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The Hamiltonicity and Hamiltonian Connectivity of L-shaped Supergrid Graphs

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***Abstract*—Supergrid graphs include grid graphs and triangular grid graphs as their subgraphs. The Hamiltonian path problem for general supergrid graphs is a well-known NP-complete problem. A graph is called Hamiltonian connected if there exists a Hamiltonian path between any two distinct vertices. In the past, we verified the Hamiltonian connectivity of some special supergrid graphs, including rectangular, triangular, parallelogram, trapezoid, and alphabet supergrid graphs, except few trivial conditions. In this paper, we will prove that every L-shaped supergrid graph always contains a Hamiltonian cycle except one trivial condition. We also present necessary and sufficient conditions for the existence of a Hamiltonian path between two given vertices in L-shaped supergrid graphs. The Hamiltonian connectivity of L-shaped supergrid graphs can be applied to compute the optimal stitching trace of computer embroidering machines while a varied-sized letter L is sewed into an object.**

***Index Terms*—Hamiltonicity, Hamiltonian connectivity, longest path, supergrid graphs, computer embroidering machines.**

I. Introduction

A *Hamiltonian path* (resp., *cycle*) in a graph is a simple path (resp., cycle) in which each vertex of the graph appears exactly once. The *Hamiltonian path* (resp., *cycle*) *problem* involves deciding whether or not a graph contains a Hamiltonian path (resp., cycle). A graph is called *Hamiltonian* if it contains a Hamiltonian cycle. A graph G is said to be *Hamiltonian connected* if for every pair of distinct vertices u and v of G, there is a Hamiltonian path from u to v in G. If (u, v) is an edge of a Hamiltonian connected graph, then there exists a Hamiltonian cycle containing edge (u, v). Thus, a Hamiltonian connected graph contains many Hamiltonian cycles, and, hence, the sufficient conditions of Hamiltonian connectivity are stronger than those of Hamiltonicity. The *longest path problem* is to find a simple path with the maximum number of vertices in a graph. The Hamiltonian path problem is clearly a special case of the longest path problem.

The Hamiltonian path and cycle problems have numerous applications in different areas, including establishing transport routes, production launching, the on-line optimization of flexible manufacturing systems [1], computing the perceptual boundaries of dot patterns [37], pattern recognition [2], [39], [42], DNA physical mapping [14], and fault-tolerant routing for 3D network-on-chip architectures [9]. It is well known that the Hamiltonian path and cycle problems are NP-complete for general graphs [11], [26]. The same holds true for bipartite graphs [32], split graphs [12], circle graphs [8], undirected path graphs [3], grid graphs [25], triangular grid graphs [13], and supergrid graphs [20].

In the literature, there are many studies for the Hamiltonian connectivity of interconnection networks, including WK-recursive network [10], recursive dual-net [34], hypercomplete network [5], alternating group graph [27], arrangement graph [36]. The popular hypercubes are Hamiltonian but are not Hamiltonian connected. However, many variants of hypercubes, including augmented hypercubes [19], generalized base-b hypercube [18], hypercube-like networks [38], twisted cubes [17], crossed cubes [16], Möbius cubes [7], folded hypercubes [15], and enhanced hypercubes [35], have been known to be Hamiltonian connected.

A supergrid graph is a graph in which vertices lie on integer coordinates and two vertices are adjacent if and only if the difference of their x or y coordinates is not greater than 1. Let $v = (v_x, v_y)$ be a vertex in a supergrid graph, where v_x and v_y represent the x and y *coordinates* of v, respectively. Then, the possible adjacent vertices of v include $(v_x, v_y - 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, $(v_x, v_y + 1)$, $(v_x - 1, v_y - 1)$, $(v_x + 1, v_y + 1)$, $(v_x + 1, v_y - 1)$, and $(v_x - 1, v_y + 1)$. Let $R(m, n)$ be the supergrid graph whose vertex set $V(R(m, n)) = \{v = (v_x, v_y) | 1 \leqslant v_x \leqslant m$ and $1 \leqslant v_y \leqslant n\}$. A *rectangular supergrid graph* is a supergrid graph which is isomorphic to $R(m, n)$. Let $L(m, n; k, l)$ be a supergrid graph obtained from a rectangular supergrid graph $R(m, n)$ by removing its subgraph $R(k, l)$ from the upper right corner. A L-shaped supergrid graph is isomorphic to $L(m, n; k, l)$. In this paper, we only consider $L(m, n; k, l)$.

