Classes of Ordinary Differential Equations Obtained for the Probability Functions of inverse Rayleigh Distribution

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***Abstract*— Differential calculus was used to obtain the ordinary differential equations (ODE) of the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of inverse Rayleigh distribution. The parameters and support that define the distribution inevitably determine the nature, existence, uniqueness and solution of the ODEs. The method can be extended to other probability distributions, functions and can serve an alternative to estimation and approximation. Computer codes and programs can be used for the implementation.**

***Index Terms*— Differentiation, quantile function, survival function, approximation, hazard function, Rayleigh.**

I. Introduction

CALCULUS is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The method of maximum likelihood is an example.

Differential equations often arise from the understanding and modeling of real life problems or some observed physical phenomena. Approximations of probability functions are one of the major areas of application of calculus and ordinary differential equations in mathematical statistics. The approximations are helpful in the recovery of the probability functions of complex distributions [1-10].

Apart from mode estimation, parameter estimation and approximation, probability density function (PDF) of probability distributions can be expressed as ODE whose solution is the PDF. Some of which are available. They include: beta distribution [11], Lomax distribution [12], beta prime distribution [13], Laplace distribution [14] and raised cosine distribution [15].

The aim of this research is to develop homogenous ordinary differential equations for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of inverse Rayleigh distribution. This will also help to provide the answers as to whether there are discrepancies between the support of the distribution and the necessary conditions for the existence of the ODEs. Similar results for other distributions have been proposed, see [16-28] for details.

Inverse or inverted Raylegh distribution was earlier studied by [29]. Estimation is one of the areas of the distribution that have been studied extensively. The details can be seen in the research outputs of [30-36]. In particular, emphasis was placed on comparison of the efficiency of different estimators [37-39]. Estimation under censoring features prominently in the works of [40-42]. Prakash [43] work was based strictly on Bayes estimation of the parameters of the distribution. Other areas already explored are acceptance sampling based on the distribution [44-45]. Rosaiah et al. [46] applied the distribution to economic reliability analysis. Recently a new approach of correctly estimating the probability density function (PDF) and cumulative distribution function (CDF) of the distribution was proposed by [47]. Process capability and system availability analysis of the distribution was described by [48]. Generalizations, compounding and modifications include: Bivariate inverse Rayleigh distribution [49], beta inverse Rayleigh distribution [50], discrete inverse Rayleigh distribution [51], transmuted inverse Rayleigh distribution [52], transmuted modified inverse Rayleigh distribution [53], modified inverse Rayleigh distribution [54], Kumaraswamy inverse Rayleigh distribution [55], mixture of inverse Rayleigh distribution [56] and others.

The ordinary differential calculus was used to obtain the results.

II. Probability Density Function

The probability density function (PDF) of the inverse Rayleigh distribution is given by;

$$f(x)=\frac{2\theta}{x^3}\mathrm{e}^{-\frac{\theta}{x^2}} \tag{1}$$

Differentiate equation (1), to obtain;

$$f'(x)=\left\{-\frac{3x^{-4}}{x^{-3}}-\frac{2\theta\,\mathrm{e}^{-\frac{\theta}{x^2}}}{x^3\,\mathrm{e}^{-\frac{\theta}{x^2}}}\right\}f(x) \tag{2}$$

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$$f'(x) = -\left(\frac{3}{x} + \frac{2\theta}{x^3}\right) f(x) \quad (3)$$

The condition necessary for the existence of the equation is $x, \theta > 0.$

The first order ordinary differential equation of the probability density function of the inverse Rayleigh distribution is given as;

$$x^3 f'(x) + (3x^2 + 2\theta) f(x) = 0 \quad (4)$$

$$f(1) = 2\theta e^{-\theta} \quad (5)$$

III. Quantile Function

The Quantile function (QF) of the inverse Rayleigh distribution is given by;

$$Q(p) = \sqrt{\frac{\theta}{\ln\left(\frac{1}{p}\right)}} \quad (6)$$

Differentiate equation (6), to obtain;

$$Q'(p) = \frac{\theta}{2p\left(\ln\left(\frac{1}{p}\right)\right)^2 \sqrt{\frac{\theta}{\ln\left(1/p\right)}}} \quad (7)$$

The condition necessary for the existence of the equation is $\theta > 0, 0 < p < 1.$

Substitute equation (6) into (7);

$$Q'(p) = \frac{\theta}{2p\left(\ln\left(\frac{1}{p}\right)\right)^2 Q(p)} \quad (8)$$

Equation (6) can also be written as;

$$Q^2(p) = \frac{\theta}{\ln\left(\frac{1}{p}\right)} \Rightarrow \ln\left(\frac{1}{p}\right) = \frac{\theta}{Q^2(p)} \quad (9)$$

$$\left(\ln\left(\frac{1}{p}\right)\right)^2 = \frac{\theta^2}{Q^4(p)} \quad (10)$$

