric method. The optimal design of unpaced assembly lines was developed by Hillier [3]. The joint optimization of both the allocation of workload and the allocation of buffer spaces simultaneously when the objective is to maximize the revenue from throughput minus the cost of work-in-process inventory was analyzed. The disposition strategy for a self-blocking queueing system consisting of two service stations with different service rate was considered by Ke and Tsai [4].

II. PROBLEM FORMULATION AND NOTATIONS

There are three independent service stations in series configuration and operates simultaneously in the queueing system. Poisson arrival process with mean arrival rate λ and the time to serve a customer in each service station is exponentially distributed with mean service time $\frac{1}{\mu}$. When the service stations are all in the idle situation, the customer must enter the terminal station to receive the service. There are no queues between each service station. A customer can finish the service at any stations then leave the system directly if the self-blocking phenomenon does not happen. The self-blocking phenomenon means that when a customer completes the service in a service station, but the another customer in the next station has not finished the service yet. The customer who is receiving the service blocks the customer who has completed the service in the previous station. The blocking phenomenon happens in the station-1, and the station-2 in this system. We assume an infinite queue in front of the first service station. In addition, the service station can only serve a customer at a time and the service rate is independent of the number of customers. The service of the system obeys the first come first serve (FCFS) discipline.

The notation P_{n_1,n_2,n_3,n_4} is used to denote the steady-state probability P_{n_1,n_2,n_3,n_4} of n_1 customer in the station-3 and n_2 customer in the station-2 and n_3 customer in the station-1 and n_4 customer in the queue.

III. MODELING FRAMEWORK

- Matrix-Geometric Method

The steady-state probability vector corresponding to the structured generator matrix Q is denoted as

$\mathbf{P}=[\mathbf{P_0},\mathbf{P_1},\mathbf{P_2},...]$. The steady-state probability vector can be obtained by solving the system of equations $\mathbf{P}Q=\mathbf{0}$, while obeying the normalization condition $\mathbf{P1}=1$. The global balance equations of the system can be written as

$$\mathbf{P_0}B_{0,0}+\mathbf{P_1}B_{1,0}+\mathbf{P}_2B_{2,0}+\mathbf{P}_3B_{3,0}=\mathbf{0}, \quad (1)$$

$$\mathbf{P_0}B_{0,1}+\mathbf{P_1}A_1+\mathbf{P}_4A_4=\mathbf{0}, \quad (2)$$

$$\mathbf{P}_iA_0+\mathbf{P}_{i+1}A_1+\mathbf{P}_{i+4}A_4=\mathbf{0}, \quad i\geq 1. \quad (3)$$

A rate matrix R is introduced to construct the following recurrence relations

$$\mathbf{P}_i=\mathbf{P}_{i-1}R=\mathbf{P}_1R^{i-1}, \quad i\geq 1. \quad (4)$$

Substituting (4) into (3), we can obtain the following characteristic equation of the recurrence relation

$$A_0+RA_1+R^4A_4=0. \quad (5)$$

Therefore, we solve (5) by iteration method for the rate matrix R.
The matrix equations of (1) and (2) can be further simplified as

$$\mathbf{P_0}B_{0,0}+\mathbf{P_1}(B_{1,0}+RB_{2,0}+R^2B_{3,0})=\mathbf{0}, \quad (6)$$

$$\mathbf{P_0}B_{0,1}+\mathbf{P_1}(A_1+R^3A_4)=\mathbf{0}. \quad (7)$$

The normalization condition equation that involves $\mathbf{P_0}$ and $\mathbf{P_1}$ is given by

$$\mathbf{P_0}\mathbf{1}+\mathbf{P_1}(I-R)^{-1}\mathbf{1}=1, \quad (8)$$

where I is the identity matrix with same size as the rate matrix R.
Taking (6), (7) and the normalization condition (8) into account, the steady-state probability vector of $\mathbf{P_0}$ and $\mathbf{P}_1$ can be obtained by solving following matrix equation

$$(\mathbf{P_0},\mathbf{P_1})\begin{pmatrix} B_{0,0} & B^*_{0,1} & \mathbf{1} \\ B_{1,0}+RB_{2,0}+R^2B_{3,0} & (A_1+R^3A_4)^* & (I-R)^{-1}\mathbf{1}\end{pmatrix}=(\mathbf{0},1). \quad (9)$$

where $(.)^*$ indicates that the last column of the included matrix is removed to avoid linear dependency.