The possible application of the Hamiltonian connectivity of L-shaped supergrid graphs is presented as follows. Consider a computerized embroidery machine to embroider the object, e.g., clothes, with a L letter. First, we produce a set of lattices to represent the letter. Then, a path is computed to visit the lattices of the set such that each lattice is visited exactly once. Finally, the software transmits the stitching trace of the computed path to the computerized embroidering machine, and the machine then performs the stitching work along the trace on the object. Since each stitch position of an embroidering machine can be moved to its eight neighboring positions (left, right, up, down, up-left, up-right, down-left, and down-right), one set of neighboring lattices forms a L-shaped supergrid graph. Note that each lattice will be represented by a vertex of a supergrid graph. The desired stitching trace of the set of adjacent lattices is the Hamiltonian path of the corresponding L-shaped supergrid graph. The width and height of L-shaped supergrid graph $L(m, n; k, l)$ can be adjusted according to the parameters m,

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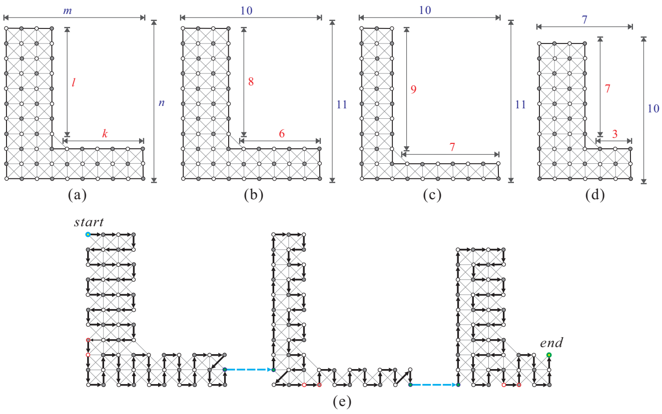

Fig. 1. (a) The structure of L-shaped supergrid graph $L(m, n; k, l)$, (b) $L(10, 11; 6, 8)$, (c) $L(10, 11; 7, 9)$, (d) $L(7, 10; 3, 7)$, and (e) a possible stitching trace for the sets of lattices in (b)–(d), where solid arrow lines indicate the computed trace and dashed arrow lines indicate the jump lines connecting two continuous letters.

n, k, and l. For example, Fig. 1(a) indicates the structure of $L(m, n; k, l)$, and Figs. 1(b)–(d) depict $L(10, 11; 6, 8)$, $L(10, 11; 7, 9)$, and $L(7, 10; 3, 7)$, respectively. Given a string with varied-sized L letters. By the Hamiltonian connectivity of L-shaped supergrid graphs, we can seek the end vertices of Hamiltonian paths in the corresponding L-shaped supergrid graphs so that the total length of jump lines connecting two L-shaped supergrid graphs is minimum. For instance, given three L-shaped supergrid graphs in Figs. 1(b)–(d), in which each L-shaped supergrid graph represents a set of lattices, Fig. 1(e) shows a such minimum stitching trace for the sets of lattices.

Previous related works are summarized as follows. Recently, Hamiltonian path (cycle) and Hamiltonian connected problems on grid, triangular grid, and supergrid graphs have received much attention. Itai *et al.* [25] showed that the Hamiltonian path problem on grid graphs is NP-complete. They also gave necessary and sufficient conditions for a rectangular grid graph having a Hamiltonian path between two given vertices. Note that rectangular grid graphs are not Hamiltonian connected. Zamfirescu *et al.* [43] gave sufficient conditions for a grid graph having a Hamiltonian cycle, and proved that all grid graphs of positive width have Hamiltonian line graphs. Later, Chen *et al.* [6] improved the Hamiltonian path algorithm of [25] on rectangular grid graphs and presented a parallel algorithm for the Hamiltonian path problem with two given endpoints in rectangular grid graphs. Also there is a polynomial-time algorithm for finding Hamiltonian cycles in solid grid graphs [33]. In [41], Salman introduced alphabet grid graphs and determined classes of alphabet grid graphs which contain Hamiltonian cycles. Keshavarz-Kohjerdi and Bagheri gave necessary and sufficient conditions for the existence of Hamiltonian paths in alphabet grid graphs, and presented linear-time algorithms for finding Hamiltonian paths with two given endpoints in these graphs [28]. They also presented a linear-time algorithm for computing the longest path between two given vertices in rectangular grid graphs [29], gave a parallel algorithm to solve the longest path problem in rectangular grid graphs [30], and solved the Hamiltonian connected problem in L-shaped grid graphs [31]. Reay and Zamfirescu [40] proved that all 2-connected, linear-convex triangular grid graphs except one special case contain Hamiltonian cycles. The Hamiltonian cycle (path) on triangular grid graphs has been shown to be NP-complete [13]. They also proved that all connected, locally connected triangular grid graphs (with one exception) contain Hamiltonian cycles. Recently, we prove that the Hamiltonian cycle and path problems on supergrid graphs are NP-complete [20]. We also showed that every rectangular supergrid graph always contains a Hamiltonian cycle. In [21], we prove linear-convex supergrid graphs, which form a subclass of supergrid graphs, to be Hamiltonian. Very recently, we verify the Hamiltonian connectivity of rectangular, shaped, and alphabet supergrid graphs [24], [22], [23].