Substitute equation (10) into (8);

$$Q'(p) = \frac{Q^3(p)}{2\theta p} \quad (11)$$

The first order ordinary differential equation of the Quantile function of the inverse Rayleigh distribution is given as;

$$2\theta p Q'(p) - Q^3(p) = 0 \quad (12)$$

$$Q(0.1) = \sqrt{\frac{\theta}{\ln 10}} = 0.659\sqrt{\theta} \quad (13)$$

IV. Survival Function

The Survival function (SF) of the inverse Rayleigh distribution is given by;

$$S(t) = 1 - e^{-\frac{\theta}{t^2}} \quad (14)$$

Differentiate equation (14), to obtain;

$$S'(t) = \frac{2\theta}{t^3} e^{-\frac{\theta}{t^2}} \quad (15)$$

The condition necessary for the existence of the equation is $t, \theta > 0.$

Substitute equation (14) into (15);

$$S'(t) = \frac{2\theta}{t^3}(1 - S(t)) \quad (16)$$

The first order ordinary differential equation of the survival function of the inverse Rayleigh distribution is given as;

$$t^3 S'(t) + 2\theta S(t) - 2\theta = 0 \quad (17)$$

$$S(1) = 1 - e^{-\theta} \quad (18)$$

V. Inverse Survival Function

The inverse survival function (ISF) of the inverse Rayleigh distribution is given by;

$$Q(p) = \sqrt{\frac{\theta}{\ln\left(\frac{1}{1-p}\right)}} \quad (19)$$

Differentiate equation (19), to obtain;

$$Q'(p) = -\frac{\theta}{2(1-p)} \sqrt{\frac{\theta}{\left(\ln\left(\frac{1}{1-p}\right)\right)^3}} \quad (20)$$

The condition necessary for the existence of the equation is $\theta > 0, 0 < p < 1.$

Equation (19) can also be written as;

$$Q^2(p) = \frac{\theta}{\ln\left(\frac{1}{1-p}\right)} \Rightarrow \ln\left(\frac{1}{1-p}\right) = \frac{\theta}{Q^2(p)} \quad (21)$$

$$\left(\ln\left(\frac{1}{1-p}\right)\right)^3 = \frac{\theta^3}{Q^6(p)} \quad (22)$$

Substitute equation (22) into (20);

$$Q'(p) = -\frac{Q^3(p)}{2\theta(1-p)} \quad (23)$$

The first order ordinary differential equation of the inverse survival function of the inverse Rayleigh distribution is given as;

$$2\theta(1-p)Q'(p) + Q^3(p) = 0 \quad (24)$$

$$Q(0.11) = \sqrt{\frac{\theta}{0.116533}} \quad (25)$$

VI. Hazard Function

The hazard function (HF) of the inverse Rayleigh distribution is given by;

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$$h(t)=\frac{2\theta\,\mathrm{e}^{-\frac{\theta}{t^2}}}{t^3(1-\mathrm{e}^{-\frac{\theta}{t^2}})} \tag{26}$$

Differentiate equation (26), to obtain;

$$h'(t)=\left\{-\frac{2\theta\,\mathrm{e}^{-\frac{\theta}{t^2}}}{t^3\,\mathrm{e}^{-\frac{\theta}{t^2}}}-\frac{3t^{-4}}{t^{-3}}-\frac{2\theta\,\mathrm{e}^{-\frac{\theta}{t^2}}(1-\mathrm{e}^{-\frac{\theta}{t^2}})^{-2}}{t^3(1-\mathrm{e}^{-\frac{\theta}{t^2}})^{-1}}\right\}h(t) \tag{27}$$

$$h'(t)=-\left\{\frac{2\theta}{t^3}+\frac{3}{t}+\frac{2\theta\,\mathrm{e}^{-\frac{\theta}{t^2}}}{t^3(1-\mathrm{e}^{-\frac{\theta}{t^2}})}\right\}h(t) \tag{28}$$

The condition necessary for the existence of the equation is $t,\theta>0$.

$$h'(t)=-\left\{\frac{2\theta}{t^3}+\frac{3}{t}+h(t)\right\}h(t) \tag{29}$$

The first order ordinary differential equation of the hazard function of the inverse Rayleigh distribution is given as;

$$t^3h'(t)+t^3h^2(t)+(3t^2+2\theta)h(t)=0 \tag{30}$$

$$h(1)=\frac{2\theta\,\mathrm{e}^{-\theta}}{t^3(1-\mathrm{e}^{-\theta})}=\frac{2\theta}{t^3(\mathrm{e}^{\theta}-1)} \tag{31}$$

VII. Reversed Hazard Function

The reversed hazard function (RHF) of the inverse Rayleigh distribution is given by;

$$j(t)=\frac{2\theta}{t^3} \tag{32}$$

Differentiate equation (32), to obtain;

$$j'(t)=-\frac{6\theta}{t^4} \tag{33}$$

The condition necessary for the existence of the equation is $t,\theta>0$.

Substitute equation (32) into equation (33);

$$j'(t)=-\frac{3j(t)}{t} \tag{34}$$

The first order ordinary differential equation of the reversed hazard function of the inverse Rayleigh distribution is given as;

$$tj'(t)+3j(t)=0 \tag{35}$$

$$j(1)=2\theta \tag{36}$$

VIII. Concluding Remarks

Ordinary differential equations (ODEs) has been obtained for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of inverse Rayleigh distribution. This differential calculus and efficient algebraic simplifications were used to derive the various classes of the ODEs. The parameter and the supports that characterize the inverse Rayleigh distribution determine the nature, existence, orientation and uniqueness of the ODEs. The results are in agreement with those available in scientific literature. Furthermore several methods can be used to obtain desirable solutions to the ODEs [57-62]. This method of characterizing distributions cannot be applied to distributions whose PDF or CDF are either not differentiable or the domain of the support of the distribution contains singular points.

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