- Stability Conditions

Neuts [2] indicated that the steady-state conditions for the ergodicity of steady-state probabilities can be denoted as

$$\mathbf{P}_AA_0\mathbf{1}<\mathbf{P}_AA_2\mathbf{1}+2\mathbf{P}_AA_3\mathbf{1}+3\mathbf{P}_AA_4\mathbf{1}, \quad (10)$$

where $\mathbf{P}_A$ is the steady-state probability vector corresponding to the generator matrix A.

Theorem 1. Stability conditions of the self-blocking queueing system consisting of three service stations.

The stability conditions for the system consisting of three service stations are shown in following

(1) For $\mu_1\neq\mu_2\neq\mu_3$

$$\lambda<\frac{N}{D}, \quad (11)$$

where

$$N=3\mu_1\mu_2\mu_3(\mu_1+\mu_2)(\mu_1+\mu_3)(\mu_2+\mu_3)(\mu_1+\mu_2+\mu_3),$$

and

$$\begin{aligned}D=&\mu_1^4(\mu_2^2+\mu_2\mu_3+\mu_3^2)+\mu_1^3(2\mu_2^3+4\mu_2^2\mu_3+4\mu_2\mu_3^2+2\mu_3^3)\\&+\mu_1^2(\mu_2^4+4\mu_2^3\mu_3+5\mu_2^2\mu_3^2+4\mu_2\mu_3^3+\mu_3^4)\\&+\mu_1(\mu_2^4\mu_3+4\mu_2^3\mu_3^2+4\mu_2^2\mu_3^3+\mu_2\mu_3^4)\\&+\mu_2^2(\mu_2^2\mu_3^2+2\mu_2\mu_3^3+\mu_3^4).\end{aligned}$$

(2) Special case: $\mu_1=\mu_2=\mu_3=\mu$

$$\lambda<\frac{18}{11}\mu. \quad (12)$$

- Performance Metrics and Disposition Strategy

Performance measures including mean number in the system, mean number in the queue, mean waiting time in the system, mean waiting time in the queue, blocking probability of the service stations in front of the terminal station for the self-blocking queueing system consisting of three service stations, and mean throughputs are defined. In addition, we propose general disposition strategies for the self-blocking queueing system consisting of the arbitrary number of service stations based on the numerical results in this section.

- Performance Measures

Performance measures for the system consisting of three service stations are defined by

(1) Mean number of customers in the system – see eq. (13)

$$\begin{aligned}L=&(P_{0,0,1,0}+P_{0,1,0,0}+P_{1,0,0,0}+P_{1,b,0,0}+P_{0,1,b,0})+2(P_{0,0,1,1}+P_{0,1,1,0}+P_{1,0,1,0}+P_{1,1,0,0})\\&+\sum_{n=3}^{\infty}(P_{0,0,1,n-1}+P_{0,1,1,n-2}+P_{1,0,1,n-2}+P_{1,1,1,n-3})\cdot n+\sum_{n=2}^{\infty}(P_{1,b,1,n-2}+P_{0,1,b,n-1}+P_{1,1,b,n-2})\cdot n\\&+\sum_{n=1}^{\infty}(P_{1,b,b,n-1})\cdot n.\end{aligned} \quad (13)$$

(2) Mean number of customer in the queue

$$L_q = (P_{0,0,1,1} + P_{0,1,1,1} + P_{1,0,1,1} + P_{0,1,b,1}) + 2(P_{0,0,1,2}) + \sum_{n=3}^{\infty}(P_{0,0,1,n})\cdot n + \sum_{n=2}^{\infty}(P_{0,1,1,n} + P_{1,0,1,n} + P_{0,1,b,n})\cdot n + \sum_{n=1}^{\infty}(P_{1,1,1,n} + P_{1,b,1,n} + P_{1,b,b,n} + P_{1,1,b,n})\cdot n \overset{\text{if } \mu_1=\mu_2=\mu_3}{=} \mathbf{P}_1[R(I-R)^{-1}(I-R)^{-1}]\cdot \mathbf{1} \quad (14)$$

(3) Mean waiting time in the system

$$W = \frac{L}{\lambda}. \quad (15)$$

(4) Mean waiting time in the queue

$$W_q = \frac{L_q}{\lambda}. \quad (16)$$

(5) Blocking probability of the customer in the station-1

$$P_{b,1} = \sum_{n=0}^{\infty} P_{1,b,b,n} + P_{0,1,b,n} + P_{1,1,b,n}. \quad (17)$$