The rest of the paper is organized as follows. In Section II, some notations and observations are given. Previous results are also introduced. Section III shows that L-shaped supergrid graphs are Hamiltonian and Hamiltonian connected. Finally, we make some concluding remarks in Section IV.

II. Terminologies and Background Results

In this section, we will introduce some terminologies and symbols. Some observations and previously established results for the Hamiltonicity and Hamiltonian connectivity of rectangular supergrid graphs are also presented. For graph-theoretic terminology not defined in this paper, the reader is referred to [4].

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let S be a subset of vertices in G, and let u and v be two vertices in G. We write $G[S]$ for the subgraph of G *induced* by S, $G - S$ for the subgraph $G[V - S]$, i.e., the subgraph induced by $V - S$. In general, we write $G - v$ instead of $G - \{v\}$. If (u, v) is an edge in G, we say that u is *adjacent* to v, and u and v are *incident* to edge (u, v). The notation $u \sim v$ (resp., $u \nsim v$) means that vertices u and v are adjacent (resp., non-adjacent). Edge $e_1 = (u_1, v_1)$ is said to be *parallel* with edge $e_2 = (u_2, v_2)$ if $u_1 \sim u_2$ and $v_1 \sim v_2$. The notation $e_1 \approx e_2$ means that edges e_1 and e_2 are parallel. A *neighbor* of v in G is any vertex that is adjacent to v. We use $N_G(v)$ to denote the set of neighbors of v in G, and let $N_G[v] = N_G(v) \cup \{v\}$. The number of vertices adjacent to vertex v in G is called the *degree* of v in G and is denoted by $deg(v)$. A path P of length $|P|$ in G, denoted by $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{|P|-1} \rightarrow v_{|P|}$, is a sequence $(v_1, v_2, \cdots, v_{|P|-1}, v_{|P|})$ of vertices such that $(v_i, v_{i+1}) \in E$ for $1 \leqslant i < |P|$, and all vertices except $v_1, v_{|P|}$ in it are distinct. By the *length* of path P we mean the number of vertices in P. The first and last vertices visited by P are called the *path-start* and *path-end* of P, denoted by $start(P)$ and $end(P)$, respectively. We will use $v_i \in P$ to denote "P visits vertex v_i" and use $(v_i, v_{i+1}) \in P$ to denote "P visits edge (v_i, v_{i+1})". A path from v_1 to v_k is denoted by (v_1, v_k)-path. In addition, we use P to refer to the set of vertices visited by path P if it is understood without ambiguity. A cycle is a path C with $|V(C)| \geqslant 4$ and $start(C) = end(C)$. Two paths (or cycles) P_1 and P_2 of graph G are called *vertex-disjoint* if $V(P_1) \cap V(P_2) = \emptyset$. Two vertex-disjoint paths P_1 and P_2 can be concatenated into a path, denoted by $P_1 \Rightarrow P_2$, when $end(P_1) \sim start(P_2)$.

Let S^∞ be the infinite graph whose vertex set consists of all points of the plane with integer coordinates and in which two vertices are adjacent if the difference of their x or y coordinates is not larger than 1. A *supergrid graph* is a finite, vertex-induced subgraph of S^{∞}. For a vertex v in a supergrid graph, let v_x and v_y denote x and y coordinates of its corresponding point, respectively. We color vertex v to be *white* if $v_x + v_y \equiv 0 \pmod 2$; otherwise, v is colored to be *black*. Then there are eight possible neighbors of vertex v including four white vertices and four black vertices. Obviously, all supergrid graphs are not bipartite. However, all grid graphs are bipartite [25].

Rectangular supergrid graphs first appeared in [20], in which the Hamiltonian cycle problem was solved. Let $R(m,n)$ be the supergrid graph whose vertex set $V(R(m,n)) = \{v = (v_x, v_y) | 1 \leqslant v_x \leqslant m \text{ and } 1 \leqslant v_y \leqslant n\}$. That is, $R(m,n)$ contains m columns and n rows of vertices in S^{∞}. A *rectangular supergrid graph* is a supergrid graph which is isomorphic to $R(m,n)$ for some m and n. Then m and n, the *dimensions*, specify a rectangular supergrid graph up to isomorphism. The size of $R(m,n)$ is defined to be mn, and $R(m,n)$ is called n-rectangle. $R(m,n)$ is called *even-sized* if mn is even, and it is called *odd-sized* otherwise. In this paper, without loss of generality we will assume that $m \geqslant n$.

Let $v = (v_x, v_y)$ be a vertex in $R(m,n)$. The vertex v is called the *upper-left* (resp., *upper-right*, *down-left*, *down-right*) *corner* of $R(m,n)$ if for any vertex $w = (w_x, w_y) \in R(m,n)$, $w_x \geqslant v_x$ and $w_y \geqslant v_y$ (resp., $w_x \leqslant v_x$ and $w_y \geqslant v_y$, $w_x \geqslant v_x$ and $w_y \leqslant v_y$, $w_x \leqslant v_x$ and $w_y \leqslant v_y$). The edge (u,v) is said to be *horizontal* (resp., *vertical*) if $u_y = v_y$ (resp., $u_x = v_x$), and is called *crossed* if it is neither a horizontal nor a vertical edge. In the figures we will assume that $(1,1)$ are coordinates of the upper-left corner in a rectangular supergrid graph $R(m,n)$. There are four boundaries in a rectangular supergrid graph $R(m,n)$ with $m,n \geqslant 2$. The edge in the boundary of $R(m,n)$ is called *boundary edge*. A path is called *boundary* of $R(m,n)$ if it visits all vertices of the same boundary in $R(m,n)$ and its length equals to the number of vertices in the visited boundary.

A *L-shaped supergrid graph*, denoted by $L(m,n;k,l)$, is a supergrid graph obtained from a rectangular supergrid graph $R(m,n)$ by removing its subgraph $R(k,l)$ from the upper right corner, where $k,l \geqslant 1$ and $m,n > 1$. Then, $m-k \geqslant 1$ and $n-l \geqslant 1$. The structure of $L(m,n;k,l)$ can be found in Fig. 1(a). The parameters $m-k$ and $n-l$ are used to adjust the width and height of $L(m,n;k,l)$, respectively.

In [20], we have showed that rectangular supergrid graphs always contain Hamiltonian cycles except 1-rectangles. Let $R(m,n)$ be a rectangular supergrid graph with $m \geqslant n$, C be a cycle of $R(m,n)$, and let H be a boundary of $R(m,n)$, where H is a subgraph of $R(m,n)$. The restriction of C to H is denoted by $C_{|H}$. If $|C_{|H}| = 1$, i.e., $C_{|H}$ is a boundary path on H, then $C_{|H}$ is called *flat face* on H. If $|C_{|H}| > 1$ and $C_{|H}$ contains at least one boundary edge of H, then $C_{|H}$ is called *concave face* on H. A Hamiltonian cycle of $R(m,3)$ is called *canonical* if it contains three flat faces on two shorter boundaries and one longer boundary, and it contains one concave face on the other boundary, where the shorter boundary consists of three vertices. And, a Hamiltonian cycle of $R(m,n)$ with $n = 2$ or $n \geqslant 4$ is said to be *canonical* if it contains three flat faces on three boundaries, and it contains one concave face on the other boundary. The following lemma states the result in [20] concerning the Hamiltonicity of rectangular supergrid graphs.

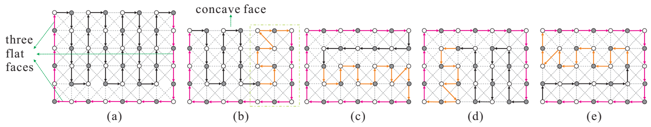

Fig. 2. A canonical Hamiltonian cycle containing three flat faces and one concave face for (a) $R(8,6)$ and (b)–(e) $R(7,5)$, where solid arrow lines indicate the edges in the cycles and $R(7,5)$ contains four distinct canonical Hamiltonian cycles in (b)–(e) such that their concave faces are placed on different boundaries.

Lemma 1. *(See [20].) Let $R(m,n)$ be a rectangular supergrid graph with $m \geqslant n \geqslant 2$. Then, the following statements hold true:*
(1) *if $n = 3$, then $R(m,3)$ contains a canonical Hamiltonian cycle;*
(2) *if $n = 2$ or $n \geqslant 4$, then $R(m,n)$ contains four canonical Hamiltonian cycles with concave faces being on different boundaries.*

Fig. 2 shows canonical Hamiltonian cycles for even-sized and odd-sized rectangular supergrid graphs found in Lemma 1. Each Hamiltonian cycle found by this lemma contains all the boundary edges on any three sides of the rectangular supergrid graph. This shows that for any rectangular supergrid graph $R(m,n)$ with $m \geqslant n \geqslant 4$, we can always construct four canonical Hamiltonian cycles such that their concave faces are placed on different boundaries. For instance, the four distinct canonical Hamiltonian cycles of $R(7,5)$ are shown in Fig. 2(b)–(e), where the concave faces of these four canonical Hamiltonian cycles are arranged on different boundaries.