(6) Blocking probability of the customer in the station-2

$$P_{b,2} = \sum_{n=0}^{\infty} P_{1,b,b,n} + P_{1,b,0,n}. \quad (18)$$

(7) Mean throughputs

$$T = \mu_1[\sum_{n=0}^{\infty} P_{0,0,1,n} + P_{0,1,1,n} + P_{1,1,1,n} + P_{1,b,1,n}] + \mu_2[P_{0,1,0,0} + P_{1,1,0,0} + \sum_{n=0}^{\infty} P_{0,1,b,n} + P_{0,1,1,n} + P_{1,1,1,n} + P_{1,1,b,n}] + \mu_3[P_{1,0,0,0} + P_{1,1,0,0} + P_{1,b,0,0} + \sum_{n=0}^{\infty} P_{1,1,1,n} + P_{1,b,1,n} + P_{1,b,b,n} + P_{1,1,b,n}]. \quad (19)$$

Proposition 3.1. Disposition strategies for the self-blocking queueing system consisting of the arbitrary number of service stations with different service rates are same.

We propose disposition strategies for the system based on previous research conducted by Ke and Tsai [4] and this work in order to increase the operational efficiency of the system.

(1) Self-blocking queueing system with **the arbitrary number** of service stations

It is better to arrange higher service rate for the terminal service station and the service stations near the terminal station compared with other service stations in the system in order to obtain the best operational efficiency for the system.

IV. NUMERICAL RESULTS

We present numerical experiments for the self-blocking queueing system consisting of three stations in this section. Performance measures including mean number in the system, mean throughput of each service station and mean waiting time in the system are illustrated. The better disposition strategies to increase operational efficiency for the system are suggested according to the simulations.

- **Same service rates for each service station**

First, we investigate the trends of mean number in the system and mean throughput of each service station as a function of mean arrival rate λ. Mean number in the system is plotted in **Fig. 2** The upper bound of the stability condition of the mean number in the system approaches to $\frac{18}{11}$ (≈ 1.636). This numerical result is consistent with the exact formula given in the Section 3. Mean throughput of each service stations as a function of mean arrival rate is shown in **Fig. 3** It is investigated that the mean throughput of the station-1 is higher than that of the station-2 and of the station-3 in the condition that all of the service stations are set in the same service rate.

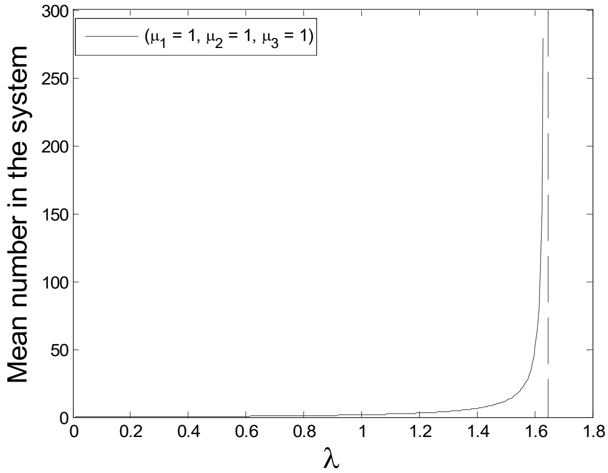

Fig 2. Mean number in the system.

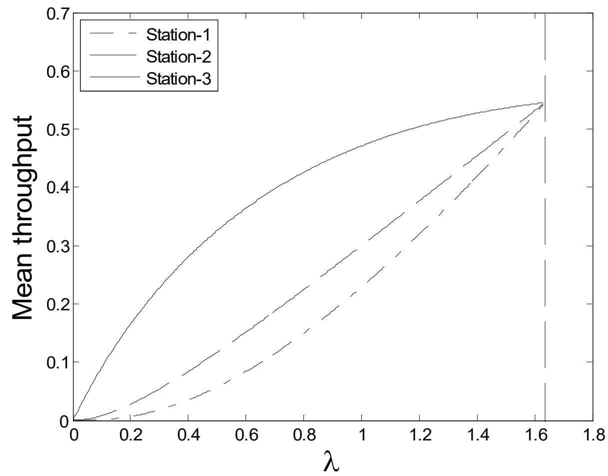

Fig 3. Mean throughput of each station.

- **Controlling the service rates of the two service stations**

We study the disposition conditions that we can concurrently control the service rates of two service stations and the service rate of only one service station for the system consisting of three service stations.