Let (G,s,t) denote the supergrid graph G with two specified distinct vertices s and t. Without loss of generality, we will assume that $s_x \leqslant t_x$. We denote a Hamiltonian path between s and t in G by $HP(G,s,t)$. We say that $HP(G,s,t)$ does exist if there is a Hamiltonian (s,t)-path in G. From Lemma 1, we know that $HP(R(m,n),s,t)$ does exist if $m,n \geqslant 2$ and (s,t) is an edge in the constructed Hamiltonian cycle of $R(m,n)$.

Recently, we verify the Hamiltonian connectivity of rectangular supergrid graphs except one condition [24]. The forbidden condition for $HP(R(m,n),s,t)$ holds only for 1-rectangle or 2-rectangle. To describe the exception condition, we define the vertex cut and cut vertex of a graph as follows.

Definition 1. Let G be a connected graph and let V_1 be a subset of the vertex set $V(G)$. V_1 is a *vertex cut* of G if $G - V_1$ is disconnected. A vertex v of G is a *cut vertex* of G if $\{v\}$ is a vertex cut of G. For an example, in Fig. 3(b) $\{s,t\}$ is a vertex cut and in Fig. 3(a) t is a cut vertex.

Then, the following condition implies $HP(R(m,1),s,t)$ and $HP(R(m,2),s,t)$ do not exist.

(F1) s or t is a cut vertex of $R(m,1)$, or $\{s,t\}$ is a vertex cut of $R(m,2)$ (see Fig. 3(a) and Fig. 3(b)). Notice that, here, s or t is a cut vertex of $R(m,1)$ if either s or t is

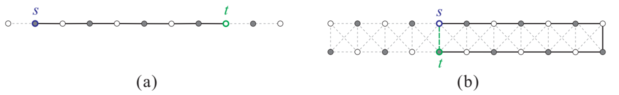

Fig. 3. Rectangular supergrid graph in which there is no Hamiltonian (s,t)-path for (a) $R(m,1)$, and (b) $R(m,2)$, where solid lines indicate the longest path between s and t.

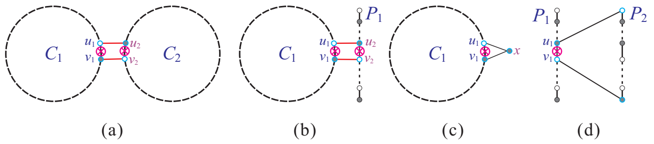

Fig. 4. A schematic diagram for (a) Proposition 1, (b) Proposition 2, (c) Proposition 3, and (d) Proposition 4, where ⊗ represents the destruction of an edge while constructing a combined cycle or a path.

not a corner vertex, and $\{s,t\}$ is a vertex cut of $R(m,2)$ if $2 \leqslant s_x(=t_x) \leqslant m-1$.

The following lemma showing that $HP(R(m,n),s,t)$ does not exist if $(R(m,n),s,t)$ satisfies condition (F1) can be verified by the same arguments in [31].

Lemma 2. *(See [31].) Let $R(m,n)$ be a rectangular supergrid graph with two distinct vertices s and t. If $(R(m,n),s,t)$ satisfies condition (F1), then $(R(m,n),s,t)$ contains no Hamiltonian (s,t)-path.*

In [24], we obtain the following lemma to show the Hamiltonian connectivity of rectangular supergrid graphs.

Lemma 3. *Let $R(m,n)$ be a rectangular supergrid graph with $m,n \geqslant 1$, and let s and t be its two distinct vertices. If $(R(m,n),s,t)$ does not satisfy forbidden condition (F1), then $HP(R(m,n),s,t)$ does exist.*

The Hamiltonian (s,t)-path P of $R(m,n)$ constructed in [24] satisfies that P contains at least one boundary edge of each boundary, and is called *canonical*.

We next give some observations on the relations among cycle, path, and vertex. These propositions will be used in proving our results and are given in [20], [21], [24]. Let C_1 and C_2 be two vertex-disjoint cycles of a graph G. If there exist two edges $e_1=(u_1,v_1)\in C_1$ and $e_2=(u_2,v_2)\in C_2$ such that $e_1 \approx e_2$, then C_1 and C_2 can be merged into a cycle of G. Thus the following proposition holds true.

Proposition 1. *Let C_1 and C_2 be two vertex-disjoint cycles of a graph G. If there exist two edges $e_1\in C_1$ and $e_2\in C_2$ such that $e_1\approx e_2$, then C_1 and C_2 can be combined into a cycle of G. (see Fig. 4(a))*

Let C_1 be a cycle and let P_1 be a path in a graph G such that $V(C_1)\cap V(P_1)=\emptyset$. If there exist two edges $e_1\in C_1$ and $e_2\in P_1$ such that $e_1\approx e_2$, then C_1 and P_1 can be combined into a path P of G with $start(P)=start(P_1)$ and $end(P)=end(P_1)$. Fig. 4(b) depicts such a construction, and hence the following proposition holds true.

Proposition 2. *(See [21].) Let C_1 and P_1 be a cycle and a path, respectively, of a graph G such that $V(C_1)\cap V(P_1)=\emptyset$. If there exist two edges $e_1\in C_1$ and $e_2\in P_1$ such that $e_1\approx e_2$, then C_1 and P_1 can be combined into a path of G. (see Fig. 4(b))*

The above observation can be extended to a vertex x, where $P_1=x$, as shown in Fig. 4(c), and we then have the following proposition.

Proposition 3. *(See [21].) Let C_1 be a cycle (path) of a graph G and let x be a vertex in $G-V(C_1)$. If there exists an edge (u_1,v_1) in C_1 such that $u_1\sim x$ and $v_1\sim x$, then C_1 and x can be combined into a cycle (path) of G. (see Fig. 4(c))*

Let P_1 and P_2 be two vertex-disjoint paths of a graph G. If there exists one edges $(u_1,v_1)\in P_1$ such that $u_1\sim start(P_2)$ and $v_1\sim end(P_2)$, then P_1 and P_2 can be combined into a path P of G with $start(P)=start(P_1)$ and $end(P)=end(P_1)$. Hence, the following observation is immediately true.

Proposition 4. *Let P_1 and P_2 be two vertex-disjoint paths of a graph G. If there exists one edge $(u_1,v_1)\in P_1$ such that $u_1\sim start(P_2)$ and $v_1\sim end(P_2)$, then P_1 and P_2 can be combined into a path of G. (see Fig. 4(d))*

III. The Hamiltonian and Hamiltonian Connected Properties of L-shaped Supergrid Graphs

In this section, we will verify the Hamiltonicity and Hamiltonian connectivity of L-shaped supergrid graphs. We begin with the following definition.

Definition 2. Let $\mathcal{L}$ be a L-shaped supergrid graph $L(m,n;k,l)$ or a rectangular supergrid graph $R(m,n)$. A *separation operation* of $\mathcal{L}$ is a partition of $\mathcal{L}$ into two vertex disjoint rectangular supergrid subgraphs $\mathcal{L}_1$ and $\mathcal{L}_2$, i.e., $V(\mathcal{L})=V(\mathcal{L}_1)\cup V(\mathcal{L}_2)$ and $V(\mathcal{L}_1)\cap V(\mathcal{L}_2)=\emptyset$. A separation is called *vertical* if it consists of a set of horizontal edges, and is called *horizontal* if it contains a set of vertical edges. For an example, the bold dashed vertical (resp., horizontal) line in Fig. 5(a) indicates a vertical (resp., horizontal) separation of $L(10,11;9,7)$ which partitions it into $R(3,11)$ and $R(7,2)$ (resp., $R(3,9)$ and $R(10,2)$).

A. The Hamiltonian Property of L-shaped Supergrid Graphs

In this subsection, we will verify the Hamiltonicity of L-shaped supergrid graphs. Obviously, $L(m,n;k,l)$ contains no Hamiltonian cycle if there exists a vertex w in $L(m,n;k,l)$ such that $deg(w)=1$. Thus, $L(m,n;k,l)$ is not Hamiltonian when the following condition is satisfied.

(F2) there exists a vertex w in $L(m,n;k,l)$ such that $deg(w)=1$.

When the above condition is satisfied, $m-k=1$ or $n-l=1$. We then show the Hamiltonicity of L-shaped supergrid graphs as follows.