First, in the cases that we are able to control two service rates of the service stations in this system. We set $\mu_1 = 2, \mu_2 = 2, \mu_3 = 1$ and $\mu_1 = 2, \mu_2 = 1, \mu_3 = 2$ and $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2$ and then vary the mean arrival rate λ from 0.01 to 2. It is observed that setting higher service rates for the station-2, and the station-3 is a better disposition strategy than that of other two cases, as shown in **Fig 4**. We suggest the case $\mu_1 = 1, \mu_2 = 2, \mu_3 = 2$ as the best disposition strategy, when we are able to control service rates of two service stations for the system.

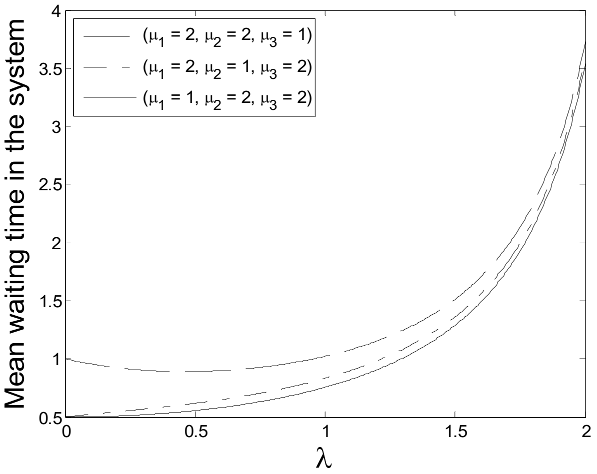

Fig 4. (Controlling two service rates)
Mean waiting time of the system with different service rate.

- **Controlling the service rate of only one service station**

Next, the cases of controlling service rate of one service station are presented. We set $\mu_1 = 2, \mu_2 = 1, \mu_3 = 1$ and $\mu_1 = 1, \mu_2 = 2, \mu_3 = 1$ and $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2$, then vary the mean arrival rate λ from 0.01 to 1.6. It is investigated that the mean waiting time is the lowest in the case of $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2$ compared with other two cases as shown in **Fig 5**. Therefore, the case $\mu_1 = 1, \mu_2 = 1, \mu_3 = 2$ is suggested as the best disposition strategy, when we can control service rates of only one service station for the system.

Note that, in both case studies, numerical computations of all cases should obey the stability conditions we derived in the section 3. in order to satisfy the ergodicity condition of the steady-state probabilities.

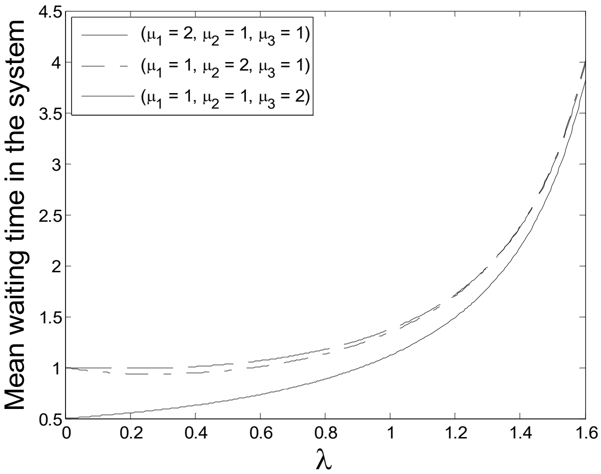

Fig 5. (Controlling only one service rate)
Mean waiting time of the system with different service rate.

V. Conclusion

In this paper, the general disposition strategies for the self-blocking queueing system with arbitrary number of service stations are suggested. We can effectively increase the operational efficiency of the system through applying the strategies. We concurrently derive the exact formulae of stability conditions for the system consisting of three service stations in order to keep the ergodicity of the steady-state probabilities established. Numerical simulations also show the consistent results with the exact formulae of the stability conditions. Steady-state probabilities of the system with infinite capacity are evaluated by matrix-geometric method. Performance measures including mean number in the system, mean number in the queue, mean throughputs, mean waiting time in the system and mean waiting time in the queue are investigated. We have also derived exact formula for the mean number in the queue in the condition that the service rate are all equivalent (e.g. $\mu_1 = \mu_2 = \mu_3$).

Theoretical analyses and propositions are expected to be validated by real experiments. Transient analysis and working breakdown conditions of the system would be considered in the future.

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