Theorem 1. *Let $L(m,n;k,l)$ be a L-shaped supergrid graph. Then, $L(m,n;k,l)$ contains a Hamiltonian cycle if it does not satisfy condition (F2).*

Proof: We first make a vertical separation on $L(m,n;k,l)$ to obtain two disjoint rectangular supergrid subgraphs $L_1=R(m-k,n)$ and $L_2=R(k,n-l)$, as depicted in Fig. 5(b). We prove this theorem by constructing

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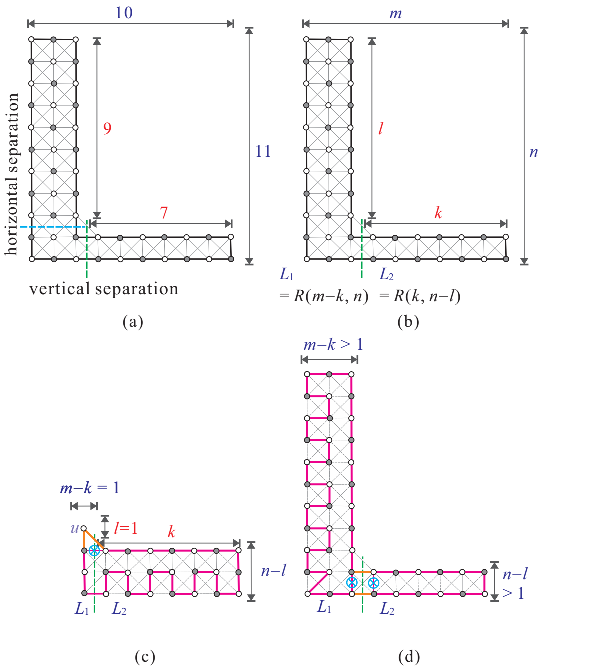

Fig. 5. (a) Separations of $L(10, 11; 9, 7)$, (b) a vertical separation for $L(m, n; k, l)$ to obtain $L_1 = R(m-k, n)$ and $L_2 = R(k, l)$, (c) a Hamiltonian cycle of $L(m, n; k, l)$ when $m-k = 1$, and (d) a Hamiltonian cycle of $L(m, n; k, l)$ when $m - k \geqslant 2$, $n - l \geqslant 2$, and $k \geqslant 2$, where the bold dashed vertical (resp., horizontal) line in (a) indicates a vertical (resp., horizontal) separation of $L(10, 11; 9, 7)$ which partitions it into $R(3, 11)$ and $R(7, 2)$ (resp., $R(3, 9)$ and $R(10, 2)$), and $\otimes$ represents the destruction of an edge while constructing a Hamiltonian cycle of $L(m, n; k, l)$.

a Hamiltonian cycle of $L(m, n; k, l)$. Depending on the sizes of L_1 and L_2, we consider the following cases:

Case 1: $m - k = 1$ or $n - l = 1$. Suppose that $m - k = 1$. Since there exists no vertex w in $L(m, n; k, l)$ such that $deg(w) = 1$, we get that $l = 1$. Consider that $n - l = 1$. Then, $k = 1$. Thus, $L(m, n; k, l)$ consists of only three vertices which forms a cycle. On the other hand, consider that $n - l \geqslant 2$. Let u be a vertex of L_1 with $deg(u) = 2$, $L_1^* = L_1 - \{u\}$, and let $L^* = L_1^* \cup L_2$. Then, $L^* = R(k + 1, n - l)$, where $k + 1 \geqslant 2$ and $n - l \geqslant 2$. By Lemma 1, L^* contains a canonical Hamiltonian cycle HC^*. Then, there exists a flat face of HC^* that is placed to face u. Thus, there exists an edge (x, y) in HC^* such that $u \sim x$ and $u \sim y$. By Proposition 3, u and HC^* can be combined into a Hamiltonian cycle of $L(m, n; k, l)$. For example, Fig. 5(c) depicts such a construction of Hamiltonian cycle of $L(m, n; k, l)$, where $m - k = 1$ and $n - l \geqslant 2$. The case of $n - l = 1$ can be proved by the same arguments. Thus, $L(m, n; k, l)$ is Hamiltonian when $m - k = 1$ or $n - l = 1$.

Case 2: $m - k \geqslant 2$ and $n - l \geqslant 2$. In this case, $L_1 = R(m - k, n)$ and $L_2 = R(k, n - l)$ satisfy that $m - k \geqslant 2$ and $n - l \geqslant 2$. Since $n - l > 1$ and $l \geqslant 1$, $n > l + 1 \geqslant 2$. Thus, $L_1 = R(m - k, n)$ satisfies that $m - k \geqslant 2$ and $n \geqslant 3$. By Lemma 1, L_1 contains a canonical Hamiltonian cycle HC_1 whose one flat face is placed to face L_2. Consider that $k = 1$. Then, $L_2 = R(k, n - l)$ is a 1-rectangle. Let $V(L_2) = \{v_1, v_2, \cdots, v_{n-l}\}$, where v_{i_y} is the y-coordinate of v_i and $v_{i+1_y} = v_{i_y} + 1$ for $n - l - 1 \geqslant i \geqslant 1$. Since HC_1 contains a flat face that is placed to face L_2, there exists an edge (u, v) in HC_1 such that $u \sim v_1$ and $v \sim v_1$. By Proposition 3, v_1 and HC_1 can be combined into a cycle HC_1^*. By the same arguments, v_2, v_3, $\cdots$, v_{n-l} can be merged into the cycle to form a Hamiltonian cycle of $L(m, n; k, l)$. On the other hand, consider that $k \geqslant 2$. Then, $L_2 = R(k, n - l)$ satisfies that $k \geqslant 2$ and $n - l \geqslant 2$. By Lemma 1, L_2 contains a canonical Hamiltonian cycle HC_2 such that one flat face of HC_2 is placed to face L_1. Then, there exist two edges $e_1 = (u_1, v_1) \in HC_1$ and $e_2 = (u_2, v_2) \in HC_2$ such that $e_1 \approx e_2$. By Proposition 1, HC_1 and HC_2 can be combined into a Hamiltonian cycle of $L(m, n; k, l)$. For instance, Fig. 5(d) shows a Hamiltonian cycle of $L(m, n; k, l)$ when $m - k \geqslant 2$, $n - l \geqslant 2$, and $k \geqslant 2$. Thus, $L(m, n; k, l)$ contains a Hamiltonian cycle in this case.

We have proved that $L(m, n; k, l)$ is Hamiltonian in any case. Thus, the lemma holds true. ■

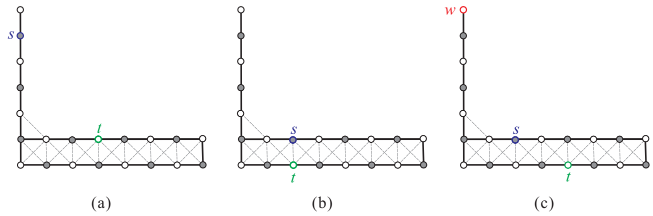

Fig. 6. L-shaped supergrid graph in which there is no Hamiltonian (s, t)-path for (a) s is a cut vertex, (b) $\{s, t\}$ is a vertex cut, and (c) there exists a vertex w such that $deg(w) = 1$, $s \neq w$, and $t \neq w$.

B. *The Hamiltonian Connected Property of L-shaped Supergrid Graphs*

In this subsection, we will verify the Hamiltonian connectivity of L-shaped supergrid graphs. By the same forbidden condition (F1) for $HP(R(m, n), s, t)$, the following condition implies $HP(L(m, n; k, l), s, t)$ does not exist.

(F3) s or t is a cut vertex of $L(m, n; k, l)$, or $\{s, t\}$ is a vertex cut of $L(m, n; k, l)$ (see Fig. 6(a) and Fig. 6(b)).

The following lemma showing that $HP(L(m, n; k, l), s, t)$ does not exist if $(L(m, n; k, l), s, t)$ satisfies condition (F3) can be verified by the same arguments in [31].

Lemma 4. *(See [31].) Let $L(m, n; k, l)$ be a L-shaped supergrid graph with two distinct vertices s and t. If $(L(m, n; k, l), s, t)$ satisfies condition (F3), then $L(m, n; k, l)$ contains no Hamiltonian (s, t)-path.*

We can easily see that $HP(L(m, n; k, l), s, t)$ does not exist if $(L(m, n; k, l), s, t)$ satisfies the following condition.

(F4) there exists a vertex w in $L(m, n; k, l)$ such that $deg(w) = 1$, $s \neq w$, and $t \neq w$ (see Fig. 6(c)).

We will prove that $HP(L(m, n; k, l), s, t)$ does exist when $(L(m, n; k, l), s, t)$ does not satisfy conditions (F3) and (F4) in Theorem 2. Due to the space limitation, we omit its proof.

Theorem 2. *Let $L(m, n; k, l)$ be a L-shaped supergrid graph with distinct vertices s and t. Then, $L(m, n; k, l)$ contains a Hamiltonian (s, t)-path, i.e., $HP(L(m, n; k, l), s, t)$ does exist, if it does not satisfy conditions (F3) and (F4).*

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IV. Concluding Remarks

Based on the Hamiltonicity and Hamiltonian connectivity of rectangular supergrid graphs, we prove L-shaped supergrid graphs to be Hamiltonian and Hamiltonian connected except one or two conditions. The result can be applied to C-shaped supergrid graphs. We leave it to interesting readers. On the other hand, the Hamiltonian cycle problem on solid grid graphs was known to be polynomial solvable. However, it remains open for solid supergrid graphs in which there exists no hole.

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